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On the Analytical Solution of Conformance Fractional Problems and Some Applications

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Declaration

I declare that the master thesis entitled (On the Analytical Solution of Conformance Fractional Problems and Some Applications) is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Dedications

I dedicate my thesis to myself and my parents, husband, childern, friends, sisters and brother who supported me on each step of the way.

Acknowledgements

In the name of Allah, the most Gracious, most Merciful.
First and foremost, I thank ALLAH for bestowing me with health, patience, and knowledge to complete this thesis and without ALLAH's grace, we couldn't have done it. So to ALLAH returns all the praise and gratitude.

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Abstract

This survey treats the new simpler and more efficient definition of fractional derivative which is called “ conformable fractional derivative ”. The new definition reflects a nature extension of normal derivative.
Also, the thesis discusses the general analytical exact solutions for some first and second conformable fractional linear/nonlinear differential equations. These equations are: Conformable fractional Bernoulli, Riccatti, Abel and Euler differential equations, some explanatory examples are presented to illustrate the proposed approach. In addition, the systems of conformable linear differential equations with constant coefficients are discussed and give full solution for homogeneous and non-homogeneous systems.

The finite difference method and it's error terms which can be applied to approximate the solution of the fractional differential equations based on conformable fractional derivative definition.
Numerical examples are given to certify the applicability of our proposed method. These numerical examples have proved good results when compared with exact solutions or other known numerical methods.

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Chapter 1

Introduction

Fractional differential equations are very important in many fields like Fluid Mechanics, Biology, Physics, Optics, Electrochemistry of Corrosion, Engineering, Viscoelasticity, Electrical Networks and Control Theory of Dynamic Systems.
Fractional Calculus deals with integrals and derivatives of real or even complex order. It is a generalization of the classical calculus and therefore preserves some of the basic properties.

The objective of the present thesis is to use conformable fractional derivative to solve fractional differential equation.
Most fractional differential equations (FDEs) don't have exact solution, so approximate and numerical techniques. Various numerical and approximate methods to solve the FDEs have been discussed as Euler method, Taylor method of order 2, Modified method and Hence method.

This thesis is divided into three chapters.
Chapter one, which consists of four sections. This chapter gives the two familiar operators of fractional calculus which are: Rieman-Liouville and Caputo operators. It focuses on a new definition of " conformable fractional derivatives " and studies the rules of differentiation and integration. The new definition reflects a nature extension of normal derivative. And we discuses some properties and theory of conformable fractional derivatives. We give some application to fractional differential equations.

Chapter two contains three sections, we present the general analytical exact so-

lutions for first and second orders of conformable fractional linear/nonlinear differential equations.
We discuss the form of the Wronskain for conformable fractional linear differential equations with variable coefficients. Further, we prove that there is an Able's formula for fractional differential equations.
We introduce the general exact solutions of the conformable fractional nonlinear Bernoulli, Riccatti, Abel and Euler differential equations and some special cases are also discussed and solved.
Finally, we discuss systems of conformable fractional linear differential equations with constant coefficients. We give full solution for homogeneous and non-homogeneous systems.

In chapter three we use the conformable fractional derivatives to derive some finite difference formulas and its error terms which are used to solve fractional differential equations. We first derive conformable fractional Euler and Taylor methods based on the fractional Taylor expansion.
Also we drive the conformable fractional Modified and Heun's methods based on Trapezoidal and Simpson's rules.
To provide the contribution of our work, some applications on finite difference formulas are given.
Finally, we compared two formulas the first formula of conformable fractional Euler and Taylor method [1^{st} (CFEM) and (CFTM)] and the second formula of conformable fractional Euler and Taylor method [2^{nd} (CFEM) and (CFTM)] that depend on the conformable fractional derivative with the conformable fractional Euler and Taylor methods that we derived trying to be self-contained.

Chapter 2

Fractional calculus

2.1 Introduction

Fractional calculus is a branch of mathematical analysis that studies the several different possibilities of defining real number powers or complex number powers of the differentiation operator D

$$Df(x) = \frac{d}{dx}f(x),$$

and of the integration operator J

$$Jf(x) = \int_0^x f(s)ds.$$

Fractional calculus developed since 17^{th} century through the pioneering works of Leibniz, Euler, Lagrange, Laplace, Abel, Liouvlle, Riemann and many others ([20],[22]). First reported attempt to generalize derivatives to fractional order is contained in the correspondence of Leibniz (1695) with l'hospital, wherein Leibniz has given interpretation of the symbol $\frac{d^n y}{dx^n}$ for $n = \frac{1}{2}$. Let us assume that $f(x)$ is a monomial of the form

$$f(x) = x^k, \ k \in I\!N.$$

The first derivative is as usual

$$f'(x) = \frac{d}{dx}f(x) = kx^{k-1}. \tag{2.1}$$

The formula for ordinary derivative,

$$\frac{d^n}{dx^n}x^k = \frac{k!}{(k-n)!}x^{k-n}, \ \ k \geq n, \ k \in I\!N. \tag{2.2}$$

For arbitrary order, it follows

$$\frac{d^n}{dx^n}x^k = \frac{\Gamma(k+1)}{(k-n+1)}x^{k-n}, \quad k \in I\!N, \tag{2.3}$$

where $\Gamma(k+1) = k!$. For $k = 1$ and $n = \frac{1}{2}$, we obtain the half-derivative of the function x as

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}x^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}} = \frac{1}{\frac{\sqrt{\pi}}{2}}x^{\frac{1}{2}}. \tag{2.4}$$

We repeat the process to get

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} = \frac{2}{\sqrt{\pi}}\frac{\Gamma(1+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}+1)}x^{\frac{1}{2}-\frac{1}{2}} = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{3}{2})}{\Gamma(1)}x^0 = \frac{2\frac{\sqrt{\pi}}{2}}{\sqrt{\pi}} = 1, \tag{2.5}$$

(since $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ and $\Gamma(1) = 1$) which is indeed the expected result of

$$(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}})x = \frac{d}{dx}x = 1.$$

Fourier (1822) defined fractional operators using integral representation of $f(x)$, namely

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)du \int_{-\infty}^{\infty} \cos(t(x-u))dt.$$

$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)du \int_{-\infty}^{\infty} t^n \cos(t(x-u)+\frac{1}{2}n\pi)dt.$$

He formally replaced n with α, where α is arbitrary numbers [23]. Abel was the first to apply fractional order derivatives to solve an integral equation that arises in tautochrone problem [18] (the problem of determining the shape of the curve such that the time of descent of a frictionless point mass sliding down the curve in the gravitational field is independent of the starting point). If the time of slide is a known constant, then Abel's integral equation is of the form

$$k = \int_0^x (x-t)^\alpha f(t)dt, \tag{2.6}$$

(with $\alpha = \frac{-1}{2}$ as a special case) which were investigated. The integral in eq (2.6), except for the multiplicative factor $\frac{1}{\Gamma(\frac{1}{2})}$, is a particular case of a definite integral defining fractional integration. In the integral equations such as eq (2.6), the function f in the integrand has to be determined. Abel wrote the right-hand side of eq (2.6) as $\sqrt{\pi}[\frac{d^{\frac{-1}{2}} f(x)}{dx^{\frac{-1}{2}}}]$ [24]. Then he operated on both sides of the equation with $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$ to obtain

$$\frac{d^{\frac{1}{2}} k}{dx^{\frac{1}{2}}} = \sqrt{\pi} f(x),$$

because the fractional operators (with suitable conditions on f) have the property

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}[\frac{d^{\frac{-1}{2}}}{dx^{\frac{-1}{2}}} f] = \frac{d^0 f}{dx^0} = f.$$

Thus when the fractional derivative of order $\frac{1}{2}$ of the constant k in eq (2.6) is computed, $f(x)$ is determined. This is a remarkable achievement of Abel in the fractional calculus. It is important to note that the fractional derivative of a constant need not be equal to zero. Fourier's integral formula and Abel's elegant solution attracted Liouville's attention.

For a general function $f(x)$ and $0 < \alpha < 1$, the complete fractional derivative is

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^\alpha} dt.$$

For arbitrary α [9], since the gamma function is undefined for arguments whose real part is a negative integer and whose imaginary part is zero, it is necessary to apply the fractional derivative after the integer derivative has been performed. For example,

$$D^{\frac{3}{2}} f(x) = D^{\frac{1}{2}} D^1 f(x) = D^{\frac{1}{2}} \frac{d}{dx} f(x).$$

2.2 Fractional derivative

There exists a many of definitions of fractional derivative, which have different origins and are not necessarily equivalent. We will report a few of them and some of their properties.

Definition 2.1. *The Riemann-Liouvill of fractional derivative definition is [18]:*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-(n-1)}} dx, \quad n-1 \leq \alpha < n, \quad t > a, \quad n \in I\!N.$$

Definition 2.2. *The Caputo of fractional derivative definition is [18]:*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-(n-1)}} dx, \quad n-1 \leq \alpha < n, \quad t > a, \quad n \in I\!N.$$

The following are properties of Riemann-Liouvill and Caputo derivatives:

1. The Riemann derivative does not satisfy $D_a^\alpha(1) = 0$ while $D_a^\alpha(1) = 0$ for the Caputo derivative, where α is not a natural number.

   Proof. Let $f(x) = 1$, then the Riemann derivative is:

$$\begin{aligned} D_a^\alpha f(x) &= D_a^\alpha(1) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{1}{(t-x)^{\alpha-(n-1)}} dx \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{-1}{y^{\alpha-(n-1)}} dy, \quad \text{since } \ y = (t-x) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left[\frac{-y^{-\alpha+(n-1)+1}}{-\alpha+(n-1)+1} \Big|_a^t \right] \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left[\frac{-(t-x)^{-\alpha+n}}{-\alpha+n} \Big|_a^t \right] \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left[\frac{(t-a)^{-\alpha+n}}{-\alpha+n} \right]. \end{aligned}$$

   When $n = 1$, $0 < \alpha < 1$,

$$\begin{aligned} D_a^\alpha(1) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{(t-a)^{-\alpha+1}}{-\alpha+1} \right] \\ &= \frac{1}{\Gamma(1-\alpha)} \left[\frac{(-\alpha+1)(t-a)^{-\alpha}}{-\alpha+1} \right] \\ &= \frac{1}{\Gamma(1-\alpha)} (t-a)^{-\alpha} \\ &\neq 0. \end{aligned}$$

   Since $0 < \alpha < 1$, $\Gamma(1-\alpha) \neq 0$ and $t > a$, $(t-a)^{-\alpha} \neq 0$.
   When $n = 2$, $1 < \alpha < 2$,

$$\begin{aligned} D_a^\alpha(1) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{1}{\Gamma(1-\alpha)} (t-a)^{-\alpha} \right] \\ &= \frac{1}{[\Gamma(1-\alpha)]^2} \left[-\alpha(t-a)^{-\alpha-1} \right] \\ &\neq 0. \end{aligned}$$

   Since $1 < \alpha < 2$, $\Gamma(1-\alpha) \neq 0$, $\alpha \neq 0$ and $t > a$, $(t-a)^{-\alpha-1} \neq 0$.

The Caputo derivative is:

$$\begin{aligned}
D_a^\alpha(1) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{(1)'}{(t-x)^{\alpha-(n-1)}}dx \\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{0}{(t-x)^{\alpha-(n-1)}}dx \\
&= \frac{1}{\Gamma(n-\alpha)}C.
\end{aligned}$$

Where C is constant. If $C = 0$, then $D_a^\alpha(1) = 0$. ∎

2. Both definitions do not satisfy the following:

   (a) $D_a^\alpha(fg) = fD_a^\alpha(g) + gD_a^\alpha(f)$.

   (b) $D_a^\alpha(f/g) = \frac{gD_a^\alpha(f) - fD_a^\alpha(g)}{g^2}$.

   (c) $D_a^\alpha(f \circ g) = f^\alpha(g(t))g^\alpha(t)$.

   (d) $D^\alpha D^\beta f = D^{\alpha+\beta} f$.

3. The Caputo definition assumes that the function f is differentiable.

4. Both definitions satisfy $D_a^\alpha(f \pm g) = D_a^\alpha(f) \pm D_a^\alpha(g)$.

   Proof. Riemann derivative:

$$\begin{aligned}
D_a^\alpha(f \pm g)(x) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{(f \pm g)(x)}{(t-x)^{\alpha-(n-1)}}dx \\
&= \frac{1}{\Gamma(n-\alpha)}\left[\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-(n-1)}}dx \pm \frac{d^n}{dt^n}\int_a^t \frac{g(x)}{(t-x)^{\alpha-(n-1)}}dx\right] \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-(n-1)}}dx \pm \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{g(x)}{(t-x)^{\alpha-(n-1)}}dx \\
&= D_a^\alpha f(x) \pm D_a^\alpha g(x).
\end{aligned}$$

Caputo derivative:

$$\begin{aligned}
D_a^\alpha(f \pm g)(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(f \pm g)'(x)}{(t-x)^{\alpha-(n-1)}} dx \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f'(x) \pm g'(x)}{(t-x)^{\alpha-(n-1)}} dx \\
&= \frac{1}{\Gamma(n-\alpha)} \left[\int_a^t \frac{f'(x)}{(t-x)^{\alpha-(n-1)}} dx \pm \int_a^t \frac{g'(x)}{(t-x)^{\alpha-(n-1)}} dx \right] \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f'(x)}{(t-x)^{\alpha-(n-1)}} dx \pm \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g'(x)}{(t-x)^{\alpha-(n-1)}} dx \\
&= D_a^\alpha f(x) \pm D_a^\alpha g(x).
\end{aligned}$$

□

However, the next definition over come the failure of satisfaction as mentioned.

Definition 2.3. *Given a function $f : [0, \infty) \to I\!R$, then the conformable fractional derivative of f of order α is defined by [16]*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0$ and $\alpha \in (n, n+1)$. If f is α-differentiable in some (0,a), $a > 0$, and $\lim_{t \to 0^+} f^\alpha(t)$ exists, the define $f^\alpha(0) = \lim_{t \to 0^+} f^\alpha(t)$.
T_α is denote the operator which is called the conformable fractional derivative of order α.

Sometime, write $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α.
Note: If the conformable fractional derivative of f of order α exists, then we say that f is α-differentiable.

Theorem 2.4. *If a function $f : [0, \infty) \to I\!R$ is α-differentiable at $t_0 > 0$, $\alpha \in (0, 1]$ then f is continuous at t_0 [20].*

Proof. Let us consider the identity

$$f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0) = \frac{f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0)}{\varepsilon} \varepsilon.$$

Applying the limit $\varepsilon \to 0$ on both sides, we get

$$\lim_{\varepsilon \to 0} (f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0)) = \lim_{\varepsilon \to 0} \frac{f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0)}{\varepsilon} \lim_{\varepsilon \to 0} \varepsilon = 0,$$

which implies that $\lim_{\varepsilon\to 0} f(t_0+\varepsilon t_0^{1-\alpha}) = f(t_0)$, let $h=\varepsilon t_0^{1-\alpha}$, then

$$\lim_{h\to 0} f(t_0+h) = f(t_0).$$

Hence f is continuous at t_0. □

Theorem 2.5. *Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t>0$, then [20]:*

1. $T_\alpha(af+bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a,b\in \mathbb{R}$.
2. $T_\alpha(t^p) = pt^{p-\alpha}$, *for all* $p\in\mathbb{R}$.
3. $T_\alpha(\lambda) = 0$, *for all constant function* $f(t)=\lambda$.
4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
5. $T_\alpha(f/g) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
6. *If in addition, f is differentiable then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Proof. 1. To prove it, we use the definition 2.3.

$$\begin{aligned}
T_\alpha(af+bg)(t) &= \lim_{\varepsilon\to 0}\frac{af(t+\varepsilon t^{1-\alpha}) + bg(t+\varepsilon t^{1-\alpha}) - [af(t)+bg(t)]}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{af(t+\varepsilon t^{1-\alpha}) - af(t) + bg(t+\varepsilon t^{1-\alpha}) - bg(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{af(t+\varepsilon t^{1-\alpha}) - af(t)}{\varepsilon} + \lim_{\varepsilon\to 0}\frac{bg(t+\varepsilon t^{1-\alpha}) - bg(t)}{\varepsilon}\\
&= a\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} + b\lim_{\varepsilon\to 0}\frac{g(t+\varepsilon t^{1-\alpha}) - g(t)}{\varepsilon}\\
&= aT_\alpha(f) + bT_\alpha(g).
\end{aligned}$$

2. Assume $f(t)=t^p$, for all $p\in\mathbb{R}$ into the definition of the conformable fractional derivative
$$T_\alpha(t^p) = \lim_{\varepsilon\to 0}\frac{(t+\varepsilon t^{1-\alpha})^p - t^p}{\varepsilon},$$
then use the Binomial theorem to expand out the first term.
$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + ... + nab^{n-1} + b^n.$$

