Adomian Decomposition Method: Numerical Implementation and Physical Applications

Prepared by

Sa'dia Sarahna

Supervisor

Dr. Tareq Amro

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By
Sa'dia Jebreel Sarahna

This thesis was defended successfully on 30/4/2017 and approved by:

Committee Members:
Signature

- Dr. Tareq Amro   Supervisor   ....

- Dr. Hasan Almanasreh   Internal Examiner   ....

- Dr. Amjad Barham   External Examiner   ....

Declaration

I declare that the master thesis entitled "Adomian Decomposition Method: Numerical Implementation and Physical Applications" is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Sa'dia Sarahna

Signature:_____   Date:_____

Dedications

To my parents, my husband and my child Mohammed

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Abstract

The Adomian decomposition method was firstly introduced in 1980 by George Adomian. This method is analytical numerical method for solving differential equations. Indeed, the Adomian decomposition method is based on splitting the given equation into linear and nonlinear parts. The nonlinear part is decomposed into a series of Adomian polynomials.

This thesis is mainly concerned with the Adomian decomposition method for both ordinary and partial differential equations. Firstly, we introduce the Adomian decomposition method and Adomian polynomials. Secondly, we use Adomian decomposition method for solving linear and nonlinear differential equations. Finally, we solve a convection between two parallel walls equation, a diffusion of oxygen in absorbing tissue equation and Burgers' equation by using Adomian decomposition method.

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Adomian Decomposition Method

1.1   Introduction

Since its introduction in the 1980s, the Adomian Decomposition Method (ADM) has proven to be an efficient and reliable method for solving many types of problems. Originally developed to solve nonlinear functional equations, the ADM has since been used for a wide range of equation types (like boundary value problems, integral equations, equations arising in flow of incompressible and compressible fluids, etc...), [11].

The ADM involves separating the equation under investigation into linear and nonlinear portions. The linear operator representing the linear portion of the equation is inverted and the inverse operator is then applied to the equation. Any given conditions are taken into consideration. The nonlinear portion is decomposed into a series of Adomian polynomials. ADM generates a solution in the form of a series whose terms are determined by a recursive relationship using these Adomian polynomials. The method provides the solution in a rapidly convergent series with components that can be elegantly computed [1].

The main advantage of the method is that it can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or

inhomogeneous, with constant coefficients or with variable coefficients. Another important advantage is that the method is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution [9].

1.2   Decomposition method and Adomian polynomials

Solution of linear and nonlinear differential equations can be carried out by using an approximation method called the decomposition method. Decomposition method can be used for solving operator equation of the form $Fu = g$ where the operator F may be partial differential operator, our attention here is the case where F is differential operator.

Basically two techniques are involved in applying this method. First, the nonlinear part in the equation to be solved is written in terms of the Adomian's polynomials. Second, the assumed solution $u = F^{-1}g$ is decomposed into components to be determined, such that the first components is the solution for the linear part of F, or of a suitable invertible part, including conditions on u, the other components are then found in terms of preceding components [3].

Definition 1.1. *[22] (Decomposition series of finite-order p) A decomposition series of finite-order p is a series $\sum C_k$, where each C_k is an E-valued function of the $p(k+1)$ variables $X_0^{(1)}, \ldots, X_k^{(1)}, \ldots, X_0^{(p)}, \ldots, X_k^{(p)}$*

The decomposition series of first order is simply called the decomposition series.

Definition 1.2. *[22] (Weak convergence of the decomposition series of finite-order p) A decomposition series of finite-order p is weakly convergent if for each*

collection of p convergent series in E $\left(\sum u_n^{(1)}, \ldots, u_n^{(p)}\right)$, *the series*

$$\sum C_k \left(u_0^{(1)}, \ldots, u_k^{(1)}, \ldots, u_0^{(p)}, \ldots, u_k^{(p)}\right)$$

in E converge.

Definition 1.3. *[22] (Strong convergence of the decomposition of nite-order p) A decomposition series of finite-order p is strongly convergent if it is weakly convergent and if its sum is depends only on the sum of the series in E, i.e.*

$$\sum_{n=0}^{\infty} u_n^{(i)} = \sum_{n=0}^{\infty} v_n^{(i)}$$

$$\Rightarrow \quad S\left(\sum u_n^{(1)}, \ldots, u_n^{(p)}\right) = S\left(\sum v_n^{(1)}, \ldots, v_n^{(p)}\right), \quad \forall i \in [1, p]$$

Definition 1.4. *[22] (Decomposition Scheme) Let* $\sum C_k(x_0, \ldots, x_k)$ *be a strongly convergent decomposition series. The decomposition scheme associated with* $\sum C_k$ *is the recurrent scheme* $u_0 = 0$, $u_{n+1} = C_n(u_0, \ldots, u_n)$, *which constructs a series* $\sum C_n$ *in a Banach space E.*

Definition 1.5. *[22] (Decomposition Method) Is the method consisting of constructing the solution of an equation with a decomposition scheme The ADM consists of decomposing the unknown function* $u(x, t)$ *of any equation into sum of infinite number of components defined by*

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t).$$

The ADM consists of splitting the given equation into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary conditions and the terms involving the independent variable alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomian polynomials and finding the successive terms of the series solution by recurrent relation using Adomian polynomials.

Adomian polynomials are the key in solving nonlinear equations, and which notion was named the Adomian polynomials by Rach [19]. The Adomian decomposition technique suggests that the unknown solution $u(x, t)$ can be represented by the following decomposition series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t),$$

with u_n being computed recursively in an elegant way. However, the nonlinear term $F(u)$, such as u^2, u^3, $\sin u$, e^u, uu_x, etc, can be expressed by an infinite series of the Adomian polynomials A_n

$$F(u) = \sum_{n}^{\infty} A_{n=0}(u_0, u_1, u_2, \ldots, u_n), \tag{1.1}$$

where the Adomian polynomials A_n can be evaluated for all forms of nonlinearity.