And we get,

$$\begin{aligned}
T_\alpha(t^p) &= \lim_{\varepsilon\to 0}\frac{[t^p + pt^{p-1}\varepsilon t^{1-\alpha} + p\frac{(p-1)}{2!}t^{p-2}(\varepsilon t^{1-\alpha})^2 + \ldots\ldots + pt(\varepsilon t^{1-\alpha})^{p-1} + (\varepsilon t^{1-\alpha})^p - t^p]}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{pt^{p-1}\varepsilon t^{1-\alpha} + p\frac{(p-1)}{2!}t^{p-2}(\varepsilon t^{1-\alpha})^2 + \ldots.. + pt(\varepsilon t^{1-\alpha})^{p-1} + (\varepsilon t^{1-\alpha})^p}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}[pt^{p-1}t^{1-\alpha} + p\frac{(p-1)}{2!}t^{p-2}\varepsilon(t^{1-\alpha})^2 + \ldots.. + pt(\varepsilon)^{p-2}(t^{1-\alpha})^{p-1} + (\varepsilon)^{p-1}(t^{1-\alpha})^p]\\
&= pt^{p-1+1-\alpha}\\
&= pt^{p-\alpha}.
\end{aligned}$$

3. Let $f(t) = \lambda$, λ is constant then

$$\begin{aligned}
T_\alpha(\lambda) &= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{\lambda - \lambda}{\varepsilon}\\
&= 0.
\end{aligned}$$

4. By using definition 2.3, we get

$$\begin{aligned}
T_\alpha(fg)(t) &= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})g(t+\varepsilon t^{1-\alpha}) - f(t)g(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})g(t+\varepsilon t^{1-\alpha}) + f(t+\varepsilon t^{1-\alpha})g(t) - f(t+\varepsilon t^{1-\alpha})g(t) - f(t)g(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})[g(t+\varepsilon t^{1-\alpha}) - g(t)]}{\varepsilon} + \lim_{\varepsilon\to 0}\frac{g(t)[f(t+\varepsilon t^{1-\alpha}) - f(t)]}{\varepsilon}\\
&= \lim_{\varepsilon\to 0} f(t+\varepsilon t^{1-\alpha})\lim_{\varepsilon\to 0}\frac{g(t+\varepsilon t^{1-\alpha}) - g(t)}{\varepsilon} + \lim_{\varepsilon\to 0} g(t)\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}\\
&= f(t)T_\alpha(g)(t) + g(t)T_\alpha(f)(t).
\end{aligned}$$

5. By using definition 2.3, we get

$$
\begin{aligned}
T_\alpha(f/g)(t) &= \lim_{\varepsilon\to 0} \frac{\frac{f(t+\varepsilon t^{1-\alpha})}{g(t+\varepsilon t^{1-\alpha})} - \frac{f(t)}{g(t)}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \frac{f(t+\varepsilon t^{1-\alpha})g(t) - f(t)g(t+\varepsilon t^{1-\alpha})}{g(t+\varepsilon t^{1-\alpha})g(t)} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \frac{f(t+\varepsilon t^{1-\alpha})g(t) - f(t)g(t) + f(t)g(t) - f(t)g(t+\varepsilon t^{1-\alpha})}{g(t+\varepsilon t^{1-\alpha})g(t)} \\
&= \lim_{\varepsilon\to 0} \frac{1}{g(t+\varepsilon t^{1-\alpha})g(t)} \left(\frac{f(t+\varepsilon t^{1-\alpha})g(t) - f(t)g(t)}{\varepsilon} + \frac{f(t)g(t) - f(t)g(t+\varepsilon t^{1-\alpha})}{\varepsilon} \right) \\
&= \lim_{\varepsilon\to 0} \left(\frac{1}{g(t+\varepsilon t^{1-\alpha})g(t)} \left[g(t)\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} - f(t)\frac{g(t+\varepsilon t^{1-\alpha}) - g(t)}{\varepsilon} \right] \right) \\
&= \frac{1}{g(t)g(t)} \left(g(t)T_\alpha(f)(t) - f(t)T_\alpha(g)(t) \right) \\
&= \frac{g(t)T_\alpha(f)(t) - f(t)T_\alpha(g)(t)}{g^2(t)}.
\end{aligned}
$$

6. Let $h = \varepsilon t^{1-\alpha}$, then $\varepsilon = t^{\alpha-1}h$

$$
\begin{aligned}
T_\alpha(f)(t) &= \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \\
&= \lim_{h\to 0} \frac{f(t+h) - f(t)}{ht^{\alpha-1}} \\
&= t^{1-\alpha} \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\
&= t^{1-\alpha} \frac{df}{dt}(t).
\end{aligned}
$$

□

Based on 6. we derive conformable fractional derivative of some special functions:

1. $T_\alpha(e^{ct}) = ct^{1-\alpha}e^{ct} \;\; c \in \mathbb{R}$.

2. $T_\alpha(\sin(bt)) = bt^{1-\alpha}\cos(bt) \;\; b \in \mathbb{R}$.

3. $T_\alpha(\cos(bt)) = -bt^{1-\alpha}\sin(bt) \;\; b \in \mathbb{R}$.

4. $T_\alpha(\frac{1}{\alpha}t^\alpha) = 1$.

Proof. 1. By using definition 2.3 and L'hopital's rule,

$$\begin{aligned}
T_\alpha(e^{ct}) &= \lim_{\varepsilon \to 0} \frac{e^{ct+c\varepsilon t^{1-\alpha}} - e^{ct}}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{e^{ct}[e^{c\varepsilon t^{1-\alpha}} - 1]}{\varepsilon} \\
&= e^{ct} \lim_{\varepsilon \to 0} \frac{e^{c\varepsilon t^{1-\alpha}} - 1}{\varepsilon} \\
&= e^{ct} \lim_{\varepsilon \to 0} \frac{ct^{1-\alpha} e^{c\varepsilon t^{1-\alpha}}}{1} \\
&= ce^{ct} t^{1-\alpha} \lim_{\varepsilon \to 0} e^{c\varepsilon t^{1-\alpha}} \\
&= ce^{ct} t^{1-\alpha} e^{0} \\
&= ce^{ct} t^{1-\alpha}.
\end{aligned}$$

2. By using definition 2.3 we get,

$$\begin{aligned}
T_\alpha(\sin(bt)) &= \lim_{\varepsilon \to 0} \frac{\sin(bt + b\varepsilon t^{1-\alpha}) - \sin(bt)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\sin(bt)\cos(b\varepsilon t^{1-\alpha}) + \cos(bt)\sin(b\varepsilon t^{1-\alpha}) - \sin(bt)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\cos(bt)\sin(b\varepsilon t^{1-\alpha})}{\varepsilon} + \lim_{\varepsilon \to 0} \frac{-\sin(bt)(1 - \cos(b\varepsilon t^{1-\alpha}))}{\varepsilon} \\
&= \cos(bt) \lim_{\varepsilon \to 0} \frac{\sin(b\varepsilon t^{1-\alpha})}{\varepsilon} - \sin(bt) \lim_{\varepsilon \to 0} \frac{(1 - \cos(b\varepsilon t^{1-\alpha}))}{\varepsilon}.
\end{aligned}$$

Let $h = b\varepsilon t^{1-\alpha}$, this implies $\varepsilon = \frac{h}{bt^{1-\alpha}}$ we get,

$$\begin{aligned}
T_\alpha(\sin(bt)) &= \cos(bt) \lim_{h \to 0} \frac{\sin h}{\frac{h}{bt^{1-\alpha}}} - \sin(bt) \lim_{h \to 0} \frac{(1 - \cos h)}{\frac{h}{bt^{1-\alpha}}} \\
&= bt^{1-\alpha} \cos(bt) \lim_{h \to 0} \frac{\sin h}{h} - bt^{1-\alpha} \sin(bt) \lim_{h \to 0} \frac{1 - \cos h}{h}.
\end{aligned}$$

Using $\lim_{x \to 0} \frac{\sin x}{x} = 1$, and use L'hopital's rule we get

$$\begin{aligned}
T_\alpha(\sin(bt)) &= bt^{1-\alpha} \cos(bt) - bt^{1-\alpha} \sin(bt) \lim_{h \to 0} \frac{\sin h}{1} \\
&= bt^{1-\alpha} \cos(bt) - 0 \\
&= bt^{1-\alpha} \cos(bt).
\end{aligned}$$

3. By using definition 2.3 we get,

$$\begin{aligned}
T_\alpha(\cos(bt)) &= \lim_{\varepsilon\to 0}\frac{\cos(bt+b\varepsilon t^{1-\alpha})-\cos(bt)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{\cos(bt)\cos(b\varepsilon t^{1-\alpha})-\sin(bt)\sin(b\varepsilon t^{1-\alpha})-\cos(bt)}{\varepsilon}\\
&= -\sin(bt)\lim_{\varepsilon\to 0}\frac{\sin(b\varepsilon t^{1-\alpha})}{\varepsilon}-\cos(bt)\lim_{\varepsilon\to 0}\frac{(1-\cos(b\varepsilon t^{1-\alpha}))}{\varepsilon}.
\end{aligned}$$

Let $h=b\varepsilon t^{1-\alpha}$, this implies $\varepsilon=\frac{h}{bt^{1-\alpha}}$ we get,

$$T_\alpha(\cos(bt)) = -bt^{1-\alpha}\sin(bt)\lim_{h\to 0}\frac{\sin h}{h}-bt^{1-\alpha}\cos(bt)\lim_{h\to 0}\frac{1-\cos h}{h}.$$

Using $\lim_{x\to 0}\frac{\sin x}{x}=1$, and use L'hopital's rule we get

$$\begin{aligned}
T_\alpha(\cos(bt)) &= -bt^{1-\alpha}\sin(bt)-bt^{1-\alpha}\cos(bt)\lim_{h\to 0}\frac{\sin h}{1}\\
&= -bt^{1-\alpha}\sin(bt).
\end{aligned}$$

4. Using definition 2.3 and L'hopital's rule, we get

$$\begin{aligned}
T_\alpha(\frac{1}{\alpha}t^\alpha) &= \lim_{\varepsilon\to 0}\frac{\frac{1}{\alpha}(t+\varepsilon t^{1-\alpha})^\alpha-\frac{1}{\alpha}t^\alpha}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{\frac{1}{\alpha}\alpha(t+\varepsilon t^{1-\alpha})^{\alpha-1}t^{1-\alpha}-0}{1}\\
&= \lim_{\varepsilon\to 0}(t+\varepsilon t^{1-\alpha})^{\alpha-1}t^{1-\alpha}\\
&= t^{\alpha-1}t^{1-\alpha}\\
&= 1.
\end{aligned}$$

□

However, below is a list of conformable fractional derivatives of certain functions:

1. $T_\alpha(e^{\frac{1}{\alpha}t^\alpha})=e^{\frac{1}{\alpha}t^\alpha}$.
2. $T_\alpha(\sin(\frac{1}{\alpha}t^\alpha))=\cos(\frac{1}{\alpha}t^\alpha)$.
3. $T_\alpha(\cos(\frac{1}{\alpha}t^\alpha))=-\sin(\frac{1}{\alpha}t^\alpha)$.

Definition 2.6. *Let $\alpha\in(n,n+1]$, and f be an n-differentiable at t where $t>0$, then the conformable fractional derivative of f of order α is defined as:*

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f^{[\alpha]-1}(t-\varepsilon t^{[\alpha]-\alpha})-f^{[\alpha]-1}(t)}{\varepsilon}.$$

Where $[\alpha]$ is the smallest integer greater than or equal to α [16].

Lemma 2.7. *If f is $(n+1)$-differentiable at $t>0$, then as a consequence of the definition one can have*

$$T_\alpha(f)(t) = t^{[\alpha]-\alpha} f^{[\alpha]}(t),$$

where $\alpha \in (n, n+1]$ [16].

Proof. f is $(n+1)$-differentiable at $t>0$, then by definition 2.6

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f^{[\alpha]}(t - \varepsilon t^{[\alpha]-\alpha}) - f^{[\alpha]}(t)}{\varepsilon}.$$

Let $h = \varepsilon t^{[\alpha]-\alpha}$, this implies $\varepsilon = \frac{h}{t^{[\alpha]-\alpha}}$ we get

$$\begin{aligned} T_\alpha(f)(t) &= t^{[\alpha]-\alpha} \lim_{h \to 0} \frac{f^{[\alpha]}(t-h) - f^{[\alpha]}(t)}{h} \\ &= t^{[\alpha]-\alpha} f^{[\alpha]}(t). \end{aligned}$$

□

Theorem 2.8. *(Roll's theorem for conformable fractional differentiable functions) Let $a>0$ and $f:[a,b] \to \mathbb{R}$ be a function with the properties that:*

1. *f is continuous on [a,b].*
2. *f is α-differentiable on (a,b) for some $\alpha \in (0,1)$.*
3. *$f(a) = f(b)$.*

Then there exist at least $c \in (a,b)$, such that $f^\alpha(c) = 0$ [17].

Proof. We prove this using contradiction, since f is continuous on $[a,b]$ and $f(a) = f(b)$, there is $c \in (a,b)$, at least one, which is a point of local extreme.
On the other hand, as f is α-differentiable on (a,b) for some $\alpha \in (0,1)$, we have

$$f^\alpha(c) = \lim_{\varepsilon \to 0^+} \frac{f(c + \varepsilon c^{1-\alpha}) - f(c)}{\varepsilon} = \lim_{\varepsilon \to 0^-} \frac{f(c + \varepsilon c^{1-\alpha}) - f(c)}{\varepsilon}.$$

But the two limits have opposite signs, the first limit is non-negative and the second limit is non-positive which is contradiction. Hence $f^\alpha(c) = 0$.
If the two limits have same sign then as $f(a) = f(b)$, we have that f is constant and the result is trivially followed. □

Theorem 2.9. *(Mean value theorem for conformable fractional differentiable functions)*
Let $a > 0$ and $f : [a, b] \to I\!R$ be a continuous function on $[a, b]$ and α-differentiable on (a, b) for some $\alpha \in (0, 1)$. Then there exist at least $c \in (a, b)$, such that $f^{\alpha}(c) = \frac{f(b)-f(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}$ [17].

Proof. Consider the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}(\frac{1}{\alpha}x^{\alpha} - \frac{1}{\alpha}a^{\alpha}).$$

$g(x)$ is continuous function on $[a, b]$ and integrable on $x \in (a, b)$.
$g(a) = f(a) - f(a) - \frac{f(b)-f(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}(\frac{1}{\alpha}a^{\alpha} - \frac{1}{\alpha}a^{\alpha}) = 0.$
$g(b) = f(b) - f(a) - \frac{f(b)-f(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}(\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}) = 0.$
This implies $g(a) = g(b)$, then the function g satisfies the conditions of the fractional Rolle's theorem. Hence there exists $c \in (a, b)$, such that $T_{\alpha}(c) = g^{\alpha}(c) = 0$.

$$g^{\alpha}(c) = f^{\alpha}(c) - \frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}T_{\alpha}(\frac{1}{\alpha}x^{\alpha}).$$

Using the fact that $T_{\alpha}(\frac{1}{\alpha}x^{\alpha}) = 1$, we have

$$g^{\alpha}(c) = f^{\alpha}(c) - \frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}} = 0.$$

This implies $f^{\alpha}(c) = \frac{f(b)-f(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}$. □

Proposition 2.10. *Let $a > 0$ and $f : [a, b] \to I\!R$ be α-differentiable for some $\alpha \in (0, 1)$ [16].*

1. *If f^{α} is bounded on $[a, b]$ where $a > 0$, then f is uniformly continuous on $[a, b]$ and hence f is bounded.*

2. *If f^{α} is bounded on $[a, b]$ and continuous at a, then f is uniformly continuous on $[a, b]$ and hence f is bounded.*

Proof. We want to prove 1.
Given f^{α} is bounded on $[a, b]$, there is a positive constant k such that $|f^{\alpha}| \leq k$ at every point of the interval (a, b).

For any $a < b$ in interval the conditions of the first mean value theorem are met on the subinterval $[a, b]$, so there is a point $c \in (a, b)$ such that

$$f^{\alpha}(c) = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}.$$

f is bounded then $|f^{\alpha}(c)| \leq k$, this implies

$$\left| \frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}} \right| \leq k$$

$$|f(b) - f(a)| \leq k \left| \frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha} \right|,$$

for $\left|\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right| < \delta$, let $\delta = \frac{\varepsilon}{k}$ we will get that

$$|f(b) - f(a)| \leq k\delta = \frac{\varepsilon}{k}$$

$$|f(b) - f(a)| \leq \varepsilon,$$

then there exists $\delta > 0$ such that if $\left|\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right| < \delta$, then $|f(b) - f(a)| \leq \varepsilon$. Therefor f is uniformly continuous on $[a, b]$ and hence f is bounded. $\square$

2.3 Fractional integral

Definition 2.11. *The Riemann-Liouvill of fractional integral of order $\alpha \geq 0$ for a continuous function f on $[a, b]$ which is defined by:*

$$J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) dt,$$

where $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$ is the gamma function [25].

Let $\alpha \in (0, \infty)$, define $J_{\alpha}(t^p) = \frac{t^{p+\alpha}}{p+\alpha}$ for any $p \in I\!R$ and $\alpha \neq -p$.
If $f(t) = \sum_{k=0}^{n} b_k t^k$, then we define $J_{\alpha}(f) = \sum_{k=0}^{n} b_k J_{\alpha}(t^k) = \sum_{k=0}^{n} b_k \frac{t^{k+\alpha}}{k+\alpha}$.
If $f(t) = \sum_{k=0}^{n} b_k t^k$, where the series is uniformly convergent then we define $J_{\alpha}(f) = \sum_{k=0}^{\infty} b_k \frac{t^{k+\alpha}}{k+\alpha}$.
Further, if $\alpha = 1$, then J_{α} is the usual integral.
For example, with $\alpha = \frac{1}{2}$

1.

$$J_{\frac{1}{2}}(\sin t) = J_{\frac{1}{2}}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}t^{2n+1}\right)$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1+\frac{1}{2}}}{(2n+1+\frac{1}{2})(2n+1)!}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+\frac{3}{2}}}{(2n+\frac{3}{2})(2n+1)!}.$$

2.

$$J_{\frac{1}{2}}(\cos t) = J_{\frac{1}{2}}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}t^{2n}\right)$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+\frac{1}{2}}}{(2n+\frac{1}{2})(2n)!}.$$

3.

$$J_{\frac{1}{2}}(e^t) = J_{\frac{1}{2}}\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\frac{t^{n+\frac{1}{2}}}{(n+\frac{1}{2})n!}.$$

Now we define $I_\alpha^a(f)(t)$ to denote to conformable α-fractional integral of a function f starting from $a \geq 0$.

Definition 2.12. $I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx$, *where the integral is the usual Riemann improper integral and* $\alpha \in (0,1)$ *[18].*

So, $I_0^{\frac{1}{2}}(\sqrt{t}\cos t) = \int_0^t \frac{\cos x}{x^{\frac{1}{2}-\frac{1}{2}}}dx = \int_0^t \cos x dx = \sin t.$
and $I_0^{\frac{1}{2}}(\cos 2\sqrt{t}) = \int_0^t \frac{\cos 2\sqrt{x}}{x^{1-\frac{1}{2}}}dx = \int_0^t \frac{\cos 2\sqrt{x}}{x^{\frac{1}{2}}}dx = \sin 2\sqrt{t}.$

Theorem 2.13. $T_\alpha I_\alpha^a(f)(t) = f(t)$, *for* $t \geq 0$ *where* f *is any contiuous function in the domain of* I_α *[22].*

Proof. Since f is continuous, then $I_\alpha^a(f)(t)$ is differentiable hence

$$\begin{aligned}
T_\alpha I_\alpha^a(f)(t) &= t^{1-\alpha}\frac{d}{dt}(I_\alpha^a(f)(t) \\
&= t^{1-\alpha}\frac{d}{dt}\int_a^t \frac{f(x)}{x^{1-\alpha}}dx \\
&= t^{1-\alpha}\frac{f(t)}{t^{1-\alpha}} \\
&= f(t).
\end{aligned}$$

□

Consider the conformable fractional linear differential equations of order α

$$T^\alpha y + a(x)y = f(x). \tag{2.7}$$

Theorem 2.14. *The homogeneous solution of the conformable differential equation (2.7) is given by*

$$y_c(x) = e^{-I_0^\alpha a(x)},$$

where $a(x)$ is any continuous function in the domain of I_0^α [26].

Proof. To prove Theorem 2.14 , we have just verified that the equation (2.7) is satisfied by getting the function $y_c(x) = e^{-I_0^\alpha a(x)}$. By replacing above candidate solution into the conformable differential equation (2.7), and using 6., we get:

$$\begin{aligned}
T_\alpha(y_c) + a(x)y_c &= x^{1-\alpha}\frac{d}{dt}\left[e^{-I_0^\alpha a(x)}\right] + a(x)e^{-I_0^\alpha a(x)} \\
&= x^{1-\alpha}\frac{d}{dt}\left[-I_0^\alpha a(x)\right] e^{-I_0^\alpha a(x)} + a(x)e^{-I_0^\alpha a(x)} \\
&= -x^{1-\alpha}\frac{d}{dt}\left[\int_0^t \frac{a(x)}{x^{1-\alpha}}\right] e^{-I_0^\alpha a(x)} + a(x)e^{-I_0^\alpha a(x)} \\
&= -x^{1-\alpha}\frac{a(x)}{x^{1-\alpha}}e^{-I_0^\alpha a(x)} + a(x)e^{-I_0^\alpha a(x)} \\
&= 0.
\end{aligned}$$

Then we conclude that the homogeneous solution of the conformable differential equation (2.7) is given by $y_c(x) = e^{-I_0^\alpha a(x)}$. □

Theorem 2.15. *The particular solution of the conformable differential equation (2.7) is given by*

$$y_p(x) = \lambda e^{-I_0^\alpha a(x)},$$

where $a(x)$ is any continuous function in the domain of I_0^α and the function λ is obtained by the following condition [26].

$$\lambda(x) = I_0^\alpha(f(x)e^{I_0^\alpha a(x)}).$$

Proof. To prove this Theorem, we have just verified that the equation (2.7) is satisfied by getting the function $y_p(x) = \lambda e^{-I_0^\alpha a(x)}$. Replacing above candidate solution into the conformable differential equation (2.7), and using 6., we have that:

$$\begin{aligned}
T_\alpha(y_p) + a(x)y_p &= T_\alpha\left[\lambda e^{-I_0^\alpha a(x)}\right] + a(x)\lambda e^{-I_0^\alpha a(x)} \\
&= T_\alpha(\lambda)e^{-I_0^\alpha a(x)} + \lambda T_\alpha(e^{-I_0^\alpha a(x)}) + a(x)\lambda e^{-I_0^\alpha a(x)} \\
&= (f(x)e^{I_0^\alpha a(x)})e^{-I_0^\alpha a(x)} + \lambda\left[-a(x)e^{-I_0^\alpha a(x)}\right] + a(x)\lambda e^{-I_0^\alpha a(x)} \\
&= f(x).
\end{aligned}$$

Then we conclude that the homogeneous solution of the conformable differential equation (2.7) is given by $y_p(x) = \lambda e^{-I_0^\alpha a(x)}$. $\square$

2.4 Examples

In this section we introduce basic examples of solving conformable fractional differential equations.

Example 2.16. *Find the general solution of the following equation*

$$y^{(\frac{1}{2})} + 2y = 2x^2 + 4x^{\frac{3}{2}}, \quad y(0) = 0.$$

Solution: Find the solution y_c of the homogeneous equation $y^{(\frac{1}{2})} + 2y = 0$.
let $y_c = e^{r\sqrt{x}}$, then

$$\begin{aligned}
y_c^{(\frac{1}{2})} + 2y_c &= 0 \\
\frac{1}{2}rx^{\frac{1}{2}-\frac{1}{2}}e^{r\sqrt{x}} + 2e^{r\sqrt{x}} &= 0 \\
\frac{1}{2}re^{r\sqrt{x}} + 2e^{r\sqrt{x}} &= 0 \\
\frac{1}{2}r + 2 &= 0
\end{aligned}$$

$\Longrightarrow r = -4$.
Hence, $y_c = e^{-4\sqrt{x}}$.

Since the right side of the equation is equal $2x^2 + 4x^{\frac{3}{2}}$, we assume a particular solution of the non-homogeneous equation $y_p(x) = Ax^2 + Bx + C$, then
$y_p^{(\frac{1}{2})} = 2Ax^{2-\frac{1}{2}} + Bx^{1-\frac{1}{2}} = 2Ax^{\frac{3}{2}} + Bx^{\frac{1}{2}}$.
Substituting into the given fractional differential equation, we have
$2Ax^{\frac{3}{2}} + Bx^{\frac{1}{2}} + Ax^2 + Bx + C = 2x^2 + 4x^{\frac{3}{2}}$.
$Ax^2 + 2Ax^{\frac{3}{2}} + Bx^{\frac{1}{2}} + Bx + C = 2x^2 + 4x^{\frac{3}{2}}$.
Thus $A = 2$, $B = 0$ and $C = 0$.
A particular solution is therefor $y_p(x) = 2x^2$, and the general solution is: $y(x) = y_c(x) + y_p(x) = c_1 e^{-4\sqrt{x}} + 2x^2$, where c_1 is constant.
Finally, the initial condition $y(0) = 0$ implies that
$y(0) = c_1 e^0 + 0 = 0 \Longrightarrow c_1 = 0$. Hence, $y(x) = 2x^2$.

Example 2.17. *Find the general solution of the following conformable fractional equation*

$$y^{(\beta)} - y = 0, \quad \beta \in (0, 1].$$

Solution: The equation is homogeneous. Let $y = e^{rx^\beta}$, then

$$y^{(\beta)} - y = 0$$
$$\beta r x^{\beta-\beta} e^{rx^\beta} - e^{rx^\beta} = 0$$
$$\beta r e^{rx^\beta} - e^{rx^\beta} = 0$$
$$\beta r - 1 = 0$$

$\Longrightarrow r = \frac{1}{\beta}$.
So that the solution is given by $y(x) = e^{\frac{1}{\beta}x^\beta}$.

Example 2.18. *Find the general solution of the following conformable fractional equation*

$$y^{(\frac{1}{4})} + \sqrt{x}y = 2xe^{\frac{-4}{3}x^{\frac{3}{4}}}.$$

Solution: If a linear differential equation is written in the standard form

$$y^{(\alpha)} + a(x)y = f(x).$$

The integrating factor is defined by the formula $u(x) = e^{I_\alpha(a(x))}$, we solve this equation by multiplying it by $e^{I_{\frac{1}{4}}(\sqrt{x})} = e^{\int_0^x \frac{\sqrt{t}}{t^{1-\frac{1}{4}}}\,dt} = e^{\int_0^x t^{\frac{-1}{4}}\,dt} = e^{\frac{4}{3}x^{\frac{3}{4}}}$.

Converts the left side into the α-derivative of the product $e^{\frac{4}{3}x^{\frac{3}{4}}}y$, then

$$e^{\frac{4}{3}x^{\frac{3}{4}}}y^{(\frac{1}{4})} + e^{\frac{4}{3}x^{\frac{3}{4}}}\sqrt{x}y = 2e^{\frac{4}{3}x^{\frac{3}{4}}}xe^{\frac{-4}{3}x^{\frac{3}{4}}} = 2x.$$

$$T_{\frac{1}{4}}(e^{\frac{4}{3}x^{\frac{3}{4}}}y) = 2x.$$

$$I_{\frac{1}{4}}T_{\frac{1}{4}}(e^{\frac{4}{3}x^{\frac{3}{4}}}y) = I_{\frac{1}{4}}(2x)$$

$$e^{\frac{4}{3}x^{\frac{3}{4}}}y = \int_0^x \frac{2t}{t^{1-\frac{1}{4}}}dt = \int_0^x 2t^{\frac{1}{4}}dt = \frac{8}{5}t^{\frac{5}{4}}|_0^x = \frac{8}{5}x^{\frac{5}{4}} + C$$

$$e^{\frac{4}{3}x^{\frac{3}{4}}}y = \frac{8}{5}x^{\frac{5}{4}} + C$$

$\Longrightarrow y = \frac{8}{5}x^{\frac{5}{4}}e^{\frac{-4}{3}x^{\frac{3}{4}}} + Ce^{\frac{-4}{3}x^{\frac{3}{4}}}$, C is constant.
The general solution of the fractional differential equation is
$y(x) = \frac{8}{5}x^{\frac{5}{4}}e^{\frac{-4}{3}x^{\frac{3}{4}}} + Ce^{\frac{-4}{3}x^{\frac{3}{4}}}$.

Example 2.19. $y^{(\frac{1}{2})} = \frac{x^{\frac{3}{2}}+y\sqrt{x}}{x+y}$.

Solution: Let $y^{(\frac{1}{2})} = \sqrt{x}\frac{dy}{dx}$, then

$$\sqrt{x}\frac{dy}{dx} = \frac{x^{\frac{3}{2}} + y\sqrt{x}}{x+y}$$

$$\frac{dy}{dx} = \frac{x+y}{x+y}$$

$$\frac{dy}{dx} = \frac{1+\frac{y}{x}}{1+\frac{y}{x}}. \tag{2.8}$$

Let $v = \frac{y}{x}$, then $y = vx$ and $\frac{dy}{dx} = xv' + v$ substitute in (2.8), we get
$xv' + v = \frac{1+v}{1+v} \Longrightarrow x\frac{dv}{dx} = 1 - v$, this implies $\frac{1}{x}dx = \frac{1}{1-v}dv$, integrate both sides we get $\ln(x) + \ln(c) = \ln(1-v)$ by taking exponential both sides $cx = (1-v) \Longrightarrow \frac{y}{x} = 1 - cx$.
The solution is $y = x(1-cx)$.

Chapter 3

Conformable fractional differential equations

3.1 First order conformable fractional differential equations

In this section we discuss the general solutions of the conformable fractional linear and nonlinear differential equations of order α.
First we will discuss conformable fractional linear differential equations of order α of the form

$$T^{\alpha}y + a(x)y = f(x). \tag{3.1}$$

By using the property $T^{\alpha}y = x^{1-\alpha}y'$, eq (3.1) becomes

$$x^{1-\alpha}y' + a(x)y = f(x).$$

$$y' + \frac{a(x)}{x^{1-\alpha}}y = \frac{f(x)}{x^{1-\alpha}}. \tag{3.2}$$

Where eq (3.2) is a first order linear ordinary differential equation with general solution

$$y = \frac{1}{\mu_{\alpha}}[\int \mu_{\alpha}\frac{f(x)}{x^{1-\alpha}}dx + C], \quad x \neq 0, \tag{3.3}$$

where $\mu_{\alpha} = e^{\int \frac{a(x)}{x^{1-\alpha}}dx}$ and C is arbitrary constant [28]. Now by using the definition of conformable fractional integral and substitution in eq (3.3) we obtain

$$y = \frac{1}{\mu_{\alpha}}[I_{\alpha}(\mu_{\alpha}f(x)) + C], \tag{3.4}$$

where $\mu_\alpha = e^{I_\alpha(a(x))}$. The result (3.4) is the general solution of eq (3.1).
Now, we will discuss the general solutions of the conformable fractional nonlinear Bernoulli, Riccatti and Abel differential equations.