Definition 1.6. *[19] (Adomian Polynomials) Let F be an analytical function and $\sum u_n$ a convergent series in a Banach space E. Then the Adomian polynomials A_n for the nonlinear term $F(u)$ can be evaluated by the following expression*

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left(F(\sum_{n=0}^{\infty} \lambda^n u_n) \right) \Big|_{\lambda=0}.$$

Example 1.1. *The Adomian polynomials for $F(u) = u^2$ are*

$$
\begin{aligned}
A_0 &= u_0^2, \\
A_1 &= 2u_0 u_1, \\
A_2 &= u_1^2 + 2u_0 u_2, \\
A_3 &= 2u_1 u_2 + 2u_0 u_3, \\
A_4 &= u_2^2 + 2u_1 u_3 + 2u_0 u_4, \\
&\vdots
\end{aligned}
$$

Example 1.2. *The Adomian polynomials for $F(u) = \sin u$ are*

$$
\begin{aligned}
A_0 &= \sin u_0, \\
A_1 &= u_1 \cos u_0, \\
A_2 &= u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0, \\
&\;\;\vdots
\end{aligned}
\tag{1.2}
$$

Remark: In ADM, the solution $u(x,t)$ is decomposed in the form of an infinite series given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Further, the nonlinear function $N(u)$ is assumed to admit the representation

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n),$$

where A'_ns are called $k - th$ order Adomian polynomials. In the linear case $N(u) = u$, A_n simply reduces to u_n. Adomian's method is simple in principle, but involves tedious calculations of Adomian polynomials. Adomian gave a method for determining these Adomian polynomials by parameterizing $u(x,t)$ as

$$u_\lambda(x,t) = \sum_{n=0}^{\infty} u_n(x,t)\lambda^n,$$

and assuming $N(u_\lambda)$ to be analytic in λ, which is decomposed as

$$N(u_\lambda) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n)\lambda^k.$$

Hence, the Adomian polynomials A_m are given by

$$A_m(u_0, u_1, \ldots, u_n) = \frac{1}{m!} \frac{\partial^m N(u_\lambda)}{\partial \lambda^m}\bigg|_{\lambda=0}, \quad \forall m \in \mathbb{N}\bigcup 0,$$

Theorem 1.1. *[15] Let ϕ and ψ be functions of the parameter λ*
$\phi = \sum_{k=0}^{\infty} u_n\lambda^n$, $\psi = \sum_{k=0}^{\infty} w_n\lambda^n$, then it holds

$$(i) \ A_m(\phi) = u_m,$$

$$(ii) \ A_m(\lambda^k \phi) = A_{m-k}(\phi),$$

$$(iii) \ A_m(\phi\psi) = \sum_{k=0}^{m} A_k(\phi)A_{m-k}(\psi) = \sum_{k=0}^{m} A_k(\psi)A_{m-k}(\phi),$$

$$(iv) \ A_m(\phi^{n+1}) = \sum_{k=0}^{m} A_k(\phi)A_{m-k}(\phi^n) = \sum_{k=0}^{m} A_{m-k}(\phi)A_k(\phi^n)$$

where $m \geq 0$ and $0 \leq k \leq m$ are integers.

Proof. (i) According to Taylor theorem, the unique coefficient u_m of the Maclaurin series of ϕ is given by

$$u_m = \frac{1}{m!} \frac{\partial^m \phi}{\partial \lambda^m} \Big|_{\lambda=0},$$

which gives (i) by means of the definition of $A_m(\phi)$.

(ii) It holds

$$\lambda^k \phi = \lambda^k \sum_{i=0}^{\infty} u_i \lambda^i = \sum_{i=0}^{\infty} u_i \lambda^{i+k} = \sum_{m=k}^{\infty} u_{m-k} \lambda^m,$$

which gives by means of (i) that

$$A_m(\lambda^k \phi) = u_{m-k} = A_{m-k}(\phi).$$

(iii) According to Leibnitz's rule for derivatives of product, it holds

$$\frac{\partial^m(\phi\psi)}{\partial \lambda^m} = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} \frac{\partial^i \phi}{\partial \lambda^i} \frac{\partial^{m-i}\psi}{\partial \lambda^{m-i}} = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} \frac{\partial^i \psi}{\partial \lambda^i} \frac{\partial^{m-i}\phi}{\partial \lambda^{m-i}},$$

which gives that

$$A_m(\phi\psi) = \frac{1}{m!} \frac{\partial^m(\phi\psi)}{\partial \lambda^m} \Big|_{\lambda=0} = \sum_{k=0}^{\infty} \left(\frac{1}{k!} \frac{\partial^k(\phi)}{\partial \lambda^i} |_{\lambda=0} \right) \left(\frac{1}{(m-k)!} \frac{\partial^{m_k}(\psi)}{\partial \lambda^{m_k}} \Big|_{\lambda=0} \right) = \sum_{k=0}^{m} A_k(\psi)A_{m-k}(\phi).$$

Similarly, it holds

$$A_m(\phi\psi) = \sum_{k=0}^{\infty} A_k(\psi)A_{m-k}(\phi).$$

(iv) Write $\Phi = \phi^n$. According to (iii), it holds

$$A_m(\phi^{n+1}) = A_m(\Phi^n \phi) = \sum_{k=0}^{\infty} A_k(\Phi)A_{m-k}(\phi).$$

Similarly, it holds

$$A_m(\phi^{n+1}) = \sum_{k=0}^{\infty} A_k(\phi) A_{m-k}(\Phi).$$

□

Theorem 1.2. *[18] For function $f(u) = u^k$, the corresponding mth-order Adomian polynomial is given by*

$$A_m(u^k) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{k-2}=0}^{r_{k-3}} u_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} u_{r_{k-2}-r_{k-1}} u_{r_{k-1}},$$

(1.3)

where $m \geq 0$ and $k \geq 0$ are positive integers.

Proof. The statement can be proved by the method of mathematical induction.
(i) According to (1.1), it is obvious that the statement holds when $\sigma = 2$.
(ii) Assume that the statement holds when $\sigma = 2$, i.e.

$$A_m(u^k) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{k-2}=0}^{r_{k-3}} u_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} u_{r_{k-2}-r_{k-1}} u_{r_{k-1}},$$

where $m \geq 0$ and $k \geq 2$ are integers. Replacing r_j by r'_{j+1} and m by r'_1, the above expression reads

$$A_{r'_1}(u^k) = \sum_{r'_2=0}^{r'_1} u_{r'_1-r'_2} \sum_{r'_3=0}^{r'_2} u_{r'_2-r'_3} \sum_{r'_4=0}^{r'_3} u_{r'_3-r'_4} \cdots \sum_{r'_k=0}^{r'_{k-1}} u_{r'_{k-1}-r'_k} u_{r'_k},$$

using the above expression and by means

$$A_m(u^{k+1}) = \sum_{r'_1=0}^{m} A_{m-r'_1}(u) A_{r'_1}(u^k)$$

$$= \sum_{r'_1=0}^{m} u_{m-r'_1} \sum_{r'_2=0}^{r'_1} u_{r'_1-r'_2} \sum_{r'_3=0}^{r'_2} u_{r'_2-r'_3} \sum_{r'_4=0}^{r'_3} u_{r'_3-r'_4} \cdots \sum_{r'_k=0}^{r'_{k-1}} u_{r'_{k-1}-r'_k} u_{r'_k}.$$

Therefor, the statement holds for $\sigma = k + 1$.
(iii) According to (i) and (ii), the statement holds for any positive integer $\sigma \geq 2$.

□

Theorem 1.3. *[18] For a parametric series $u(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n$, it holds*

$$\frac{1}{m!}\frac{\partial^m f(u(\lambda))}{\partial \lambda^m} = \frac{1}{m!}\frac{\partial^m}{\partial \lambda^m} f\left(\sum_{i=0}^{m} u_i \lambda^i\right), \tag{1.4}$$

where f is a smooth function.

Proof. Suppose $f(u)$ is a nonlinear function, since

$$u = \sum_{i=0}^{\infty} u_i \lambda^i = \sum_{i=0}^{m} u_i \lambda^i + \sum_{i=m+1}^{\infty} u_i \lambda^i,$$

we have such result as following:

$$\begin{aligned}
\frac{\partial^m f(u(\lambda))}{\partial \lambda^m} &= \frac{\partial^m}{\partial \lambda^m} f\left(\sum_{i=0}^{\infty} u_i \lambda^i\right) \\
&= \frac{\partial^m}{\partial \lambda^m} f\left(\sum_{i=0}^{m} u_i \lambda^i + \sum_{i=m+1}^{\infty} u_i \lambda^i\right) \\
&= \frac{\partial^m}{\partial \lambda^m} f\left(\sum_{i=0}^{m} u_i \lambda^i\right).
\end{aligned}$$

Therefore, we obtain

$$\frac{\partial^m f(u(\lambda))}{\partial \lambda^m} = \frac{\partial^m}{\partial \lambda^m} f\left(\sum_{i=0}^{\infty} u_i \lambda^i\right) = \frac{\partial^m}{\partial \lambda^m} f\left(\sum_{i=0}^{m} u_i \lambda^i\right).$$

$\square$

Corollary 1.1. *From Thm. (1.2), we find*

$$u^k(\lambda) = \left(\sum_{n=0}^{\infty} u_n \lambda^n\right)^k = u_0^k + \sum_{m=1}^{\infty} A_m(u^k) \lambda^m, \tag{1.5}$$

Example 1.3. *For $F(u) = u^2$*

we first set

$$u = \sum_{n=0}^{\infty} u_n. \tag{1.6}$$

Substitute equation (1.6) into $F(u) = u^2$ gives

$$
\begin{aligned}
F(u) &= (u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \ldots)^2 \\
&= u_0^2 + 2u_0u_1 + 2u_0u_2 + u_1^2 + 2u_0u_3 + 2u_1u_2 + \ldots \\
&= \underbrace{u_0^2}_{A_0} + \underbrace{2u_0u_1}_{A_1} + \underbrace{2u_0u_2 + u_1^2}_{A_2} + \\
&\quad \underbrace{2u_0u_3 + 2u_1u_2}_{A_3} + \underbrace{2u_0u_4 + 2u_1u_3 + u_2^2}_{A_4}.
\end{aligned}
$$

This is consistent with the results obtained before using Adomians algorithm.

Theorem 1.4. *[18] Assume that $f(u)$ has the Taylor expansion with respect to u_0, then*

$$
A_m(f(u)) = \sum_{k=1}^{m} \frac{f^{(k)}(u_0)}{k!} \frac{1}{m!} \frac{\partial^m (\sum_{i=1}^{m} u_i \lambda^i)^k}{\partial \lambda^m}\bigg|_{\lambda=0}. \tag{1.7}
$$

Proof. Expanding $f(u)$ in Taylor series with respect to u_0, one has

$$
f(u) = f(u_0) + \sum_{k=1}^{\infty} \frac{f^{(k)}(u_0)}{k!} (u - u_0)^k. \tag{1.8}
$$

From (1.8), we have

$$
A_m(f(u)) = \frac{1}{m!} \frac{\partial^m \left(\sum_{k=1}^{\infty} \frac{f^{(k)}(u_0)}{k!} (u(\lambda) - u_0)^k \right)}{\partial \lambda^m}\bigg|_{\lambda=0}.
$$

$\square$

Corollary 1.2. *From Thm. (1.4), we find*

$$
f(u(\lambda)) = f(u_0) + \sum_{n=1}^{\infty} A_m(f(u)) \lambda^m.
$$

Example 1.4. *Take $F(u) = \sin u$.*

Note that it is impossible to perform algebraic operations here. Therefore, our main aim is to separate $A_0 = F(u_0)$ from other terms. To achieve this goal, we first substitute

$$
u = \sum_{n=0}^{\infty} u_n \tag{1.9}
$$

into $F(u) = \sin u$ to obtain

$$F(u) = \sin(u_0 + u_1 + u_2 + \ldots) = \sin(u_0 + [u_1 + u_2 + \ldots]).$$

Thus

$$\sin(u_0 + [u_1 + u_2 + \ldots]) = \sin u_0 \cos(u_1 + u_2 + \ldots) + \cos u_0 \sin(u_1 + u_2 + \ldots)$$

Applying the Taylor expansion for $\sin(u_1 + u_2 + \ldots)$ and $\cos(u_1 + u_2 + \ldots)$.

$$
\begin{aligned}
F(u) &= \sin u_0 \left[1 - \frac{(u_1 + u_2 + \ldots)^2}{2!} + \frac{(u_1 + u_2 + \ldots)^4}{4!} - \ldots \right] + \\
&\quad \cos u_0 \left[(u_1 + u_2 + \ldots) - \frac{(u_1 + u_2 + \ldots)^3}{3!} + \ldots \right] \\
&= \sin u_0 \left[1 - \frac{1}{2!}(u_1^2 + 2u_1 u_2 + \ldots) + \ldots \right] + \\
&\quad \cos u_0 [(u_1 + u_2 + \ldots) - \frac{1}{3!}(u_1^3 + 3u_1^2 u_2 + 3u_1^2 u_3 + \ldots) + \ldots] \\
&= \underbrace{\sin u_0}_{A_0} + \underbrace{u_1 \cos u_0}_{A_1} + \underbrace{u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0}_{A2} + \ldots
\end{aligned}
\tag{1.10}
$$

When we compare the Adomian polynomials found in eq. (1.10) with the ones found in eq. (1.2) we see that we have the same Adomian polynomials computed using two different methods.

1.3 ADM and Taylor series method

In this section an important observation can be made here. If we substitute Adomian polynomials into eq. (1.1) we obtain

$$
\begin{aligned}
F(u) &= A_0 + A_1 + A_2 + A_3 + A_4 + \cdots \\
&= u_0 + (u_1 + u_2 + u_3)F'(u_0) + \\
&\quad \frac{1}{2!}(u_1^2 + 2u_1u_2 + 2u_1u_3 + u_2^2 + \cdots)F''(u_0) + \\
&\quad \frac{1}{3!}(u_1^3 + 3u_1^2u_2 + 3u_1^2u_3 + 6u_1u_2u_3 + \cdots)F'''(u_0) \\
&= F(u_0) + (u - u_0)F'(u_0) + \frac{1}{2!}(u - u_0)^2 F''(u_0) + \\
&\quad \frac{1}{3!}(u - u_0)^3 F'''(u_0) + \cdots \\
&= \sum_{n=0}^{\infty} \frac{F^{(n)}(u_0)}{n!}(u - u_0)^n.
\end{aligned}
$$

The last expansion confirms that the series of A_n polynomials is a Taylor series expansion about a function u_0 and not about a point as usually used.