• **Conformable fractional nonlinear Bernoulli's differential equation:**
The general formula of the conformable fractional nonlinear Bernoulli's differential equation (CFNB) can be written as

$$T^\alpha y + p(x)y = f(x)y^n, \quad \alpha \in (0,1], \quad n \in \mathbb{N}. \tag{3.5}$$

Where $p(x)$, $f(x)$ are α-differentiable functions and y is an unknown function to be solved [13]. When $n = 0$ or 1 the equation is a linear, otherwise it is nonlinear.
By using the property $T^\alpha y = x^{1-\alpha}y'$, eq (3.5) becomes

$$x^{1-\alpha}y' + p(x)y = f(x)y^n, \quad \alpha \in (0,1].$$

$$\Rightarrow y' + x^{\alpha-1}p(x)y = x^{\alpha-1}f(x)y^n, \quad \alpha \in (0,1]. \tag{3.6}$$

Where eq (3.6) is the Bernoulli equation [28], it can be reduced to a linear equation for any other value of n by the change of dependent variable $u = y^{1-n}$, the solution as follows

$$y^{1-n} = \frac{1}{\mu(x)}[\int((1-n)x^{\alpha-1}f(x)\mu(x))dx + C]. \tag{3.7}$$

Where C is an arbitrary constant and $\mu(x) = e^{\int(1-n)x^{\alpha-1}p(x)dx}$. Finally the general solution is given by

$$y = (\frac{1}{\mu(x)}[\int((1-n)x^{\alpha-1}f(x)\mu(x))dx + C])^{\frac{1}{1-n}}. \tag{3.8}$$

In the special case [7], eq (3.5) takes the following formula

$$T^\alpha y + p(x)T^\beta y = f(x)y^n, \quad \alpha, \beta \in (0,1], \quad n \in \mathbb{N}. \tag{3.9}$$

To find the general solution of (3.9), we have three cases:

1. Case one: When $\beta = 1$, then eq (3.9) becomes

$$T^\alpha y + p(x)y' = f(x)y^n, \quad \alpha \in (0,1], \quad n \in \mathbb{N}.$$

Using the property $T^\alpha y = x^{1-\alpha}y'$, we get

$$x^{1-\alpha}y' + p(x)y' = f(x)y^n, \quad \alpha \in (0,1].$$

$$\Rightarrow y' = \frac{f(x)y^n}{x^{\alpha-1}+p(x)}, \quad \alpha \in (0,1]. \tag{3.10}$$

By applying eq (3.7), then the solution of (3.10) as follows

$$y^{1-n} = \int (1-n)\frac{f(x)}{x^{\alpha-1}+p(x)}dx + C. \tag{3.11}$$

Where $\mu(x) = e^c$, c is constant. Then the general solution is given by

$$y = [\int (1-n)\frac{f(x)}{x^{\alpha-1}+p(x)}dx + C]^{\frac{1}{1-n}}. \tag{3.12}$$

2. Case two: When $\beta \neq 1$ and $\alpha = \beta \in (0,1]$ in eq (3.9), then we get

$$T^\alpha y + p(x)T^\alpha y = f(x)y^n, \quad \alpha \in (0,1].$$

$$\Rightarrow T^\alpha y = \frac{f(x)}{1+p(x)}y^n, \quad p(x) \neq -1.$$

Use the property $T^\alpha y = x^{1-\alpha}y'$, we get

$$y' = \frac{x^{\alpha-1}f(x)}{1+p(x)}y^n. \tag{3.13}$$

By applying eq (3.7), then the solution of (3.13) as follows

$$y^{1-n} = \int (1-n)\frac{x^{\alpha-1}f(x)}{1+p(x)}dx + C, \tag{3.14}$$

where $\mu(x) = e^c$, c is constant. Then the general solution of eq (3.9) is given by

$$y = [\int (1-n)\frac{x^{\alpha-1}f(x)}{1+p(x)}dx + C]^{\frac{1}{1-n}}. \tag{3.15}$$

3. Case three: When $\alpha \neq \beta$ in eq (3.9), then we get

$$T^\alpha y + p(x)T^\beta y = f(x)y^n, \quad \alpha, \beta \in (0,1], \quad n \in \mathbb{N}.$$

Use the property $T^\alpha y = x^{1-\alpha}y'$, we get

$$x^{1-\alpha}y' + x^{1-\beta}p(x)y' = f(x)y^n.$$

$$\Rightarrow y' = \frac{f(x)y^n}{x^{1-\alpha}+x^{1-\beta}p(x)}, \quad p(x) \neq -x^{-\alpha+\beta}.$$

By applying eq (3.7), then we get

$$y^{1-n} = \int (1-n)\frac{f(x)y^n}{x^{1-\alpha}+x^{1-\beta}p(x)}dx + C,$$

where $\mu(x) = e^c$, c is constant. Then the general solution of this case is given by

$$y = [\int (1-n)\frac{f(x)y^n}{x^{1-\alpha}+x^{1-\beta}p(x)}dx + C]^{\frac{1}{1-n}}. \tag{3.16}$$

Example 3.1. *Find the general solution of the following equation*

$$T^{\frac{1}{2}}y + \sqrt{x}y = x\sqrt{x}y^2.$$

Solution: We have $p(x) = \sqrt{x}$, $f(x) = x\sqrt{x}$, $\alpha = \frac{1}{2}$, $\beta = 1$ and $n = 2$.
Then the general solution of this equation is

$$y = (\frac{1}{\mu(x)}[\int ((1-2)x^{\frac{1}{2}-1}x\sqrt{x}\mu(x))dx + C])^{\frac{1}{1-2}},$$

where $\mu(x) = e^{\int (1-2)x^{\frac{1}{2}-1}x^{\frac{1}{2}}dx} = e^{-x}$, this implies

$$y = (e^x[-\int x^{\frac{-1}{2}}x\sqrt{x}e^{-x}dx + C])^{-1} = [e^x(xe^{-x}-e^{-x}+C)]^{-1} = (x-1+C)^{-1} = \frac{1}{x-K}.$$

Such that $K = 1+C$, C and K are constant.

Example 3.2. *Find the general solution of the fractional differential equation*

$$T^{\frac{1}{2}}y + \sqrt{x}T^{\frac{1}{2}}y = \sqrt{x}y^{\frac{1}{4}}.$$

Solution: We have $p(x) = \sqrt{x}$, $f(x) = \sqrt{x}$, $\alpha = \beta = \frac{1}{2}$ and $n = \frac{1}{4}$. Then the general solution of this equation is

$$y = [\frac{3}{4}\int \frac{1}{1+\sqrt{x}}dx + C]^{\frac{4}{3}} = [\frac{3}{4}\int 2\frac{u-1}{u}du + C]^{\frac{4}{3}}.$$

Such that $u = 1+\sqrt{x}$.

$$\rightarrow y = [\frac{3}{2}u - \frac{3}{2}\ln(u) + C]^{\frac{4}{3}} = [\frac{3}{2}(1+\sqrt{x}) - \frac{3}{2}\ln(1+\sqrt{x}) + C]^{\frac{4}{3}}.$$

Example 3.3. *Find the general solution of the fractional differential equation*

$$T^{\frac{1}{2}}y + x^{\frac{-1}{4}}T^{\frac{1}{4}}y = x^2y^5.$$

Solution: We have $p(x) = x^{\frac{-1}{4}}$, $f(x) = x^2$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$ and $n = 5$. Then the general solution of this equation is

$$y = [-4\int \frac{x^2}{2x^{\frac{-1}{2}}}dx + C]^{\frac{-1}{4}} = [-2\int x^{\frac{3}{2}}dx + C]^{\frac{-1}{4}} = [\frac{-4}{5}x^{\frac{5}{2}} + C]^{\frac{-1}{4}}.$$

• **Conformable fractional nonlinear Riccatti's differential equation:**
The conformable fractional nonlinear Riccatti differential equation of order α (CFNR) can be represented by

$$T^{\alpha}y = f_2(x)y^2 + f_1(x)y + f_0(x), \quad \alpha \in (0,1]\ x > 0. \tag{3.17}$$

Where $f_0(x)$, $f_1(x)$ and $f_2(x)$ are α-differentiable functions ([13],[15]).
There exist three cases:

1. Case one: If $f_0(x) = 1$, then eq (3.17) becomes to be (CFNB) and the general solution of this case is given by

$$y = (\mu(x)[\int -x^{\alpha-1}f_2(x)\frac{1}{\mu(x)}dx + C])^{-1}.$$

With $n = 2$, $\mu(x) = e^{\int x^{\alpha-1}f_1(x)dx}$.

2. Case two: If $f_2(x) = \frac{a}{x^{\alpha+1}}$, $f_1(x) = \frac{b}{x^{\alpha}}$, $f_0(x) = \frac{c}{x^{\alpha-1}}$ and $a, b, c \in I\!R$, then the following homogeneous (CFNR)

$$T^{\alpha}y = \frac{a}{x^{\alpha+1}}y^2 + \frac{b}{x^{\alpha}}y + \frac{c}{x^{\alpha-1}}.$$

By apply the property $T^{\alpha}y = x^{1-\alpha}y'$, we get

$$y' = a(\frac{y}{x})^2 + b\frac{y}{x} + c, \tag{3.18}$$

eq (3.18) is homogenous first ordinary differential equation. To solving it let $u = \frac{y}{x}$, $y' = u'x + u$, then the eq (3.18) becomes

$$u + xu' = au^2 + bu + c \;\rightarrow x\frac{du}{dx} = au^2 + (b-1)u + c \;\rightarrow \frac{1}{x}dx = \frac{1}{au^2 + (b-1)u + c}du.$$

The general solution of this case is given by

$$\ln(x) = \int \frac{1}{au^2 + (b-1)u + c}du.$$

3. Case three: If a particular solution y_1 is known, then general solution has the form $y = y_1 + z$.
   Using the property $T^\alpha y = x^{1-\alpha} y'$, then eq (3.17) becomes

$$y' = x^{\alpha-1} f_2(x) y^2 + x^{\alpha-1} f_1(x) y + x^{\alpha-1} f_0(x). \tag{3.19}$$

   Since $y' = y_1' + z'$, we have

$$y_1' + z' = x^{\alpha-1} f_2(x) y_1^2 + 2x^{\alpha-1} f_2(x) y_1 z + x^{\alpha-1} f_2(x) z^2 + x^{\alpha-1} f_1(x) y_1 + x^{\alpha-1} f_1(x) z + x^{\alpha-1} f_0(x).$$

$$= [x^{\alpha-1} f_2(x) y_1^2 + x^{\alpha-1} f_1(x) y_1 + x^{\alpha-1} f_0(x)] + [2x^{\alpha-1} f_2(x) y_1 + x^{\alpha-1} f_1(x)] z + x^{\alpha-1} f_2(x) z^2. \tag{3.20}$$

   Now, from (3.19) and (3.20) we get

$$y_1' + z' = y_1' + [2x^{\alpha-1} f_2(x) y_1 + x^{\alpha-1} f_1(x)] z + x^{\alpha-1} f_2(x) z^2.$$

$$\rightarrow z' - [2x^{\alpha-1} f_2(x) y_1 + x^{\alpha-1} f_1(x)] z = x^{\alpha-1} f_2(x) z^2.$$

   This equation is (CFNB) with $n = 2$, $p(x) = -[2f_2(x) y_1 + f_1(x)]$ and $f(x) = f_2(x)$ then the solution is

$$z = (\frac{1}{\mu(x)}[\int -x^{\alpha-1} f_2(x) \mu(x)) dx + C])^{-1}, \tag{3.21}$$

   since that $\mu(x) = e^{\int [2x^{\alpha-1} f_2(x) y_1 + x^{\alpha-1} f_1(x)] dx}$. Since $y = y_1 + z$, then the general solution of eq (3.17) is given by

$$y = y_1 + (\frac{1}{\mu(x)}[\int -x^{\alpha-1} f_2(x) \mu(x)) dx + C])^{-1}.$$

Example 3.4. *Find the general solution of the (CFNR)*

$$T^{\frac{1}{2}} y = -x\sqrt{x} + \frac{1}{2\sqrt{x}} y + \sqrt{x} y^2.$$

Solution: Since $y = \sqrt{x}$ is a particular solution

$$\text{The left side} (\sqrt{x})^{(\frac{1}{2})} = \sqrt{x}(\frac{1}{2\sqrt{x}}) = \frac{1}{2}.$$

$$\text{The right side} - x\sqrt{x} + \frac{1}{2\sqrt{x}} \sqrt{x} + \sqrt{x}(\sqrt{x})^2 = -x\sqrt{x} + \frac{1}{2} + x\sqrt{x} = \frac{1}{2},$$

then the general solution has a form $y' = y_1' + z'$.
z is the general solution of the following (CFNB)

$$z' - (2\sqrt{x} + \frac{1}{2x})z = z^2.$$

By using eq (3.21), we get

$$z = (\frac{1}{\mu(x)}[\int -x^{\frac{-1}{2}} x^{\frac{1}{2}} \mu(x))dx + C])^{-1},$$

such that $\mu(x) = \sqrt{x} e^{\frac{4}{3}x^{\frac{3}{2}}}$.

$$\rightarrow z = (\frac{1}{\sqrt{x}} e^{\frac{-4}{3}x^{\frac{3}{2}}}[-\frac{1}{2} e^{\frac{4}{3}x^{\frac{3}{2}}} + C])^{-1} \quad = \frac{2\sqrt{x}}{2e^{\frac{-4}{3}x^{\frac{3}{2}}} C - 1}.$$

Therefore the general solution of equation is

$$y = \sqrt{x} + \frac{2\sqrt{x}}{2e^{\frac{-4}{3}x^{\frac{3}{2}}} C - 1}.$$

Example 3.5. *Find the general solution of the (CFNR)*

$$T^{\frac{1}{3}} y = -4x^{\frac{2}{3}} + x^{\frac{-1}{3}} y + x^{\frac{-4}{3}} y^2.$$

Solution:

$$x^{\frac{2}{3}} y' = -4x^{\frac{2}{3}} + x^{\frac{-1}{3}} y + x^{\frac{-4}{3}} y^2 \quad \rightarrow y' = (\frac{y}{x})^2 + \frac{y}{x} - 4.$$

Let $u = \frac{y}{x}$, then the equation becomes

$$u + xu' = u^2 + u - 4 \quad \rightarrow \frac{1}{x} dx = \frac{1}{u^2 - 4} du \quad \rightarrow \ln(x) = \frac{1}{2}[\ln(u-2) - \ln(u+2)] \quad \rightarrow x^2 = \frac{u-2}{u+2}.$$

$$\rightarrow y = \frac{-2x - 2x^3}{x^2 - 1}.$$

- **Conformable fractional nonlinear Abel's differential equation:**

1. The first kind of the conformable fractional nonlinear Abel's differential equation (CFNA) can be represented by

$$T^{\alpha} y = f_3(x) y^3 + f_2(x) y^2 + f_1(x) y + f_0(x), \quad \alpha \in (0, 1], \ x > 0, \qquad (3.22)$$

where $f_3(x) \neq 0$. If $f_3(x) = 0$, then eq (3.22) becomes to be (CFNR). We will discuss two special cases:

(a) Case one: If $f_2(x) = f_0(x) = 0$, then eq (3.22) becomes to be (CFNB) with $n = 3$ and it is easy to find the general solution of this case as we discussed it previously.

(b) Case two: If $f_3(x) = ax^{3n-m+\alpha-1}$, $f_2(x) = bx^{2n+\alpha-1}$, $f_1(x) = (m-n)x^{\alpha-2}$ and $f_0(x) = dx^{-n+\alpha-3}$ where $a, b, d \in I\!R$ and $m, n \in I\!N$, we have

$$T^\alpha y = ax^{3n-m+\alpha-1}y^3 + bx^{2n+\alpha-1}y^2 + (m-n)x^{\alpha-2}y + dx^{-n+\alpha-3}. \quad (3.23)$$

By using the property $T^\alpha y = x^{1-\alpha}y'$ and substituting $y = x^{m-n}z$, we get the following separable ordinary differential equation

$$(x^{m-n}z)' = ax^{3n-m}(x^{m-n}z)^3 + bx^{2n}(x^{m-n}z)^2 + (m-n)x^{-1}x^{m-n}z + dx^{-n-2}.$$

$$\to x^{-m-n}z' = az^3 + bz^2 + d.$$

The general solution of this case is given by

$$\frac{x^{n+m+1}}{n+m+1} = \int \frac{1}{az^3 + bz^2 + d}dz.$$

2. The second kind of the (CFNA) is given by

$$[y + g(x)]T^\alpha y = f_2(x)y^2 + f_1(x)y + f_0(x), \quad \alpha \in (0, 1], \quad (3.24)$$

where $g(x) \neq 0$. We have two cases:

(a) Case one: If $g(x)$ is a constant function, then eq (3.24) becomes to the separable conformable fractional differential equation and it is easy to find the general solution.

(b) Case two: If $g(x) = kx^n$, $f_2(x) = ax^{\alpha-2}$, $f_1(x) = bx^{n+\alpha-2}$ and $f_0(x) = cx^{2n+\alpha-2}$ where $a, b, c, k \in I\!R$ and $n \in I\!N$, then we get the following equation

$$(y + kx^n)T^\alpha y = ax^{\alpha-2}y^2 + bx^{n+\alpha-2}y + cx^{2n+\alpha-2}, \quad \alpha \in (0, 1]. \quad (3.25)$$

By substituting $z = x^{-n}y$ in eq (3.25), we get

$$x^{1-\alpha}(zx^n + kx^n)' = ax^{\alpha-2}(zx^n)^2 + bx^{n+\alpha-2}zx^n + cx^{2n+\alpha-2}.$$

$$(nx^{n-1}z + x^nz')(z+k) = x^{n-1}[az^2 + bz + c] \to x(z+k)z' = (a-n)z^2 + (b-kn)z + c.$$

By applying the property $T^{\alpha}y = x^{1-\alpha}y'$, then the equation becomes

$$xz' = az^2 + (b+n+1)z + c.$$

The general solution of this case is given by

$$\ln(x) = \frac{1}{az^2 + (b+n+1)z + c}dz.$$

Example 3.6. *Find the general solution of the fractional differential equation*

$$T^{\frac{1}{2}}y = -\sqrt{x} + 4x^{\frac{2}{2}}y - 3x^{\frac{-3}{2}}y^2 + x^{\frac{-5}{2}}y^3.$$

Solution: By using the property $T^{\alpha}y = x^{1-\alpha}y'$, the equation becomes to the homogenous ordinary differential equation

$$y' = -1 + 4\frac{y}{x} - 3(\frac{y}{x})^2 + (\frac{y}{x})^3,$$

let $z = \frac{y}{x} \to zx = y \to y' = z + xz'$, the equation becomes

$$xz' = -1 + 3z - 3z^2 + z^3 \ = (z-1)^3.$$

$$\to \ln(x) = \int \frac{1}{(z-1)^3}dz.$$

The solution of equation is given by

$$2(\ln(x) + c)(\frac{y}{x} - 1)^2 + 1 = 0.$$

Example 3.7. *Find the general solution of the fractional differential equation*

$$(y + x^2)T^{\frac{1}{2}}y = x^{\frac{7}{2}} + x^{\frac{3}{2}}y + x^{\frac{-1}{2}}y^2.$$