Proposition 1.1. *[5] Consider the differential equation*

$$\frac{du}{dx} = N(u(x)), \tag{1.11}$$

together the initial condition

$$u(x_0) = u_0. \tag{1.12}$$

Then, the general solution given by the Taylors series method is precisely the ADM, where

$$u_k(x) = \frac{u^{(k)}(x_0)}{k!}(x - x_0)^k, \qquad k = 0, 1, 2, \ldots$$

and u_k, k=0,1,..., come determined by the iterative scheme:

$$
\begin{aligned}
u_0 &= u(x_0), \\
u_n(x) &= \int_{x_0}^{x} A_{n-1}(s)ds, \qquad n = 1, 2, 3, \ldots
\end{aligned}
$$

where A_k, $k=0,1,2,\ldots$, satisfies

$$A_k(x) = \frac{1}{k!} \frac{d^k}{dx^k} \left(N(u(x)) \right) \Big|_{x=x_0} (x - x_0)^{k-1}, \quad k = 1, 2, \ldots$$

Proof. Replacing the initial condition (1.12) into eq. (1.11) to get

$$u'(x_0) = f(x_0, u_0) = N(u_0),$$

so that

$$A_0 = u'(x_0).$$

Now, by differentiating eq. (1.11) with respect to x, we obtain

$$u''(x) = \frac{d}{dx}[N(u(x))] = N'(u(x))u'(x), \tag{1.13}$$

by using the initial conditions: $u(x_0) = u_0$ and $u'(x_0) = N(u_0)$ we obtain

$$u''(x_0) = N'(u_0)u'(x_0). \tag{1.14}$$

Then, by multiplying $(x - x_0)$ both sides of eq.(1.14), we have

$$u''(x_0)(x - x_0) = u'(x_0)(x - x_0)N'(u_0) = u_1(x)N'(u_0) =: A_1(x). \tag{1.15}$$

Now, the next step is to integrate eq.(1.15) over $[x_0, x]$

$$\int_{x_0}^x u''(x_0)(s - x_0)ds = \int_{x_0}^x A_1(s)ds.$$

That is, since

$$\int_{x_0}^x u''(x_0)(s - x_0)ds = \frac{u''(x_0)}{2!}(s - x_0)^2 \Big|_{x_0}^x = u_2(x),$$

we have

$$u_2(x) = \int_{x_0}^x A_1(s)ds.$$

By differentiating eq. (1.13) again, we obtain

$$\begin{aligned}
u'''(x) &= \frac{d^2}{dx^2} N(u(x)) \\
&= N''(u(x))(u'(x))^2 + N'(u(x))u'(x)u''(x), \tag{1.16}
\end{aligned}$$

Let $x = x_0$ in eq. (1.16) and divide by 2!, then multiplying by $(x - x_0)^2$ we have

$$
\begin{aligned}
\frac{1}{2!}u'''(x_0)(x - x_0)^2 &= \frac{1}{2!}\left(N''(u(x_0))[u'(x_0)]^2 + N'(u(x_0))u'(x_0)u''(x_0)\right)(x - x_0)^2 \\
&= \frac{1}{2!}[u'(x_0)(x - x_0)]^2 N''(u_0) + \left(\frac{u''(x_0)}{2!}(x - x_0)^2\right)N'(u_0).
\end{aligned}
$$

Now,

$$
\frac{1}{2!}u'''(x_0)(x - x_0)^2 = \frac{1}{2!}u_1^2(x)N''(u_0) + u_2(x)N'(u_0) =: A_2(x), \qquad (1.17)
$$

and integrating both sides of eq. (1.17) over $[x_0, x]$, we obtain

$$
u_3(x) = \frac{u'''(x_0)}{3!}(x - x_0)^3 = \int_{x_0}^{x} \frac{u'''(x_0)}{2!}(s - x_0)^2 ds = \int_{x_0}^{x} A_2(s)ds.
$$

Then,

$$
u_3(x) = \int_{x_0}^{x} A_2(s)ds.
$$

By continuing of the same way this process, one gets

$$
\frac{u^{(n+1)}(x_0)}{n!}(x - x_0)^n = \frac{1}{n!}\frac{d^n}{dx^n}N(u(x))\Big|_{x=x_0}(x - x_0)^n = A_n(x). \qquad (1.18)
$$

Integrate both sides of eq. (1.18) over $[x_0, x]$, we have

$$
u_n(x) = \int_{x_0}^{x} A_{n-1}(s)ds.
$$

Therefore,

$$
\begin{aligned}
u_0 &= u(x_0), \\
u_n(x) &= \int_{x_0}^{x} A_{n-1}(s)ds, \quad n = 1, 2, 3, \ldots
\end{aligned}
$$

where $A_i(x)$, $i = 1, 2, 3, \ldots$ verifies

$$
A_k(x) = \frac{1}{k!}\frac{d^k}{dx^k}N(u(x))\Big|_{x=x_0}(x - x_0)^{k-1}. \quad k = 1, 2, \ldots
$$

$\square$

Chapter 2

ADM for Ordinary Differential Equations

2.1 Analysis of ADM

The discussion of decomposition technique for solving nonlinear differential equation will be discuss in this section.

Consider equation

$$Fu(t) = g(t), \tag{2.1}$$

where F represents a general nonlinear ordinary or partial differential operator including both linear and non linear terms. The linear terms are decomposed into $L + R$, where L is easily invertible (usually the highest order derivative) and R is the remained term of the linear operator. Thus, the equation can be written as

$$Lu + Nu + Ru = g, \tag{2.2}$$

where Nu presents the nonlinear term. By solving this equation for Lu, since L is

invertible, we can write

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{2.3}$$

$$u = h + L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{2.4}$$

where h is the solution of the homogeneous equation $Lu = 0$, with the prescribed initial or boundary conditions in some suitable way. The problem now is the decomposition of the nonlinear term Nu. To do so, Adomain develop a technique in which he parametrized λ in a suitable way using

$$u = \sum_{i=0}^{n} \lambda^i u_i, \tag{2.5}$$

then Nu will be a function of $\lambda, u_0, u_1, \dots$. Suppose the nonlinearity term is of the form $Nu = f(u)$ which is analytic in λ, expanding Nu with respect to λ to obtain

$$f(u(\lambda)) = \sum_{i=0}^{n} \lambda^i A_i, \tag{2.6}$$

then A_n are polynomials defined such that each A_i depends only on $u_0, \dots, u_n$, $A_n = A_n(u_0, \dots, u_n)$ and they can be calculated from the following expression

$$A_n = \frac{1}{n!} \left(\frac{d^n}{d\lambda^n} N \left(\sum_{k=0}^{\infty} \lambda^k u_k \right) \right) = \frac{1}{n!} \frac{d^n}{d\lambda^u} f(u(\lambda)) \bigg|_{\lambda}, \tag{2.7}$$

using that

$$\frac{d}{d\lambda} = \frac{du}{d\lambda} \frac{d}{du}, \ f = f(u), \ u = u(\lambda),$$

then each

$$\frac{du}{d\lambda}$$

is evaluated at $\lambda = 0$ and dividing by $n!$. Hence,

$$
\begin{aligned}
\frac{d}{d\lambda}f(u) &= \frac{df}{du}\frac{du}{d\lambda}, \\
\frac{d^2}{d\lambda^2}f(u) &= \frac{d^2f}{du^2}\left(\frac{du}{d\lambda}\right)^2 + \frac{df}{du}\frac{d^2u}{d\lambda^2}, \\
\frac{d^3}{d\lambda^3}f(u) &= \frac{d^3f}{du^3}\left(\frac{du}{d\lambda}\right)^3 + 3\frac{d^2f}{du^2}\frac{du}{d\lambda}\frac{d^2u}{d\lambda^2} + \frac{df}{du}\frac{d^3u}{d\lambda^3},
\end{aligned}
$$

for the nth derivatives

$$
\frac{d^j f}{d\lambda^j} = \sum_{i=0}^{j} c(i,j)\frac{d^i f}{du^i},
$$

where

$$
c(i,j) = \frac{d}{d\lambda}(c(i,j-1)) + \frac{du}{d\lambda}(c(i-1,j-1)),
$$

such that $c(0,0) = 1$, $c(0,1) = 0$, and noting that $c(i,j) = 0$, $i > j$, and $c(0,j) = 0$, $j > 0$.

If $i = j = 2$, then $c(2,2) = (\frac{du}{d\lambda})^2 = u_1^2$.

$c(2,3) = \frac{3du}{d\lambda}\frac{d^2u}{d\lambda^2} = 3u_1u_2$ Now, by (2.5)

$$
u = u_0 + \lambda u_1 + \lambda^2 u_2 + \cdots,
$$

the following are useful relations

$$
\left(\frac{d^n}{d\lambda^n}u(\lambda)\right)_{\lambda=0} = n!u,
$$

and

$$
\left(\frac{d^n}{du^n}f(u(\lambda))\right)_{\lambda=0} = \frac{d^u f}{du^n},
$$

hence, by eq. (2.7)

$$
\begin{aligned}
A_0 &= f(u_0), \\
A_1 &= \left(\frac{d}{d\lambda} f(u) \right)_{\lambda=0} = \left(\frac{df}{du} \frac{du}{d\lambda} \right)_{\lambda=0} = u_1 f'(u_0), \\
A_2 &= \frac{1}{2} \left(\frac{d^2}{d\lambda^2} f(u) \right)_{\lambda=0} = \left(\frac{d^2 f}{du^2} \left(\frac{du}{d\lambda} \right)^2 + \frac{df}{du} \frac{d^2}{d\lambda^2} \right)_{\lambda=0} = u_2 f'(u_0) + \frac{u_1^2}{2} f''(u_0), \\
A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f'''(u_0).
\end{aligned}
$$

In general, a convenient computational form for $A'_n s$ polynomials is

$$
A_n = \frac{1}{n!} \left(\sum_{v=1}^{n} c(v,n) \frac{d^v f}{du^v} \right) \Bigg|_{\lambda=0}.
$$

Parameterize eq. (2.4) in the form

$$
u = h + L^{-1} g - \lambda L^{-1} R u - \lambda L^{-1} N u, \tag{2.8}
$$

where λ is just an identifier for collection the terms in a suitable way such that u_n depends on $u_0, u_1, ..., u_{n-1}$

$$
\sum_{n=0}^{\infty} \lambda^n u_n = h + L^{-1} g - \lambda L^{-1} R \sum_{n=0}^{\infty} \lambda^n A_n - \lambda L^{-1} \sum_{n=0}^{\infty} \lambda^n u_n. \tag{2.9}
$$

Equating the coefficients of equal powers of λ, we obtain

$$
u_0 = h + L^{-1} g,
$$

$$
u_n = -L^{-1} R u_{n-1} - L^{-1} A_{n-1}.
$$

Hence, u_n is calculable for $n \geq 1$, as well $u = \sum_{n=0}^{\infty} u_n$. But when we tried to solve the equation in analytical form, the process is longer. However, all the terms of (2.9) can be determined and the solution is approximated by the truncated series $u = \sum_{n=0}^{N} u_n$, see [3].

2.2   Examples

Example 2.1. *As a simple example, consider the nonlinear, initial value problem*

$$\frac{dy}{dx} = y^2, \qquad\qquad (2.10)$$

with the initial condition $y(0) = 1$.

This differential equation has the exact solution of

$$y(x) = \frac{1}{1-x},$$

following the method described above, we define a linear operator

$$L = \frac{d}{dx},$$

the inverse operator is then

$$L^{-1} = \int_0^x (.)dx,$$

rewriting the differential equation (2.10) in operator form, we have

$$Ly = Ny,$$

where N is a nonlinear operator such that

$$Ny = y^2,$$

next we apply the inverse operator for L to the equation. On the left hand side of the equation, this gives

$$L^{-1}Ly = y(x) - y(0),$$

using the initial condition, this becomes

$$L^{-1}Ly = y(x) - 1,$$

returning this to equation (2.5), we now have

$$y(x) - 1 = L^{-1}(Ny),$$

or

$$y(x) = 1 + L^{-1}(Ny).$$

Next, we need to generate the Adomian polynomials, A_n. Let y be expanded as an infinite series

$$y(t) = \sum_{n=0}^{\infty} y_n(t),$$

and define

$$N_y = \sum_{n=0}^{\infty} A_n.$$

To find A_n, we introduce the scalar λ such that,

$$\sum_{n=0}^{\infty} y_n(t) = 1 + L^{-1}(\sum_{n=0}^{\infty} A_n), \qquad (2.11)$$

$$y(\lambda) = \sum_{n=0}^{\infty} \lambda^n y_n,$$

From the definition of the Adomian polynomials,

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} (Ny(\lambda)) \mid_{\lambda=0},$$

we find the Adomian polynomials.

$$
\begin{aligned}
A_0 &= y_0^2, \\
A_1 &= 2y_0 y_1, \\
A_2 &= 2y_0 y_2 + y_2, \\
A_3 &= 2y_0 y_3 + 2y_1 y_2, \\
A_4 &= 2y_0 y_4 + 2y_1 y_3 + y_2^2, \\
&\vdots
\end{aligned}
$$