Solution: By using the property $T^{\alpha}y = x^{1-\alpha}y'$ and substitute $y = zx^2$, we get

$$xz'(z+1) = 1 - z - z^2 \ \to xz' = z^2 + 4z + 1 \ \to \ln(x) = \int \frac{1}{z^2 + 4z + 1}dz.$$

The general solution of equation is given by

$$[\ln(x\sqrt{(\frac{y}{x})^2 + \frac{y}{x} - 1 + c})](2\frac{y}{x} + 1) = 1, \ \ x > 0.$$

3.2 Second order conformable fractional differential equations

In this section we will discuss the second order of conformable fractional linear and nonlinear differential equations.
First, consider the conformable fractional linear differential equation of order 2α

$$T^{\alpha}T^{\alpha}y + P(x)T^{\alpha}y + Q(x)y = 0, \quad 0 < \alpha \leq 1. \tag{3.26}$$

Definition 3.8. *[2] Let y_1 and y_2 be the solutions of eq (3.26), then the conformable fractional Wronskian of the solutions is defined by*

$$W_{\alpha}[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ T^{\alpha}y_1 & T^{\alpha}y_2 \end{vmatrix}.$$

Theorem 3.9. *[12](Abel's Theorem)*
If y_1, y_2 are solutions of (3.26), then

$$W_{\alpha}[y_1, y_2] = e^{-I_{\alpha}(P(x))}.$$

Proof. Since y_1 and y_2 are solutions of (3.26), then

$$T^{\alpha}T^{\alpha}y_1 + P(x)T^{\alpha}y_1 + Q(x)y_1 = 0. \tag{3.27}$$

And

$$T^{\alpha}T^{\alpha}y_2 + P(x)T^{\alpha}y_2 + Q(x)y_2 = 0. \tag{3.28}$$

Multiply (3.27) by y_2 and (3.28) by y_1 and subtract them, we get

$$y_2T^{\alpha}T^{\alpha}y_1 - y_1T^{\alpha}T^{\alpha}y_2 + P(x)[y_2T^{\alpha}y_1 - y_1T^{\alpha}y_2] = 0. \tag{3.29}$$

But,

$$W_{\alpha}[y_1, y_2] = y_1T^{\alpha}y_2 - y_2T^{\alpha}y_1,$$

and

$$\begin{aligned} T^{\alpha}(W_{\alpha}[y_1, y_2]) &= y_1T^{\alpha}T^{\alpha}y_2 + T^{\alpha}y_1T^{\alpha}y_2 - y_2T^{\alpha}T^{\alpha}y_1 - T^{\alpha}y_2T^{\alpha}y_1 \\ &= y_1T^{\alpha}T^{\alpha}y_2 - y_2T^{\alpha}T^{\alpha}y_1. \end{aligned}$$

To become an eq (3.29)

$$-T^{\alpha}(W_{\alpha}[y_1, y_2]) - P(x)W_{\alpha}[y_1, y_2] = 0.$$

$$\Rightarrow T^{\alpha}(W_{\alpha}[y_1, y_2]) + P(x)W_{\alpha}[y_1, y_2] = 0.$$

This implies,

$$\frac{T^{\alpha}(W_{\alpha}[y_1, y_2])}{W_{\alpha}[y_1, y_2]} = -P(x),$$

we take fractional integral for both sides, we get

$$I_{\alpha}\frac{T^{\alpha}(W_{\alpha}[y_1, y_2])}{W_{\alpha}[y_1, y_2]} = I_{\alpha}(-P(x)).$$

$$\Rightarrow I_{\alpha}\frac{W_{\alpha}^{1-\alpha}[y_1, y_2]W_{\alpha}'[y_1, y_2]}{W_{\alpha}[y_1, y_2]} = I_{\alpha}(-P(x)).$$

$$\Rightarrow \int_0^x \frac{W_{\alpha}^{1-\alpha}[y_1, y_2]W_{\alpha}'[y_1, y_2]}{W_{\alpha}[y_1, y_2]W_{\alpha}^{1-\alpha}[y_1, y_2]} = I_{\alpha}(-P(x)).$$

$$\Rightarrow \ln W_{\alpha}[y_1, y_2] = I_{\alpha}(-P(x)).$$

Therefore,

$$W_{\alpha}[y_1, y_2] = e^{-I_{\alpha}(P(x))}.$$

□

Now, let y_1 be a solution of eq (3.26). To find a second solution y_2 of eq (3.26), use theorem 3.9.

We have $W_{\alpha}[y_1, y_2] = e^{-I_{\alpha}(P(x))}$, then

$$y_1 T^{\alpha} y_2 - y_2 T^{\alpha} y_1 = e^{-I_{\alpha}(P(x))}.$$

$$\Rightarrow T^{\alpha} y_2 - y_2 \frac{T^{\alpha} y_1}{y_1} = \frac{e^{-I_{\alpha}(P(x))}}{y_1}. \tag{3.30}$$

Eq(3.30) is fractional linear equation with $a(x) = \frac{-T^{\alpha} y_1}{y_1}$ and $f(x) = \frac{e^{-I_{\alpha}(P(x))}}{y_1}$, the integrating factor is

$\mu_{\alpha} = e^{I_{\alpha}(\frac{-T^{\alpha} y_1}{y_1})} = e^{I_{\alpha}(\frac{-y_1^{1-\alpha} y_1'}{y_1})} = e^{\int \frac{-y_1^{1-\alpha} y_1'}{y_1 y_1^{1-\alpha}}} = e^{-\int \frac{y_1'}{y1}} = e^{-\ln y_1} = y_1^{-1}.$

The solution is

$$y_2 = \frac{1}{\mu_{\alpha}}[I_{\alpha}(\mu_{\alpha}\frac{e^{-I_{\alpha}(P(x))}}{y_1})].$$

$$\Rightarrow y_2 = y_1[I_{\alpha}(\frac{e^{-I_{\alpha}(P(x))}}{y_1^2})].$$

• **Conformable fractional nonlinear Euler's differential equation:**

The conformable fractional nonlinear Eular's differential equation (CFNE) of order 2α is given by

$$x^{2\alpha}T^{2\alpha}y(x) + ax^{\alpha}T^{\alpha}y(x) + by(x) = 0. \tag{3.31}$$

Where $\alpha \in (0,1]$, $x > 0$ and $a, b \in I\!R$ ([3],[7]). Clearly if $\alpha = 1$, then eq (3.31) reduces to the Euler's ordinary differential equation.
Let $x = e^t$, then $t = \ln(x)$ by using chain rule we get

$$\frac{d^\alpha y}{dx^\alpha} = \frac{d^\alpha y}{dt^\alpha}\frac{d^\alpha t}{dx^\alpha} = \frac{d^\alpha y}{dt^\alpha}(\frac{1}{x^\alpha}). \tag{3.32}$$

The second derivative will be in the form

$$\frac{d^{2\alpha} y}{dx^{2\alpha}} = \frac{d^\alpha y}{dx^\alpha}(\frac{1}{x^\alpha}) = \frac{1}{x^{2\alpha}}\frac{d^{2\alpha} y}{dt^{2\alpha}} - \frac{\alpha}{x^{2\alpha}}\frac{d^\alpha y}{dt^\alpha}. \tag{3.33}$$

Substituting eq (3.32) and eq (3.33) into eq (3.31), we get

$$x^{2\alpha}[\frac{1}{x^{2\alpha}}\frac{d^{2\alpha} y}{dt^{2\alpha}} - \frac{\alpha}{x^{2\alpha}}\frac{d^\alpha y}{dt^\alpha}] + ax^\alpha\frac{d^\alpha y}{dt^\alpha}(\frac{1}{x^\alpha}) + by(t) = 0.$$

$$\rightarrow \frac{d^{2\alpha} y}{dt^{2\alpha}} + (a-\alpha)\frac{d^\alpha y}{dt^\alpha} + by(t) = 0. \tag{3.34}$$

To solving eq (3.34), let $y = e^{rt^\alpha}$ where r is a parameter to be determined, then the general solution of eq (3.31) depends on the type of roots for the corresponding auxiliary equation of eq (3.34) [21].
There exist three cases to be determined the general solution of eq (3.31):

1. Case one: If the roots of the auxiliary equation (3.34) are real and different, let them be denoted by r_1 and r_2 where $r_1 \neq r_2$, then the general solution of eq (3.31) is given by

$$y(x) = c_1 e^{r_1(\ln x)^\alpha} + c_2 e^{r_2(\ln x)^\alpha}.$$

2. Case two: If the roots of the auxiliary equation (3.34) are repeated real numbers $r_1 = r_2 = r$, then the general solution of eq (3.31) is given by

$$y(x) = c_1 e^{r(\ln x)^\alpha} + c_2(\ln x)^\alpha e^{(\ln x)^\alpha}.$$

3. Case three: If the roots of the auxiliary equation (3.34) are conjugate complex numbers, denoted by $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, then the general solution of eq (3.31) is given by

$$y(x) = e^{\lambda(\ln x)^\alpha}[c_1\sin(\mu(\ln x)^\alpha) + c_2\cos(\mu(\ln x)^\alpha)].$$

Example 3.10. *Find the general solution of the fractional differential equation*

$$xT^{\frac{1}{2}}T^{\frac{1}{2}}y(x) + \frac{1}{2}\sqrt{x}T^{\frac{1}{2}}y(x) - y(x) = 0.$$

Solution: By change of variables $x = e^t$ leads to the equation

$$T^{\frac{1}{2}}T^{\frac{1}{2}}y(t) - y(t) = 0.$$

Let $y = e^{t^{\frac{1}{2}}}$, then the equation becomes

$$\frac{1}{4}r^2 e^{t^{\frac{1}{2}}} - e^{t^{\frac{1}{2}}} = 0, \quad e^{t^{\frac{1}{2}}} \neq 0.$$

$$\rightarrow r^2 - 4 = 0 \quad \rightarrow r_1 = -2, \quad r_2 = 2.$$

The solution is

$$y(t) = c_1 e^{-2(\sqrt{t})} + c_2 e^{2(\sqrt{t})}.$$

Then the following general solution as follows

$$y(t) = c_1 e^{-2(\sqrt{\ln t})} + c_2 e^{2(\sqrt{\ln t})}.$$

Example 3.11. *Find the general solution of the fractional differential equation*

$$x^{\frac{2}{3}}T^{\frac{1}{3}}T^{\frac{1}{3}}y(x) - x^{\frac{1}{3}}T^{\frac{1}{3}}y(x) + 4y(x) = 0.$$

Solution: By change of variables $x = e^t$ leads to the equation

$$T^{\frac{1}{3}}T^{\frac{1}{3}}y(t) - \frac{4}{3}T^{\frac{1}{3}}y(t) + 4y(t) = 0.$$

Let $y = e^{t^{\frac{1}{3}}}$, this implies $r_1 = r_2 = 2$. Then the general solution is given by

$$y(x) = c_1 e^{2(\ln x)^{\frac{1}{3}}} + c_2 (\ln x)^{\frac{1}{3}} e^{2(\ln x)^{\frac{1}{3}}}.$$

3.3 System of conformable fractional linear differential equations

In this section we will discuss the nonhomogeneous system

$$Y^{\alpha} = PY + G. \tag{3.35}$$

Where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad G = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}, \quad P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}.$$

We denoted the solutions of the system (3.35).

$$Y^{(1)} = \begin{pmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{pmatrix}, \quad Y^{(2)} = \begin{pmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{pmatrix}, \cdots Y^{(n)} = \begin{pmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{pmatrix}.$$

System (3.35) has two solutions, the homogenous solution complementary denoted by Y_c, which the solution of (3.35) with $G = 0$ and the particular solution which is any solution of (3.35) denoted by Y_p ([4],[14],[30]).
To find the homogenous solution of the system

$$Y^{\alpha} = PY, \tag{3.36}$$

let $Y = \xi e^{r\frac{t^{\alpha}}{\alpha}}$ and substituting in the equation (3.36) gives $r\xi e^{r\frac{t^{\alpha}}{\alpha}} = P\xi e^{r\frac{t^{\alpha}}{\alpha}}$.
Canceling the nonzero scalar factor $e^{r\frac{t^{\alpha}}{\alpha}}$, we get

$$(P - rI)\xi = 0, \tag{3.37}$$

where I is the identity matrix. Thus to solve the system of differential equation (3.36), we must solve the system of algebraic equations (3.37). Determine the eigenvalues r and eigenvectors ξ of the coefficient matrix P.
Therefore, the vector Y is given by equation $Y = \xi e^{r\frac{t^{\alpha}}{\alpha}}$, is a solution of eq(3.36).
There are three cases of eigenvalues:

1. Case one: If all eigenvalues are distinct real values ($r_1 \neq r_2 \neq .... \neq r_n$), hence the solutions $Y^{(1)} = \xi^{(1)} e^{r_1\frac{t^{\alpha}}{\alpha}}$, $Y^{(2)} = \xi^{(2)} e^{r_2\frac{t^{\alpha}}{\alpha}}$,... and $Y^{(n)} = \xi^{(n)} e^{r_n\frac{t^{\alpha}}{\alpha}}$ form a fundamental solution and the general solution is

$$Y(t) = c_1 Y^{(1)}(t) + c_2 Y^{(2)}(t) + ... + c_n Y^{(n)}(t).$$

2. Case two: If some eigenvalues occur in complex conjugate pairs $r_1 = \lambda + i\mu$ and $r_2 = \overline{r_1}$(the complex conjugate of r_1) corresponding eigenvectors $\xi^{(1)}$ and

$\xi^{(2)} = \overline{\xi^{(1)}}$.
The corresponding solutions $Y^{(1)} = \xi^{(1)} e^{r_1 \frac{t^\alpha}{\alpha}}$, $Y^{(2)} = \overline{\xi^{(1)}} e^{\overline{r_1} \frac{t^\alpha}{\alpha}}$.
Let us write $\xi^{(1)} = a + ib$, where a and b are real then we have

$$Y^{(1)} = (a+ib)e^{(\lambda+i\mu)\frac{t^\alpha}{\alpha}} = e^{\lambda\frac{t^\alpha}{\alpha}}[a\cos(\mu\frac{t^\alpha}{\alpha})-b\sin(\mu\frac{t^\alpha}{\alpha})]+ie^{\lambda\frac{t^\alpha}{\alpha}}[a\sin(\mu\frac{t^\alpha}{\alpha})+b\cos(\mu\frac{t^\alpha}{\alpha})],$$

if we write $Y^{(1)}(t) = u(t) + iv(t)$, then the vectors
$u(t) = e^{\lambda\frac{t^\alpha}{\alpha}}[a\cos(\mu\frac{t^\alpha}{\alpha}) - b\sin(\mu\frac{t^\alpha}{\alpha})]$ and $v(t) = ie^{\lambda\frac{t^\alpha}{\alpha}}[a\sin(\mu\frac{t^\alpha}{\alpha}) + b\cos(\mu\frac{t^\alpha}{\alpha})]$ are real valued solutions.
Then the general solution is

$$Y = c_1 u(t) + c_2 v(t).$$

3. Case three: If some eigenvalues are repeated. In a special case, consider a 3×3 system ($r_1 = r_2 = r_3 = r$), there are also three cases

   (a) suppose first that the triple eigenvalue r has three linearly independent eigenvectors $\xi^{(1)}, \xi^{(2)}$ and $\xi^{(3)}$. Then the solution is $Y^{(1)} = \xi^{(1)} e^{r_1\frac{t^\alpha}{\alpha}}$, $Y^{(2)} = \xi^{(2)} e^{r_2\frac{t^\alpha}{\alpha}}$ and $Y^{(3)} = \xi^{(3)} e^{r_3\frac{t^\alpha}{\alpha}}$.

   (b) The second case is only one corresponding eigenvectors, then the first solution is $Y^{(1)} = \xi e^{r\frac{t^\alpha}{\alpha}}$ where ξ satisfies $(P - rI)\xi = 0$.
   A second solution is $Y^{(2)} = \xi \frac{t^\alpha}{\alpha} e^{r\frac{t^\alpha}{\alpha}} + \eta e^{r\frac{t^\alpha}{\alpha}}$ where η is determined from $(P - rI)\eta = \xi$.
   And the third solution is of the form $Y^{(3)} = \xi \frac{t^{2\alpha}}{2!\alpha} e^{r\frac{t^\alpha}{\alpha}} + \eta \frac{t^\alpha}{\alpha} e^{r\frac{t^\alpha}{\alpha}} + \gamma e^{r\frac{t^\alpha}{\alpha}}$, where γ is determined from $(P - rI)\gamma = \eta$.

   (c) The final possibility is that there are two linearly independent eigenvectors $\xi^{(1)}$ and $\xi^{(2)}$ corresponding to the eigenvalue r. Then two solutions are $Y^{(1)} = \xi^{(1)} e^{r_1\frac{t^\alpha}{\alpha}}$ and $Y^{(2)} = \xi^{(2)} e^{r_2\frac{t^\alpha}{\alpha}}$, a third solution is of the form $Y^{(3)} = \xi \frac{t^\alpha}{\alpha} e^{r\frac{t^\alpha}{\alpha}} + \eta e^{r\frac{t^\alpha}{\alpha}}$ where ξ represents as a linear combination of the eigenvectors $\xi^{(1)}$ and $\xi^{(2)}(\xi = c_1\xi^{(1)} + c_2\xi^{(2)})$.