Returning the Adomian polynomials to equation (2.11), we can determine the recursive relationship that will be used to generate the solution

$$
\begin{aligned}
y_0(x) &= 1, \\
y_{n+1}(x) &= L^{-1}(A_n),
\end{aligned}
$$

solving this yields

$$
\begin{aligned}
y_0 &= 1, \\
y_1 &= x, \\
y_2 &= x^2, \\
y_3 &= x^3, \\
y_4 &= x^4, \\
&\vdots
\end{aligned}
$$

we can see that the series solution generated by this method is

$$
y(x) = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n,
$$

which we recognize as the Taylor series for the exact solution

$$y(x) = \frac{1}{1-x}$$

Example 2.2. *If we consider the anharmonic oscillator described by*

$$\frac{d^2\theta}{dt^2} + k^2 \sin(\theta) = 0 \qquad (2.12)$$

with $k^2 = g\backslash l$ and large amplitude motion and assuming $\theta(0) = \gamma$ and $\theta'(0) = 0$.

we write

$$L\theta + N\theta = 0.$$

We obtain

$$\theta = \theta(0) - L^{-1}N\theta = \theta(0) - L^{-1}\sum_{n=0}^{\infty} A_n,$$

where

$$N\theta = k^2 \sin\theta,$$

since for

$$N\theta = \sin\theta,$$

we have

$$
\begin{aligned}
A_0 &= \sin\theta_0, \\[4pt]
A_1 &= \theta_1 \cos\theta_0, \\[4pt]
A_2 &= -\frac{\theta_1^2}{2}\sin\theta_0 + \theta_2 \cos\theta_0, \\[4pt]
A_3 &= -\frac{\theta_1^3}{6}\cos\theta_0 - \theta_1\theta_2 \sin\theta_0 + \theta_3 \cos\theta_0, \\[4pt]
&\;\;\vdots
\end{aligned}
$$

we get

$$\begin{aligned}
\theta_0 &= \gamma, \\
\theta_1 &= -L^{-1}k^2 A_0, \\
\theta_2 &= -L^{-1}k^2 A_1, \\
&\vdots
\end{aligned}$$

Since L^{-1} represents a twofold definite integration from 0 to t,

$$\begin{aligned}
\theta_1 &= -\left(\frac{k^2 t^2}{2!}\right)\sin\gamma, \\
\theta_2 &= \left(\frac{k^4 t^4}{4!}\right)\sin\gamma\cos\gamma, \\
\theta_3 &= -\left(\frac{k^6 t^6}{6!}\right)\left(\sin\gamma\cos^2\gamma - 3\sin^3\gamma\right), \\
&\vdots
\end{aligned}$$

For more example see [11].

2.3   A comparison between ADM and Taylor series method

In this section, we will compare the performance of the ADM and the Taylor series method applied to the solution of linear ordinary differential equation.

Example 2.3. *For comparison purposes, consider the linear initial value problem*

$$e^x u'' + xu = 0, \tag{2.13}$$

subject to the initial conditions

$$u(0) = \alpha, \ u'(0) = \beta. \tag{2.14}$$

We will use two different methods to solve this example.

ADM method:-

Eq. (2.14) can be written in an operator form as

$$L_{xx}u = -xe^{-x}u, \qquad (2.15)$$

where $L_{xx}(.) = \frac{d^2}{dx^2}(.)$. Then the inverse of L_{xx} is, $L_{xx}^{-1}(.) = \int_0^x \int_0^x (.)dxdx$. Applying L_{xx}^{-1} to both sides of (2.15) we find that

$$u(x) = \alpha + \beta x - L_{xx}^{-1}(xe^{-x}u). \qquad (2.16)$$

The decomposition method consists of decomposing $u(x)$ into a sum of components given by the infinite series

$$u(x) = \sum_{n=0}^{\infty} u_n. \qquad (2.17)$$

Substituting (2.17) into (2.16) yields

$$\sum_{n=0}^{\infty} u_n(x) = \alpha + \beta x - L_{xx}^{-1}\left(xe^{-x}\sum_{n=0}^{\infty} u_n\right). \qquad (2.18)$$

Next, we equate selected components on both sides using the following recursive relationship:

$$u_0 = \alpha + \beta x,$$

$$u_{k+1} = -L_{xx}^{-1}(xe^{-x}\sum_{n=0}^{\infty} u_k(x)), \quad (k \geq 0).$$

Accordingly, we find

$$u_0 = \alpha + \beta x$$

$$
\begin{aligned}
u_1 &= -L_{xx}^{-1}(xe^{-x}u_0) = -L_{xx}^{-1}\left(\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^{n+1} + \beta \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^{n+2}\right) \\
&= \alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+3)(n+2)n!}x^{n+3} - \beta \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+4)(n+3)n!}x^{n+4} \\
&= \alpha\left(\frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 + \ldots\right) + \beta\left(\frac{1}{12}x^4 - \frac{1}{20}x^5 + \ldots\right).
\end{aligned}
$$

So,

$$
u(x) = \alpha\left(1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{40}x^5 + \ldots\right) + \beta\left(1 - \frac{1}{12}x^4 + \frac{1}{20}x^5 + \ldots\right). \tag{2.19}
$$

As can be verified by the above computation, two components only were used to obtain the approximation. Furthermore, the accuracy level of the approximation can be increased by evaluating further components.

The Taylor series method:-

The Taylor series method introduces the solution by an infinite series given by

$$
u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{2.20}
$$

Substituting eq. (2.20) into eq. (2.13) gives

$$
e^x \left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) = -\sum_{n=0}^{\infty} a_n x^{n+1},
$$

or, equivalently

$$
\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}\right) = -\sum_{n=0}^{\infty} a_n x^{n+1}. \tag{2.21}
$$

The coefficients a_n, $n \geq 0$, are determined by equating coefficients of like powers of x through determining a formal recurrence relation. It is obvious that an explicit recurrence relation is difficult to derive. Alternatively, we multiply the series involved, term by term, to find $a_0 = \alpha$, $a_1 = \beta$, $a_2 = 0$, $a_3 = -\frac{1}{6}\alpha$,

$a_4 = \frac{1}{12}\alpha - \frac{1}{12}\beta$ and $a_5 = -\frac{1}{40}\alpha + \frac{1}{20}\beta$. In view of eq. (2.21), the series solution eq. (2.20) follows immediately. At this point, it should be noted that using the Taylor series method, six iterations were evaluated to obtain the same result provided by the decomposition method where two components only were computed.

The two series methods were applied separately to linear and nonlinear ordinary differential equations. The study showed that the decomposition method is simple and easy to use and produces reliable results with few iterations used. The method also minimizes the computational difficulties of the Taylor series in that the components are determined elegantly by using simple integrals [5].

2.4   Convection between two parallel walls

In many physical applications two parallel walls are maintained at uniform temperatures. The transport phenomenon occurring as a result of a convective flow between the vertical walls is given by the following differential equation:

$$\frac{d^4 u}{dx^4} - Rau = \epsilon \left(\frac{du}{dx}\right)^2, \quad \epsilon << 1 \tag{2.22}$$

where u represent the velocity of the particles' between the parallel walls and Ra is Rayleigh number, associated with the boundary conditions

$$u(0) = u(1) = 0, u''(0) = u''(1) = 1. \tag{2.23}$$

Method of solution

We first write (2.22) in the form

$$Lu = \epsilon \left(\frac{du}{dx}\right)^2, \tag{2.24}$$

where L denotes the linear operator

$$L = \frac{d^4}{dx^4} - Ra, \tag{2.25}$$

we choose the linear operator to be (2.25) rather than

$$\frac{d^4}{dx^4}$$

as it is usually done in this method, since we are interested in oscillatory solutions and these are generated by (2.25) more easily.

The operator L is invertible and its inverse is given by

$$L^{-1}[.] = \int_0^1 g(x,s)[.]ds, \tag{2.26}$$

where $g(x,s)$ is the Green's function which satisfies the boundary value problem

$$Lg = \delta(x-s), \tag{2.27}$$

$$g(0,s) = g(1,s) = 0, \ g''(0,s) = g''(1,s) = 0. \tag{2.28}$$

The homogeneous equation

$$\frac{d^4u}{dx^4} - Rau = 0$$

has the four linearly independent solution $\sinh((Ra)^{1/4}x)$, $\sin((Ra)^{1/4}x)$, $\cosh((Ra)^{1/4}x)$ and $\cos((Ra)^{1/4}x)$, therefor we take the value of $g(x,s)$ to be

$$g(x,s) = \begin{cases} c_1\cosh(bx) + c_2\sinh(bx) + c_3\sin(bx) + c_4\cos(bx), & x < s \\ a_1\cosh(bx) + a_2\sinh(bx) + a_3\sin(bx) + a_4\cos(bx), & x > s \end{cases} \tag{2.29}$$

where

$$b = (Ra)^{1/4},$$

applying the boundary conditions

$$g(0, s) = 0 \text{ gives } c_1 + c_4 = 0,$$

$$g(1, s) = 0 \text{ gives } a_1 \cosh(b) + a_2 \sinh(b) + a_3 \sin(b) + a_4 \cos(b) = 0,$$

$$g''(0, s) = 0 \text{ gives } c_1 + c_4 = 0,$$

$$g''(1, s) = 0 \text{ gives } a_1 \cosh(b) + a_2 \sinh(b) + a_3 \sin(b) + a_4 \cos(b) = 0,$$

which gives $c_1 = c_4 = 0$, $a_3 = -a_4 \frac{\cos(b)}{\sin(b)}$ and $a_1 = -a_2 \frac{\sinh(b)}{\cosh(b)}$ thus,the relation (2.29)
becomes

$$g(x, s) = \begin{cases} c_2 \sinh(bx) + c_3 \sin(bx) & , x < s \\ \frac{a_2}{\cosh(b)} \sinh(b(x-1)) + \frac{a_4}{\sin(b)} \sin(b(1-x)) & , x > s \end{cases} \quad (2.30)$$

The remaining constants are determined by applying the matching conditions at
$x = s$, continuity of g, $\frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial^2 x}$ at $x = s$,

$$c_2 \sinh(bs) + c_3 \sin(bs) = a_2 \frac{\sinh(b(s-1))}{\cosh(b)} + a_4 \frac{\sin(b(1-s))}{\sin(b)},$$

$$c_2 \cosh(bs) + c_3 \cos(bs) = a_2 \frac{\cosh(b(s-1))}{\cosh(b)} - a_4 \frac{\cos(b(1-s))}{\sin(b)},$$

$$c_2 \sinh(bs) - c_3 \sin(bs) = a_2 \frac{\sinh(b(s-1))}{\cosh(b)} - a_4 \frac{\sin(b(1-s))}{\sin(b)},$$

and the value of the jump in the third derivative g is

$$a_2 \frac{\cosh(b(s-1))}{\cosh(b)} + a_4 \frac{\cos(b(1-s))}{\sin(b)} - c_2 \cosh(bs) + c_3 \cos(bs) = \frac{1}{b^3},$$

solving these four equations gives

$$c_2 = \frac{\sinh(b(s-1))}{2b^3 \sinh(b)},$$

$$c_3 = \frac{\sin(b(s-1))}{2b^3 \sin(b)},$$

$$a_2 = \frac{\sinh(bs)cosh(b)}{2b^3 \sinh(b)},$$

$$a_4 = \frac{\sin(bs)}{2b^3},$$

hence,

$$g(x,s) = \frac{1}{2b^3} \begin{cases} \frac{\sinh(b(s-1))}{sinh(b)} \sinh(bx) + \frac{\sin(b(1-s))}{\sin(b)} \sin(bx) & , x < s \\ \frac{\sinh(bs)}{sinh(b)} \sinh(b(x-1)) + \frac{\sin(bs)}{\sin(b)} \sin(b(1-x)) & , x > s \end{cases} \tag{2.31}$$

Clearly, $g(x,s)$ is symmetric and not defined for $Ra = (k\pi)^4$, which are known as the critical frequencies. In this section, we treat only the case $Ra \neq (k\pi)^4$ for which $g(x,s)$ is defined and unique.To find the inverse, L^{-1}, of the operator L, solving the homogenous differential equation of (2.22) with prescribed boundary conditions (2.23)

$$\frac{d^4u}{dx^4} - Rau = 0$$

with $u(0) = u(1) = 0$, $u''(0) = u''(1) = 1$ gives,

$$u_c(x) = \frac{1}{2b^2} \left(\frac{\sinh(bx) - \sinh(b(x-1))}{sinh(b)} - \frac{\sin(b(1-x)) + \sin(bx)}{\sin(b)} \right)$$

applying L^{-1} on both sides of (2.24) and using the solution of homogenous equation with given boundary conditions gives

$$\begin{aligned} u(x) &= \frac{1}{2b^2} \left(\frac{\sinh(bx) - \sinh(b(x-1))}{\sinh(b)} - \frac{\sin(b(1-x)) + \sin(bx)}{\sin(b)} \right) \\ &+ \varepsilon \int_0^1 g(x,s) \left(\frac{du}{ds} \right) ds, \end{aligned} \tag{2.32}$$