Example 3.12. *Consider the system*

$$Y^{(\alpha)} = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix} Y.$$

Solution: Let $Y = \xi e^{r\frac{t^\alpha}{\alpha}}$, then

$$\begin{pmatrix} 3-r & 2 & 2 \\ 1 & 4-r & 1 \\ -2 & -4 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.38}$$

Eq (3.38) have a nontrivial solution if and only if the determinant of coefficients is zero, thus

$$\begin{vmatrix} 3-r & 2 & 2 \\ 1 & 4-r & 1 \\ -2 & -4 & -1-r \end{vmatrix} = 0.$$

$$-r^3 + 6r^2 - 11r + 6 = 0 \tag{3.39}$$

Solving eq (3.39), the eigenvalues are $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$. If $r_1 = 1$ this implies

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefor, $\xi_3 = -\xi_1$ and $\xi_2 = 0$. The eigenvector corresponding to $r_1 = 1$ can be taken as

$$\xi^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

and the first solution is

$$Y^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{\frac{t^\alpha}{\alpha}}.$$

If $r_2 = 2$, then

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefor, $\xi_1 = -2\xi_2$ and $\xi_3 = 0$. The eigenvector corresponding to $r_2 = 2$ is

$$\xi^{(2)} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix},$$

and the second solution is

$$Y^{(2)} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} e^{2\frac{t^\alpha}{\alpha}}.$$

If $r_3 = 3$, then

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefor, $\xi_3 = -\xi_2$ and $\xi_1 = 0$. The eigenvector corresponding to $r_2 = 3$ is

$$\xi^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

and the third solution is

$$Y^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{3\frac{t^\alpha}{\alpha}}.$$

The general solution is

$$Y = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{\frac{t^\alpha}{\alpha}} + c_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} e^{2\frac{t^\alpha}{\alpha}} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{3\frac{t^\alpha}{\alpha}}$$

Example 3.13. *Consider the system*

$$Y^{(\alpha)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y.$$

Solution: Let $Y = \xi e^{r\frac{t^\alpha}{\alpha}}$, then

$$\begin{pmatrix} -r & -1 \\ 1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.40}$$

Eq (3.40) have a nontrivial solution if and only if the determinant of coefficients is zero, thus

$$\begin{vmatrix} -r & -1 \\ 1 & -r \end{vmatrix} = 0.$$

The eigenvalues are $r_1 = i$ and $r_2 = \bar{r_1} = -i$. If $r_1 = i$ this implies

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefor, $\xi_2 = i\xi_1$. The eigenvector corresponding to $r_1 = i$ can be taken as

$$\xi^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

So, the corresponding solution is

$$\begin{aligned} Y^{(1)} &= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\frac{t^\alpha}{\alpha}} \\ &= \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos\frac{t^\alpha}{\alpha} + i\sin\frac{t^\alpha}{\alpha}) \\ &= \begin{pmatrix} \cos\frac{t^\alpha}{\alpha} \\ i\cos\frac{t^\alpha}{\alpha} \end{pmatrix} + \begin{pmatrix} i\sin\frac{t^\alpha}{\alpha} \\ -\sin\frac{t^\alpha}{\alpha} \end{pmatrix} \\ &= \begin{pmatrix} \cos\frac{t^\alpha}{\alpha} \\ -\sin\frac{t^\alpha}{\alpha} \end{pmatrix} + i\begin{pmatrix} \sin\frac{t^\alpha}{\alpha} \\ \cos\frac{t^\alpha}{\alpha} \end{pmatrix}. \end{aligned}$$

Then, the vectors $u(t) = \begin{pmatrix} \cos\frac{t^\alpha}{\alpha} \\ -\sin\frac{t^\alpha}{\alpha} \end{pmatrix}$ and $v(t) = i\begin{pmatrix} \sin\frac{t^\alpha}{\alpha} \\ \cos\frac{t^\alpha}{\alpha} \end{pmatrix}$ are real valued solution. The general solution is

$$Y = c_1u(t) + c_2v(t).$$

Example 3.14. *Consider the system*

$$Y^{(\alpha)} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} Y.$$

Solution: Let $Y = \xi e^{r\frac{t^\alpha}{\alpha}}$, then

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.41}$$

Eq (3.41) have a nontrivial solution if and only if the determinant of coefficients is zero, thus

$$\begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = 0.$$

The eigenvalues are $r = r_1 = r_2 = 2$, this implies

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefor, $\xi_1 = -\xi_2$. The eigenvector corresponding to $r = 2$ is

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

and the first solution is

$$Y^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2\frac{t^\alpha}{\alpha}}.$$

A second solution is $Y^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{t^\alpha}{\alpha} e^{2\frac{t^\alpha}{\alpha}} + \eta e^{2\frac{t^\alpha}{\alpha}}$ where η is determined from

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefor, $\eta_1 = 1 - \eta_2$. This implies,

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \eta.$$

And the second solution is

$$Y^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{t^\alpha}{\alpha} e^{2\frac{t^\alpha}{\alpha}} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2\frac{t^\alpha}{\alpha}}.$$

The general solution is

$$Y = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2\frac{t^\alpha}{\alpha}} + c_2[\begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{t^\alpha}{\alpha} e^{2\frac{t^\alpha}{\alpha}} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2\frac{t^\alpha}{\alpha}}].$$

To find a particular solution of the nonhomogeneous system (3.35), we assume that $Y_p = \psi(t)\kappa(t)$ and substituting in eq (3.35) we obtain $\psi^\alpha(t)\kappa(t) + \psi(t)\kappa^\alpha(t) = P\psi(t)\kappa(t) + G(t)$, since $\psi(t)$ is a fundamental matrix [11].
$\psi^\alpha(t) = P\psi(t)$, hence reduces to $\psi(t)\kappa^\alpha(t) = G(t)$.
Recall that $\psi(t)$ is nonsingular on any interval where P is continues, hence $\psi^{-1}(t)$ exists and therefor, $\kappa^\alpha(t) = \psi^{-1}(t)G(t)$.

Therefore we denote $\kappa(t)$ by $\kappa(t) = I_\alpha[\psi^{-1}(t)G(t)]$.
Finally, substituting for $\kappa(t)$ in equation $Y_p = \psi(t)\kappa(t)$ gives the particular solution Y_p of the systems (3.35)

$$Y_p = \psi(t)I_\alpha[\psi^{-1}(t)G(t)],$$

then the general solution of (3.35) will be $Y = Y_p + Y_c$.

Example 3.15. *Consider the system*

$$Y^{(\alpha)} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} Y - \begin{pmatrix} 15 \\ 4 \end{pmatrix} te^{-2\frac{t^\alpha}{\alpha}}.$$

Solution: Let $Y = \xi e^{r\frac{t^\alpha}{\alpha}}$, then

$$\begin{pmatrix} 4-r & 2 \\ 3 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.42}$$

Eq (3.42) have a nontrivial solution if and only if the determinant of coefficients is zero, thus

$$\begin{vmatrix} 4-r & 2 \\ 3 & -1-r \end{vmatrix} = 0.$$

The eigenvalues are $r_1 = -2$ and $r_2 = 5$. If $r_1 = -2$, the eigenvector corresponding to $r_1 = -2$ is

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix},$$

and the first solution is

$$Y^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2\frac{t^\alpha}{\alpha}}.$$

If $r_2 = 5$, then the eigenvector is

$$\xi^{(2)} = \begin{pmatrix} 2 \\ -3 \end{pmatrix},$$

and the second solution is

$$Y^{(2)} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{5\frac{t^\alpha}{\alpha}}.$$

The homogenous solution of the system is

$$Y = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2\frac{t^\alpha}{\alpha}} + c_2 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{5\frac{t^\alpha}{\alpha}}.$$

To find the particular solution, let $Y_p = \psi(t)\kappa(t)$ such that $\kappa(t) = I_\alpha[\psi^{-1}(t)G(t)]$. Now, the fundamental matrix is

$$\psi(t) = \begin{pmatrix} e^{-2\frac{t^\alpha}{\alpha}} & 2e^{5\frac{t^\alpha}{\alpha}} \\ -3e^{-2\frac{t^\alpha}{\alpha}} & -3e^{5\frac{t^\alpha}{\alpha}} \end{pmatrix}.$$

The inverse of $\psi(t)$ is

$$\psi^{-1}(t) = \frac{1}{3}e^{-3\frac{t^\alpha}{\alpha}} \begin{pmatrix} -3e^{5\frac{t^\alpha}{\alpha}} & -2e^{5\frac{t^\alpha}{\alpha}} \\ 3e^{-2\frac{t^\alpha}{\alpha}} & e^{-2\frac{t^\alpha}{\alpha}} \end{pmatrix}.$$

Therefor,

$$\kappa(t) = I_\alpha[\frac{1}{3}e^{-3\frac{t^\alpha}{\alpha}} \begin{pmatrix} -3e^{5\frac{t^\alpha}{\alpha}} & -2e^{5\frac{t^\alpha}{\alpha}} \\ 3e^{-2\frac{t^\alpha}{\alpha}} & e^{-2\frac{t^\alpha}{\alpha}} \end{pmatrix} \begin{pmatrix} -15te^{-2\frac{t^\alpha}{\alpha}} \\ -4te^{-2\frac{t^\alpha}{\alpha}} \end{pmatrix}] = I_\alpha[\begin{pmatrix} -t \\ -7te^{-7\frac{t^\alpha}{\alpha}} \end{pmatrix}].$$

Hence,

$$Y_p = \psi(t)I_\alpha[\begin{pmatrix} -t \\ -7te^{-7\frac{t^\alpha}{\alpha}} \end{pmatrix}] = \psi(t)\int_0^t \begin{pmatrix} -s^\alpha \\ -7s^\alpha e^{-7\frac{s^\alpha}{\alpha}} \end{pmatrix} ds.$$

Chapter 4

Finite difference methods of conformable fractional differential equations

Fractional differential equations (FDEs) have received significant importance because of their wide-range of uses. Also, several problems in physics, biology, chemistry, applied science and engineering are mathematically modeled by systems of ordinary and fractional differential equations ([27]-[8]). Finding the analytical and numerical approximate solutions of different types of (FDEs) became an exciting topic for many researches ([10],[18],[22]). Different numerical and analytical techniques have been investigated and developed for solving (FDEs), especially the nonlinear problems since most of these equations don't have exact solutions.
The major goal of this subject is to find accurate approximate solutions for conformable fractional differential equations. Hence, we carry out this goal by preparing a new method called fractional fintie difference method (FFDM).
(FFDM) was applied to approximate the fractional differential equations. Some basic definitions and mathematical preliminaries of the fractional calculus, fractional Euler method, fractional Taylor method of order two for solving initial value (FDEs) and the fractional Modified and Heun's method for approximating fractional integrals are introduced. We will use the Taylor method to derive fractional Eular method and higher order, fractional Eular method is derived upon assuming h is small enough. Fractional Eular method is fractional Taylor method of order one. Fractional Taylor method of order two gives more accurate approximate values and closer to the exact

values more than fractional Euler method.
Khalil et al.[16] define the conformable fractional derivative of order $\alpha \in (0,1]$ of a function $f : [a, \infty) \to I\!R$ by

$$(T_\alpha f)(t) = \lim_{h \to 0} \frac{f(t + h(t)^{1-\alpha}) - f(t)}{h}. \tag{4.1}$$

An easy consequence of this definition is that if f has the classical derivative, then we have the following relation [16]

$$(T_\alpha f)(t) = t^{1-\alpha} f'(t), \tag{4.2}$$

where $f'(t)$ is the classical derivative of f. We immediately see that the conformable fractional derivative of a constant function is zero. Some basic properties of this conformable fractional derivative can be found in ([5],[6],[1],[16]) in details. This new definition intuitively is a natural extension of standard derivative to non-integer order. Unlike the existing definitions of fractional derivative, there are no special functions such as the Gamma, Beta and Mittag-leffler functions that are not easy to evaluate and implement in the solutions. This conformable derivative has the physical interpretation as a modification of the classical derivative in direction and magnitude of physical quantity [32]. In this chapter, we consider the following conformable fractional differential equation (FDEs).

$$\begin{cases} (T_\alpha y)(t) = f(t, y(t)), \quad t \in [a,b], \; b > a \geq 0, \\ y(a) = y_0. \end{cases} \tag{4.3}$$

4.1 Finite difference derivative of conformable fractional differential equations

In this section, we derived the formulas of four numerical methods for solving eq (4.3), Euler, Taylor, Modified and Heun's methods by using the conformable fractional derivative definition as a personal diligence. We also defined two new formulas for Euler and Taylor methods that depend on their derivation on the conformable fractional derivative definition to compare them with the Euler and Taylor formulas that we derived.

• **Conformable fractional Euler method (CFEM):**

Fractional Euler method is the most elementary approximation technique for solving initial value problems. The goal of this section is to obtain on approximation

solution of the eq (4.3).
Let $a = t_0 < t_1 < ... < t_{k-1} < t_k = b$ be a partition of $[a, b]$ such that $t_n = a + nh$, $\forall n = 0, ..., k$.
The common distance between the points $h = \frac{b-a}{k} = t_{n+1} - t_n$ is called the step size, we will use the Taylor series of $y(t_{n+1})$ about $t = t_n$ to derive conformable fractional Euler method for each $n = 0, 1, ..., k$.

$$y(t_{n+1}) = y(t_n) + y'(t_n)(t_{n+1} - t_n) + y''(\xi_n)\frac{(t_{n+1} - t_n)^2}{2},$$

for some number $\xi_n \in (t_n, t_{n+1})$. Since $y^\alpha(t) = t^{1-\alpha}y'(t)$ and $h = t_{n+1} - t_n$, we have

$$y(t_{n+1}) = y(t_n) + ht_n^{\alpha-1}y^\alpha(t_n) + \frac{h^2}{2}y''(\xi_n).$$

Conformable fractional Euler method is derived upon assuming h is small enough, so that h^2 can be neglected.
Hence, the conformable fractional Euler method is

$$y_{n+1} = y_n + ht_n^{\alpha-1}y_n^\alpha, \tag{4.4}$$

where the notation $y_n = y(t_n)$.
• **Conformable fractional Taylor method of order 2 (CFTM):**
Consider the initial value problem

$$y^{(\alpha)}(t) = f(t, y(t)), \quad y(a) = y_0, \quad t \in [a, b].$$

Expand $y(t)$ in the nth Taylor polynomial about t_n, evaluate at t_{n+1}.

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + ... + \frac{h^n}{n!}y^{(n)}(t_n) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_n).$$

Using Taylor method of order 2 given by

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n), \tag{4.5}$$

to derive conformable fractional Taylor method of order 2 for each $n = 0, 1, ..., k$. Since $y^\alpha(t) = t^{1-\alpha}y'(t)$, we want to find a formula for $y''(t_n)$ to replace it in the second term, we obtain

$$\begin{aligned}
T^{\alpha}(y^{\alpha}(t_n)) &= T^{\alpha}(t_n^{1-\alpha}y'(t_n)) \\
&= t_n^{1-\alpha}T^{\alpha}(y'(t_n)) + y'(t_n)T^{\alpha}(t_n^{1-\alpha}) \\
&= t_n^{1-\alpha}t_n^{1-\alpha}y''(t_n) + y'(t_n)(1-\alpha)t_n^{1-2\alpha} \\
&= t_n^{2-2\alpha}y''(t_n) + (1-\alpha)t_n^{1-2\alpha}t_n^{\alpha-1}y^{\alpha}(t_n) \\
&= t_n^{2-2\alpha}y''(t_n) + (1-\alpha)t^{-\alpha}y^{\alpha}(t_n) \\
&= t_n^{2-2\alpha}y''(t_n) + (1-\alpha)t^{-\alpha}f(t_n, y_n).
\end{aligned}$$

This leads to

$$y''(t_n) = t_n^{2\alpha-2}[T^{\alpha}(f(t_n, y_n)) + (\alpha-1)t^{-\alpha}f(t_n, y_n)].$$

Substituting these results into eq (4.5), we gives

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}t_n^{2\alpha-2}[T^{\alpha}(f(t_n, y_n)) + (\alpha-1)t^{-\alpha}f(t_n, y_n)]. \quad (4.6)$$

Eq (4.6) represents the formula of conformable fractional Taylor method of order 2. Conformable fractional Euler method is conformable fractional Taylor method of order one.

Now, we defined two formulas for conformable fractional Euler and Taylor methods that used the conformable fractional derivative definition in it's derivation.
B. Xin et al.[31] define the first formula of conformable fractional Euler method by

$$y(t_{n+1}) = y(t_n) + \frac{h^{\alpha}}{\alpha}f(t_n, y(t_n)), \quad (4.7)$$

denoted by 1^{st} (CFEM).
And Mohammad Nezhad et al.[19] define the first formula of conformable fractional Taylor method of order 2 by

$$y(t_{n+1}) = y(t_n) + \frac{h^{\alpha}}{\alpha}f(t_n, y(t_n)) + \frac{h^{2\alpha}}{2\alpha^2}T^{\alpha}f(t_n, y(t_n)), \quad (4.8)$$

denoted by 1^{st} (CFTM).
Toprakseven.[29] define the second formula of conformable fractional Euler method by

$$y(t_{n+1}) = y(t_n) + \frac{h^{\alpha}}{\alpha}b_n f(t_n, y(t_n)), \quad b_n = (n+1)^{\alpha} - n^{\alpha}, \quad (4.9)$$

denoted by 2^{nd} (CFEM).
And the second formula of conformable fractional Taylor of order 2 method by

$$y(t_{n+1}) = y(t_n)+\frac{h^\alpha}{\alpha}b_n f(t_n, y(t_n))+\frac{h^{2\alpha}}{2\alpha^2}b_n^2 T^\alpha f(t_n, y(t_n)), \ \ b_n^2 = (n+1)^{2\alpha}-n^{2\alpha}-2n^\alpha b_n, \tag{4.10}$$

denoted by 2^{nd} (CFTM).
We want to compare these formulas with the Euler and Taylor formulas that we derived at the beginning of the section through numerical examples in the next section.

• **Conformable fractional Modified method (CFMM):**
Consider the initial value problem

$$T^\alpha y(t) = f(t, y(t)), \ \ t \in [a, b].$$

To find the conformable fractional Modified method. Take fractional integral for both side over $[t_0, t_1]$, we get

$$\int_{t_0}^{t_1} \frac{T^\alpha y(t)}{t^{1-\alpha}}dt = \int_{t_0}^{t_1} \frac{f(t, y(t))}{t^{1-\alpha}}dt.$$

By Trapezoidal rule $\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}$, we get

$$\int_{t_0}^{t_1} \frac{t^{1-\alpha}y'(t)}{t^{1-\alpha}}dt = \frac{t_1-t_0}{2}[\frac{f(t_0, y(t_0))}{t_0^{1-\alpha}} + \frac{f(t_1, y(t_1))}{t_1^{1-\alpha}}]$$
$$\int_{t_0}^{t_1} y'(t)dt = \frac{h}{2}[\frac{f(t_0, y(t_0))}{t_0^{1-\alpha}} + \frac{f(t_1, y(t_1))}{t_1^{1-\alpha}}]$$
$$y(t_1) - y(t_0) = \frac{h}{2}[\frac{f(t_0, y(t_0))}{t_0^{1-\alpha}} + \frac{f(t_1, y(t_1))}{t_1^{1-\alpha}}].$$

By conformable fractional Euler method $y_1 = y_0 + ht_0^{\alpha-1}f(t_0, y(t_0))$, we get

$$y(t_1) = y(t_0) + \frac{h}{2}[\frac{f(t_0, y(t_0))}{t_0^{1-\alpha}} + \frac{f(t_1, y_0 + ht_0^{\alpha-1}f(t_0, y(t_0)))}{t_1^{1-\alpha}}].$$

In general the conformable fractional Modified method

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}[\frac{f(t_n, y(t_n))}{t_n^{1-\alpha}} + \frac{f(t_{n+1}, y_n + ht_n^{\alpha-1}f(t_n, y(t_n)))}{t_{n+1}^{1-\alpha}}], \tag{4.11}$$

for $n = 0, 1, ..., k$.

• **Conformable fractional Heun's method (CFHM):**
Consider the initial value problem

$$T^\alpha y(t) = f(t, y(t)), \ \ t \in [a, b].$$

To find the conformable fractional Heun's method. Take fractional integral for both side over $[t_0, t_1]$, we get

$$\int_{t_0}^{t_1} \frac{T^\alpha y(t)}{t^{1-\alpha}} dt = \int_{t_0}^{t_1} \frac{f(t, y(t))}{t^{1-\alpha}} dt.$$

By Simpson's rule $\int_a^b f(x)dx \approx \frac{\Delta}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 4f(x_{n-1}) + f(x_n)]$, we get

$$\begin{aligned}
\int_{t_0}^{t_1} \frac{t^{1-\alpha} y'(t)}{t^{1-\alpha}} dt &= \frac{t_1 - t_0}{3}[\frac{f(t_0, y(t_0))}{t_0^{1-\alpha}} + \frac{4f(t_1, y(t_1))}{t_1^{1-\alpha}}] \\
\int_{t_0}^{t_1} y'(t)dt &= \frac{h}{3}[\frac{f(t_0, y(t_0))}{t_0^{1-\alpha}} + \frac{4f(t_1, y(t_1))}{t_1^{1-\alpha}}] \\
y(t_1) - y(t_0) &= \frac{h}{3}[\frac{f(t_0, y(t_0))}{t_0^{1-\alpha}} + \frac{4f(t_1, y(t_1))}{t_1^{1-\alpha}}].
\end{aligned}$$

By conformable fractional Euler method $y_1 = y_0 + ht_0^{\alpha-1} f(t_0, y(t_0))$, we get

$$y(t_1) = y(t_0) + \frac{h}{3}[\frac{f(t_0, y(t_0))}{t_0^{1-\alpha}} + \frac{4f(t_1, y_0 + ht_0^{\alpha-1} f(t_0, y(t_0)))}{t_1^{1-\alpha}}].$$

In general the conformable fractional Heun's method

$$y(t_{n+1}) = y(t_n) + \frac{h}{3}[\frac{f(t_n, y(t_n))}{t_n^{1-\alpha}} + \frac{4f(t_{n+1}, y_n + ht_n^{\alpha-1} f(t_n, y(t_n)))}{t_{n+1}^{1-\alpha}}], \tag{4.12}$$

for $n = 0, 1, ..., k$.

4.2 Numerical tests

In this section, we worked on two examples, one linear and the other nonlinear. In each example we found the approximate solution using all the formulas that we derived in the first section and compared them with the exact solution and found the absolute error for each of them.
First example is linear example.

Example 4.1. *Consider the fractional linear differential equation*

$$y^{(\frac{1}{2})} = t^2 + 2t^{\frac{3}{2}} - y, \quad 0.1 \le t \le 1.1, \tag{4.13}$$

with $y(0.1) = 0.01$ *and* $h = 0.1$.

1. *Find the approximate solution by using*

   (a) *Conformable fractional Euler method (CFEM).*

   (b) *Conformable fractional Taylor method of order* 2 *(CFTM).*

   (c) *First formula of conformable fractional Euler method* 1^{st} *(CFEM) and second formula of conformable fractional Euler method* 2^{nd} *(CFEM).*

   (d) *First formula of conformable fractional Taylor method* 1^{st} *(CFTM) and second formula of conformable fractional Taylor method* 2^{nd} *(CFTM).*

   And compare it with the exact solution given by $y(t) = t^2$.

2. *Compare the results in part (a) with the results in part (c).*

3. *Compare the results in part (b) with the results in part (d).*

4. *Find the approximate solution by using*

   (a) *Conformable fractional Modified method (CFMM).*

   (b) *Conformable fractional Heun's method (CFHM).*

   And compare it with the exact solution.

Solution:

1. (a) For $n = 0, 1, ..., 9,$

$$y_{n+1} = y_n + ht_n^{\alpha-1}y_n^{\alpha} = y_n + ht_n^{\frac{-1}{2}}[t_n^2 + 2t_n^{\frac{3}{2}} - y_n].$$

So,
$y_1 = 0.01 + (0.1)(0.1)^{\frac{-1}{2}}[(0.1)^2 + 2(0.1)^{\frac{3}{2}} - 0.01] = 0.03,$
$y_2 = 0.03 + (0.1)(0.2)^{\frac{-1}{2}}[(0.2)^2 + 2(0.2)^{\frac{3}{2}} - 0.03] = 0.072236,$
and so on Table 4.1 shows the comparison between the approximate values at t_n, $n = 0, 1, ..., 10$ and the exact values.

t_n	Approximation	Exact	Absolute error
0.1	0.01	0.01	0
0.2	0.03	0.04	0.01
0.3	0.072236	0.09	0.017764
0.4	0.135479	0.16	0.024521
0.5	0.219356	0.25	0.030644
0.6	0.323689	0.36	0.036311
0.7	0.448377	0.49	0.041623
0.8	0.593352	0.64	0.046648
0.9	0.758567	0.81	0.051433
1	0.943989	1	0.056011
1.1	1.149590	1.21	0.06041

Table 4.1: Approximation of $y(t)$ by using conformable fractional Euler method.

(b) We need the conformable fractional derivative of $f(t, y(t)) = t^2 + 2t^{\frac{3}{2}} - y$ with respect to the variable t.

$$\begin{aligned} T^\alpha f(t, y(t)) &= T^\alpha[t^2 + 2t^{\frac{3}{2}} - y] \\ &= 2t^{1-\alpha}t + (2)(\frac{3}{2})t^{1-\alpha}t^{\frac{1}{2}} - t^{1-\alpha}y^{1-\alpha}y' \\ &= 2t^{2-\alpha} + t^{\frac{3}{2}-\alpha} - t^{1-\alpha}y^{1-\alpha}t^{1-\alpha}f(t, y(t)) \\ &= 2t^{2-\alpha} + t^{\frac{3}{2}-\alpha} - y^{1-\alpha}[t^2 + 2t^{\frac{3}{2}} - y]. \end{aligned}$$

So, with $\alpha = \frac{1}{2}$

$$T^\alpha f(t_n, y_n(t)) = 2t_n^{\frac{3}{2}} + 3t_n - y_n^{\frac{1}{2}}[t_n^2 + 2t_n^{\frac{3}{2}} - y_n].$$

The conformable fractional Taylor method of this example is

$$y_{n+1} = y_n + ht_n^{\frac{-1}{2}}[t_n^2 + 2t_n^{\frac{3}{2}} - y_n] + \frac{h^2}{2}t_n^{-1}[T^\alpha f(t_n, y_n(t)) - \frac{1}{2}t_n^{\frac{-1}{2}}(t_n^2 + 2t_n^{\frac{3}{2}} - y_n)],$$

for $n = 0, 1, ..., 9$, so
$y_1 = 0.03 + 0.05[0.063246 + 0.3 - 0.0063245 - 0.099999915] = 0.042846$,
$y_2 = 0.08220961 + 0.025[0.1788854 + 0.6 - 0.03643888 - 0.196818075] = 0.09585$,

t_n	Approximation	Exact	Absolute error
0.1	0.01	0.01	0
0.2	0.042846	0.04	0.002846
0.3	0.09585	0.09	0.00585
0.4	0.16868	0.16	0.00868
0.5	0.26116	0.25	0.01116
0.6	0.3732	0.36	0.0132
0.7	0.5046	0.49	0.0146
0.8	0.6554	0.64	0.0154
0.9	0.8255	0.81	0.0155
1	1.015	1	0.015
1.1	1.224	1.21	0.014

Table 4.2: Approximation of $y(t)$ by using conformable fractional Taylor method.

and so on Table 4.2 shows the comparison between the approximate values at t_n and the exact values.

(c) On Table 4.3 shows the comparison between the approximate values at t_n by using 1^{st} (CFEM) and 2^{nd} (CFEM) and the exact values. We see that the 2^{nd} (CFEM) is much closer to the exact solution while the 1^{st} (CFEM) gets bigger.
(d) On Table 4.4 shows the comparison between the approximate values at t_n by using 1^{st} (CFTM) and 2^{nd} (CFTM) and the exact values. We see that the 2^{nd} (CFEM) is closer to the exact solution while the 1^{st} (CFEM) gets bigger.

2. Table 4.5 shows the comparison between (CFEM), 1^{st} (CFEM) and 2^{nd} (CFEM) through the absolute error of each value with exact solution.

3. Table 4.6 shows the comparison between (CFTM), 1^{st} (CFTM) and 2^{nd} (CFTM) through the absolute error of each value with exact solution.

4. (a) Table 4.7 shows the comparison between the approximate values at t_n by using (CFMM) and the exact values.

   (b) Table 4.8 shows the comparison between the approximate values at t_n

t_n	1^{st} (CFEM)	2^{nd} (CFEM)	Exact
0.1	0.01	0.01	0.01
0.2	0.05	0.05	0.04
0.3	0.15681	0.09424	0.09
0.4	0.32240	0.15946	0.16
0.5	0.53969	0.24529	0.25
0.6	0.80369	0.35156	0.36
0.7	1.11095	0.47817	0.49
0.8	1.45904	0.62503	0.64
0.9	1.84613	0.79209	0.81
1	2.27082	0.97934	1
1.1	2.73199	1.18673	1.21

Table 4.3: Approximation of $y(t)$ by using 1^{st} (CFEM) and 2^{nd} (CFEM).

t_n	1^{st} (CFTM)	2^{nd} (CFTM)	Exact
0.1	0.01	0.01	0.01
0.2	0.12138	0.12138	0.04
0.3	0.33203	0.43866	0.09
0.4	0.62255	0.45973	0.16
0.5	0.98435	0.51717	0.25
0.6	1.41395	0.60392	0.36
0.7	1.91075	0.71648	0.49
0.8	2.47622	0.85279	0.64
0.9	3.11371	1.01161	0.81
1	3.82861	1.19207	1
1.1	4.62881	1.39356	1.21

Table 4.4: Approximation of $y(t)$ by using 1^{st} (CFTM) and 2^{nd} (CFTM).

by using (CFHM) and the exact values.

t_n	Absolute errors (CFEM)	Absolute errors 1^{st} (CFEM)	Absolute errors 2^{nd} (CFEM)	Exact
0.1	0	0	0	0.01
0.2	0.01	0.01	0.01	0.04
0.3	0.017764	0.06681	0.00424	0.09
0.4	0.024521	0.1624	0.00054	0.16
0.5	0.030644	0.28969	0.00471	0.25
0.6	0.036311	0.44369	0.00844	0.36
0.7	0.041623	0.62095	0.01183	0.49
0.8	0.046648	0.81904	0.01497	0.64
0.9	0.051433	1.03613	0.01791	0.81
1	0.056011	1.27082	0.02066	1
1.1	0.06041	1.52199	0.02327	1.21

Table 4.5: Numerical values of $y(t)$ according to (CFEM), 1^{st} (CFEM) and 2^{nd} (CFEM).

t_n	Absolute errors (CFTM)	Absolute errors 1^{st} (CFTM)	Absolute errors 2^{nd} (CFTM)	Exact
0.1	0	0	0	0.01
0.2	0.002846	0.08138	0.08138	0.04
0.3	0.00585	0.24203	0.34866	0.09
0.4	0.00868	0.46255	0.29973	0.16
0.5	0.01116	0.73435	0.26717	0.25
0.6	0.0132	1.05395	0.24392	0.36
0.7	0.0146	1.42075	0.22648	0.49
0.8	0.0154	1.83622	0.21279	0.64
0.9	0.0155	2.30371	0.20161	0.81
1	0.015	2.82861	0.19207	1
1.1	0.014	3.41881	0.18356	1.21

Table 4.6: Numerical values of $y(t)$ according to (CFTM), 1^{st} (CFTM) and 2^{nd} (CFTM).

t_n	Approximation	Exact	Absolute error
0.1	0.01	0.01	0
0.2	0.041118	0.04	0.001118
0.3	0.09183	0.09	0.00183
0.4	0.16635	0.16	0.00635
0.5	0.25618	0.25	0.00618
0.6	0.36605	0.36	0.00605
0.7	0.49594	0.49	0.00594
0.8	0.64585	0.64	0.00585
0.9	0.815776	0.81	0.005776
1	1.00571	1	0.00571
1.1	1.21566	1.21	0.00566

Table 4.7: The absolute errors between the approximate values obtained by CFMM and exact solutions.

t_n	Approximation	Exact	Absolute error
0.1	0.01	0.01	0
0.2	0.07298	0.04	0.03298
0.3	0.16007	0.09	0.07007
0.4	0.272505	0.16	0.112505
0.5	0.41060	0.25	0.1606
0.6	0.57435	0.36	0.21435
0.7	0.763636	0.49	0.273636
0.8	0.97012	0.64	0.33012
0.9	1.21135	0.81	0.40135
1	1.47738	1	0.47738
1.1	1.76812	1.21	0.55812

Table 4.8: The absolute errors of $y(t)$ between the approximate values obtained by CFHM and exact solutions.

Example 4.2. *Consider the fractional nonlinear differential equation*

$$y^{(\frac{1}{2})} = y^{\frac{1}{4}} + \sqrt{\frac{y}{t}}, \quad t \in [1,2],$$

with $y(1) = 2$ *and* $h = 0.1$

1. *Find the approximate solution by using*

   (a) *Conformable fractional Euler method (CFEM).*

   (b) *Conformable fractional Taylor method of order* 2 *(CFTM).*

   (c) *First formula of conformable fractional Euler method* 1^{st} *(CFEM) and second formula of conformable fractional Euler method* 2^{nd} *(CFEM).*

   (d) *First formula of conformable fractional Taylor method* 1^{st} *(CFTM) and second formula of conformable fractional Taylor method* 2^{nd} *(CFTM).*

   And compare it with the exact solution given by $y(t) = 2t(\frac{t^{\frac{1}{4}}-1}{2^{\frac{1}{4}}} + 1)^4$.

2. *Compare the results in part (a) with the results in part (c).*

3. *Compare the results in part (b) with the results in part (d).*

4. *Find the approximate solution by using*

   (a) *Conformable fractional Modified method (CFMM).*

   (b) *Conformable fractional Heun's method (CFHM).*

   And compare it with the exact solution.

Solution:

1. (a) For $n = 0, 1, ..., 9,$

$$y_{n+1} = y_n + ht_n^{\alpha-1}y_n^{\alpha} = y_n + ht_n^{\frac{-1}{2}}[y_n^{\frac{1}{4}} + \sqrt{\frac{y_n}{t_n}}].$$

So,
$y_1 = 2 + 0.1(1)^{\frac{-1}{2}}[(2)^{\frac{1}{4}} + \sqrt{\frac{2}{1}}] = 2.260342,$
$y_2 = 2.260342 + 0.1(1.1)^{\frac{-1}{2}}[(2.260342)^{\frac{1}{4}} + \sqrt{\frac{2.260342}{1.1}}] = 2.513925,$
and so on Table 4.9 shows the comparison between the approximate values at t_n and the exact values.

t_n	Approximation	Exact	Absolute error
1	2	2	0
1.1	2.26034	2.38394	0.1236
1.2	2.51393	2.79919	0.28526
1.3	2.76100	3.24549	0.48449
1.4	3.00188	3.72257	0.72069
1.5	3.23688	4.23022	0.99334
1.6	3.46634	4.76821	1.30187
1.7	3.78534	5.33638	1.55104
1.8	4.11500	5.93454	1.81954
1.9	4.33533	6.56252	2.22719
2	4.54961	7.22019	2.67058

Table 4.9: Approximation of $y(t)$ by using conformable fractional Euler method.

(b) Because $y^{(\alpha)} = y^{\frac{1}{4}} + \sqrt{\frac{y}{t}}$, we have

$$\begin{aligned}
T^{\alpha} f(t, y(t)) &= T^{\alpha}[y^{\frac{1}{4}} + \sqrt{\frac{y}{t}}] \\
&= \frac{1}{4} t^{1-\alpha} y^{\frac{1}{4}-\alpha} y' + \frac{1}{2} t^{\frac{1}{2}-\alpha} y^{\frac{1}{2}-\alpha} y' + \frac{-1}{2} t^{\frac{-1}{2}-\alpha} y^{\frac{1}{2}} \\
&= [\frac{1}{4} t^{1-\alpha} y^{\frac{1}{4}-\alpha} + \frac{1}{2} t^{\frac{1}{2}-\alpha} y^{\frac{1}{2}-\alpha}] y' - \frac{1}{2} t^{\frac{-1}{2}-\alpha} y^{\frac{1}{2}} \\
&= [\frac{1}{4} t^{1-\alpha} y^{\frac{1}{4}-\alpha} + \frac{1}{2} t^{\frac{1}{2}-\alpha} y^{\frac{1}{2}-\alpha}] t^{\alpha-1} y^{(\alpha)} - \frac{1}{2} t^{\frac{-1}{2}-\alpha} y^{\frac{1}{2}} \\
&= [\frac{1}{4} y^{\frac{1}{4}-\alpha} + \frac{1}{2} t^{\frac{-1}{2}} y^{\frac{1}{2}-\alpha}] y^{(\alpha)} - \frac{1}{2} t^{\frac{-1}{2}-\alpha} y^{\frac{1}{2}}.
\end{aligned}$$

So, with $\alpha = \frac{1}{2}$

$$T^{\alpha} f(t_n, y_n(t)) = [\frac{1}{4} y^{\frac{-1}{4}} + \frac{1}{2} t^{\frac{-1}{2}}][y_n^{\frac{1}{4}} + \sqrt{\frac{y_n}{t_n}}] - \frac{1}{2} t^{-1} y^{\frac{1}{2}}.$$

The conformable fractional Taylor method of this example is

$$y_{n+1} = y_n + h t_n^{\frac{-1}{2}} [y_n^{\frac{1}{4}} + \sqrt{\frac{y_n}{t_n}}] + \frac{h^2}{2} t_n^{-1} [T^{\alpha} f(t_n, y_n(t)) - \frac{1}{2} t_n^{\frac{-1}{2}} (y_n^{\frac{1}{4}} + \sqrt{\frac{y_n}{t_n}})],$$

for $n = 0, 1, ..., 9$, so
$y_1 = 2.26034 + 0.005[1.141905334 - 1.301710339] = 2.25954$,
$y_2 = 2.25954{+}0.095346258(2.6592226){+}0.005(1.1)^{-1}[1.09332213{-}1.2677346] =$
2.51229,
and so on Table 4.10 shows the comparison between the approximate values at t_n and the exact values.

t_n	Approximation	Exact	Absolute error
1	2	2	0
1.1	2.25954	2.38394	0.1244
1.2	2.51229	2.79919	0.2869
1.3	2.75879	3.24549	0.4867
1.4	2.99918	3.72257	0.72339
1.5	3.23378	4.23022	0.99644
1.6	3.46290	4.76821	1.30531
1.7	3.68686	5.33638	1.64952
1.8	3.90594	5.93454	2.0286
1.9	4.12042	6.56252	2.4421
2	4.33055	7.22019	2.88964

Table 4.10: Approximation of $y(t)$ by using conformable fractional Taylor method.

(c) On Table 4.11 shows the comparison between the approximate values at t_n by using 1^{st} (CFEM) and 2^{nd} (CFEM) and the exact values. We see that the 2^{nd} (CFEM) is much closer to the exact solution while the 1^{st} (CFEM) gets bigger and the values oscillate frequently and not stable.

(d) On Table 4.12 shows the comparison between the approximate values at t_n by using 1^{st} (CFTM) and 2^{nd} (CFTM) and the exact values. We see that the 2^{nd} (CFTM) is much closer to the exact solution while the 1^{st} (CFTM) gets bigger and the values oscillate frequently and not stable.

t_n	1^{st} (CFEM)	2^{nd} (CFEM)	Exact
1	2	2	2
1.1	3.64655	3.64655	2.38394
1.2	5.67205	4.48554	2.79919
1.3	8.02311	5.16673	3.24549
1.4	10.65873	5.76007	3.72257
1.5	13.54659	6.29421	4.23022
1.6	16.66058	6.78451	4.76821
1.7	19.97922	7.24044	5.33638
1.8	23.48453	7.66839	5.93454
1.9	27.16126	8.07294	6.56252
2	30.99637	8.45749	7.22019

Table 4.11: Approximation of $y(t)$ by using 1^{st} (CFEM) and 2^{nd} (CFEM).

t_n	1^{st} (CFTM)	2^{nd} (CFTM)	Exact
1	2	2	2
1.1	4.20331	4.20331	2.38394
1.2	4.93283	5.25705	2.79919
1.3	7.41174	6.00817	3.24549
1.4	10.23249	6.65619	3.72257
1.5	13.35024	7.23568	4.23022
1.6	16.73009	7.76495	4.76821
1.7	20.34415	8.25516	5.33638
1.8	24.16958	8.71376	5.93454
1.9	28.18739	9.14607	6.56252
2	32.38144	9.55603	7.22019

Table 4.12: Approximation of $y(t)$ by using 1^{st} (CFTM) and 2^{nd} (CFTM).

2. Table 4.13 shows the comparison between (CFEM), 1^{st} (CFEM) and 2^{nd} (CFEM) through the absolute error of each value with exact solution.

3. Table 4.14 shows the comparison between (CFTM), 1^{st} (CFTM) and 2^{nd} (CFTM) through the absolute error of each value with exact solution.

t_n	Absolute errors (CFEM)	Absolute errors 1^{st} (CFEM)	Absolute errors 2^{nd} (CFEM)	Exact
1	0	0	0	2
1.1	0.1236	1.26261	1.26261	2.38394
1.2	0.28525	2.87286	1.68635	2.79919
1.3	0.48449	4.77762	1.92124	3.24549
1.4	0.72069	6.93616	2.0375	3.72257
1.5	0.99334	9.31637	2.06399	4.23022
1.6	1.30187	11.89237	2.0163	4.76821
1.7	1.55104	14.64284	1.90406	5.33638
1.8	1.81954	17.54999	1.73385	5.93454
1.9	2.22719	20.59874	1.51042	6.56252
2	2.67058	23.77618	1.2373	7.22019

Table 4.13: Numerical values of $y(t)$ according to (CFEM), 1^{st} (CFEM) and 2^{nd} (CFEM).

t_n	Absolute errors (CFTM)	Absolute errors 1^{st} (CFTM)	Absolute errors 2^{nd} (CFTM)	Exact
1	0	0	0	2
1.1	0.1244	1.81937	1.81937	2.38394
1.2	0.2869	2.13364	2.45786	2.79919
1.3	0.4867	4.16625	2.76268	3.24549
1.4	0.72339	6.50992	2.93362	3.72257
1.5	0.99644	9.12002	3.00546	4.23022
1.6	1.30531	11.96188	2.99674	4.76821
1.7	1.64952	15.00777	2.91878	5.33638
1.8	2.0286	18.23504	2.77922	5.93454
1.9	2.4421	21.62487	2.58355	6.56252
2	2.88964	25.16125	2.33584	7.22019

Table 4.14: Numerical values of $y(t)$ according to (CFTM), 1^{st} (CFTM) and 2^{nd} (CFTM).

In general, we notice through the values presented in the tables in the part 2 and 3 of the two examples that the formulas of conformable fractional Euler and Taylor methods that we derived in the first section is closer to exact solution and has the lowest and more stability error rate, while the 2^{nd} (CFEM) and 2^{nd} (CFTM) formulas in the two examples have low error rates and approximately close to the exact solution but not stable.
The 1^{st} (CFEM) and 1^{st} (CFTM) in the two examples have very large error rates, and it is used for special cases and can not be applied in general.

4. (a) Table 4.15 shows the comparison between the approximate values at t_n by using (CFMM) and the exact values.

   (b) Table 4.16 shows the comparison between the approximate values at t_n by using (CFHM) and the exact values.

t_n	Approximation	Exact	Absolute error
1	2	2	0
1.1	2.25696	2.38394	0.12698
1.2	2.50715	2.79919	0.29204
1.3	2.75126	3.24549	0.49423
1.4	2.98888	3.72257	0.73369
1.5	3.22074	4.23022	1.00948
1.6	3.44717	4.76821	1.32104
1.7	3.66848	5.33638	1.6679
1.8	3.88498	5.93454	2.04956
1.9	4.09695	6.56252	2.46557
2	4.30464	7.22019	2.91555

Table 4.15: The absolute errors between the approximate values obtained by CFMM and exact solutions.

t_n	Approximation	Exact	Absolute error
1	2	2	0
1.1	2.42489	2.38394	0.04095
1.2	2.84963	2.79919	0.05044
1.3	3.27218	3.24549	0.02669
1.4	3.69126	3.72257	0.03131
1.5	4.10605	4.23022	0.12417
1.6	4.51607	4.76821	0.25214
1.7	4.92103	5.33638	0.41535
1.8	5.32079	5.93454	0.61375
1.9	5.71529	6.56252	0.84723
2	6.10455	7.22019	1.11564

Table 4.16: The absolute errors between the approximate values obtained by CFHM and exact solutions.

Conclusion

The objective of the this thesis is to use conformable fractional derivative which is simpler and more efficient. The new definition reflects a natural extension of normal derivative to solve fractional differential equations.
In this thesis we found analytic expressions for the finite difference method of the fractional differential equations based on the new definition of the conformable fractional derivative.

Bibliography

[1] T. Abdeljawad, *On conformable fractional calculus,* J. Comput. Appl. Math, 279:57-66, 2015.

[2] M. Abu Hammad and R. Khalil, *Abel's formula and Wronskian for conformable fractional differential equations*, International Journal of differential equations and applictions, 13:177-183, 2014.

[3] M. Abu Hammad and R. Khalil, *Conformable fractional heat differential equation*, Inter. J. Pure Appl. Math, 94(2):215-221, 2014.

[4] M. Abu Hammad and R. Khalil, *System of linear fractional differential equations authors contributions*, Jordan. 2016.

[5] H.F. Ahmed, *Fractional Euler method an effective tool for solving fractional differential equation*, Journal of the Egyptian Mathematical Society, 26(1):38-43, 2018.

[6] R.B. Albadarneh, *Fractional Euler method and finite difference formula using conformable fractional derivative*, ISER 10th International conference At: Kuala Lumpur, Malaysia, 2015.

[7] Naeem M.H. Alkoumi, Z. Al-Zhour and S. Kumar. *On the analytical solutions of some conformable nonlinear fractional phsical problems*, 2019.

[8] T.M. Atanackovic and B. Stankovic, *On a system of differential equations with fractional derivatives arising in Rod theory*, J. Phys. A, 37:1241-1250, 2004.

[9] A. Atangana, D. Baleanu and A. Alsaedi.*New properties of conformable derivative*. Open Mathematics, 13:889-898, 2015.

[10] V. Daftardar-Gejji and A. Bakahani, *Analysis of a system of fractional differential equations.* J. Math. Anal. Appl, 293:511-522, 2004.

[11] U. Ghosh, M. Sarkar and D. Shantanu, *Solution of linear fractional non-homogeneous differential equations with Jumarie fractional derivative and evaluation of particular integral*, American J. Math., Anal. 3(3):54-64, 2015.

[12] A. Gokdogan, E. Unal and E. Celik, *Existence and uniqueness theorems for sequential linear conformable fractional differential equations*, to appear in miskolc mathematical notes.

[13] M. Ilei, J. Biazar and Z. Ayati, *General solution of Bernoulli and Riccati fractional differential equations based on conformable fractional derivative*, Inter. J. Appl. Math., Research 6(2):49-51, 2017.

[14] O.S. Iyiola and G.O. Ojo, *On the analytical solution of Fornberg-Whitham equation with the new fractional derivative,* pramana j. of physics. V85:567-575, 2015.

[15] K. Jaber and S. Al-Tarawneh, *Exact solution of Riccati fractional differential equation*, Universial J. Appl. Math, 4(3):51-54, 2016.

[16] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics 264:65-70, 2014.

[17] A. Kilbas, H. Srivastava and J. Trujillo, *Theory and applications of fractional differential equations*, Math. Studies, North-Holland, New York, 2006.

[18] K.S. Miller, *An introduction to fractional calculus and fractional differential equations,* J. Wiley and Sons, New York, 1993.

[19] V.M. Nezhad, M. Eslami and H. Rezazadeh, *Stability analysis of linear conformable fractional differential equations system with time delays*, Boletim da Sociedade Paranaense de Matematica, 38(6):159-171, 2020.

[20] K. Oldham and J. Spanier, *The fractional calculus, theory and applications of differentiation and integration of arbitrary order*, Academic Press, USA, 1974.

[21] I. Pondlubny, *Fractional differential equations: An Introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, USA. 1998.

[22] I. Podlubny, *Fractional differential equations*, Academic Press, USA, 1999.

[23] S. Samko, O. Marichev and A. Kikbas, *Fractional integrals and derivatives and some of their applications*, Sci and Tech. Minsk, Russian, 1987.

[24] S. Samko, A. Kilbas and O. Marichev, *Fractional integrals and derivatives*, Gordon and Breach, 1993.

[25] M.Z. Sarikaya, Z. Dahmani and F. Ahmad, *(k,s)-Riemann-Liouville fractional integral and applications*, Hacettepe Journal of Mathematics and Statistics, 45(1):77-89, 2016.

[26] N. Sene, *Solutions for some conformable differential equations*, Progr. Fract. Diff. Appl 4(4):493-501, 2018.

[27] N. Shimizu and W. Zhang, *Fractional calculus approach to dynamic problems of Viscoelastic materials*, JSME series, C-Mechanical Systems, Machine Elements and Manufacturing, 42:825-837, 1999.

[28] George F. Simmons, *Differential equations whit applications and historical notes,* McGraw-Hill, Inc. New York, 1974.

[29] Suayip. Toprakseven, *Numerical solutions of conformable fractional differential equations by Taylor and finite difference methods*, Journal of Natural and Applied Sciences, 850-863, 2019.

[30] L. Wang and J. Fu J. *Non-noether symmetries of Hamiltonian systems with conformable fractional derivatives*, Chinese Phys. B, 25(1):4501, 2016.

[31] B. Xin, W. Peng, Y. Kwon and Y. Liu, *Modeling, discretization and hyperchaos detection of conformable derivative approach to a financial system with market confidence and ethics risk*, Advances in Difference Equations, 138, 2019.

[32] D. Zhao and M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo, 54:903-917, 2017.