write u in the decomposition form

$$u = \sum_{n=0}^{\infty} u_n \tag{2.33}$$

and expand the nonlinear term $\left(\frac{du}{dx}\right)^2$ as

$$\left(\frac{du}{dx}\right)^2 = \sum_{n=0}^{\infty} A_n, \tag{2.34}$$

as follows

$$
\begin{aligned}
A_0(x) &= \left(\frac{du_0}{dx}\right)^2, \\
A_1(x) &= 2\frac{du_0}{dx}\frac{du_1}{dx}, \\
A_2(x) &= 2\frac{du_0}{dx}\frac{du_2}{dx} + \left(\frac{du_1}{dx}\right)^2, \\
A_3(x) &= 2\frac{du_0}{dx}\frac{du_3}{dx} + 2\frac{du_1}{dx}\frac{du_2}{dx}, \\
A_4(x) &= 2\frac{du_0}{dx}\frac{du_4}{dx} + 2\frac{du_1}{dx}\frac{du_3}{dx} + \left(\frac{du_2}{dx}\right)^2,
\end{aligned}
$$

next, substituting (2.33), (2.34) into (2.32) we get:

$$u_0(x) = \frac{1}{2b^2}\left(\frac{\sinh(bx) - \sinh(b(x-1))}{\sinh(b)} - \frac{\sin(b(1-x)) + \sin(bx)}{\sin(b)}\right),$$

$$u_n(x) = \varepsilon \int_0^1 g(x,s)A_{n-1}(s)ds, \quad n \geq 1.$$

See [3]

Chapter 3

ADM for Partial Differential Equations

3.1 ADM for linear partial differential equations

For instance, in order to solve a linear PDE with two operators

$$L_x u + L_y u = g,$$

three general algorithms can be used. The first of them inverses the operator L_x:

$$u_n = -L_x^{-1} L_y u_{n-1}, \quad \forall n \geq 1,$$

the second inverses L_y:

$$u_n = -L_y^{-1} L_x u_{n-1}, \quad \forall n \geq 1,$$

and the third uses a double inversion:

$$u_n = -\frac{1}{2}(L_x^{-1} L_y + L_y^{-1} L_x) u_{n-1}, \quad \forall n \geq 1,$$

we can implement these general algorithms with or without calculating integral constants. The algorithm choice doesn't depend only on the considered equation but also on the boundary or initial conditions. When someone is unfamiliar with

this method, solutions of equations often calculated don't verify the conditions and so they can make believe that the method is not efficient. This is because most of the algorithms don't use all the conditions and so don't directly impose them on the solution. Of course, this problem doesn't exist when the equation is a differential equation because the integration constants only have to be correctly identified. In the following we present the illustration of the implementation of several scheme on a simple case [7].

Consider the equation:

$$L_x u + L_{yy} u = 2x + y^2,$$

where u is a function of the two variables x and y, L_x is the first order derivation operator concerning the variable x and L_{yy} is the second order derivation operator associated to the variable y. Consider also the initial conditions:

$$u(x = 0) = 0, u(y = 0) = 0, \frac{\partial u}{\partial y}(y = 0) = 0.$$

One of the algorithms consists in inverting the operator L_x^1 and to calculate the integration constant created by this operation. So we can implement this scheme:

$$u_0 = L_x^{-1}(2x + y^2) + a_0(y),$$
$$u_{n+1} = L_x^{-1} L_{yy} u_n + a_{n+1}(y), \quad \forall n \geq 0,$$

the first term of the series is

$$u_0 = x^2 + xy^2 + a_0(y),$$

where $a_0(y)$ is a constant function that is calculated with the first condition. We obtain:

$$u_0 = x^2 + xy^2.$$

Then we have

$$u_1 = -L_x^{-1}(2x) = -x^2 + a_l(y),$$

and the integration constant is still null,

$$u_1 = -x^2,$$

the next term is

$$u_2 = a_2 = 0.$$

So the terms of the series are null after the second rank:

$$u_n = 0, \quad \forall n \geq 2.$$

The final result is:

$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 = xy^2,$$

which is actually the solution of our equation. A similar method consists in inverting the other differential operator L_{yy} and to implement the following scheme:

$$
\begin{aligned}
u_0 &= L_{yy}^{-1}(2x + y^2) + a_0(x)y + b_0(x), \\
u_{n+1} &= L_{yy}^{-1}L_x u_n + a_{n+1}(x)y + b_{n+1}(x), \quad \forall n \geq 0.
\end{aligned}
$$

The calculation is a bit longer because a double integration has to be made at each step and because the two integration constants generated have to be identified

with the two last conditions. We successively obtain:

$$
\begin{aligned}
u_0 &= xy^2 + \frac{y^4}{12} + a_0 y + b_0 = xy^2 + \frac{y^4}{12}, \\
u_1 &= -\frac{y^4}{12} + a_1 y + b_1 = -\frac{y^4}{12}, \\
u_2 &= a_2 y + b_2 = 0, \\
u_n &= 0, \quad \forall n > 2,
\end{aligned}
$$

and finally the accurate solution is obtained:

$$
u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 = xy^2.
$$

A third method consists in simultaneously inverting the two derivation operators without calculating any integration constants:

$$
\begin{aligned}
u_0 &= \frac{1}{2}(L_x^{-1} + L_{yy}^{-1})(2x + y^2), \\
u_{n+1} &= -\frac{1}{2}(L_x^{-1} L_{yy} + L_{yy}^{-1} L_x u_n), \quad \forall n \geq 0.
\end{aligned}
$$

Simple calculations lead to:

$$
\begin{aligned}
u_0 &= \frac{x^2}{2} + xy^2 + \frac{y^4}{24}, \\
u_1 &= -\frac{x^2}{2} - \frac{1}{2}xy^2 - \frac{y^4}{24}, \\
u_2 &= \frac{x^2}{4} + \frac{1}{2}xy^2 + \frac{y^4}{48}, \\
u_3 &= -\frac{x^2}{4} - \frac{1}{4}xy^2 - \frac{y^4}{48}, \\
&\vdots
\end{aligned}
$$

and with an immediate recursion:

$$\varphi_{2p} \;=\; \sum_{n=0}^{2p} u_n = \frac{x^2}{2^{p+1}} + xy^2 + \frac{y^4}{3 \times 2^{p+3}},$$

$$\varphi_{2p+1} \;=\; \sum_{n=0}^{2p+1} u_n = (1 - \frac{1}{2^{p+1}})xy^2,$$

so we have:

$$\lim \varphi_{2p} = \lim \varphi_{2p+1} = xy^2,$$

giving the expected result:

$$u = \sum_{n=0}^{\infty} u_n = \lim \varphi_n = xy^2.$$

3.2   ADM for second order linear partial differential equations

Hyperbolic Equation

Consider the hyperbolic equation

$$L_{tt}u = 9L_{xx}u, \tag{3.1}$$

with the associated conditions

$$u(x = 0) = 0 = u(x = 1) = 0, \; u(t = 0) = \sin(\pi x), \; \frac{\partial u}{\partial t}(t = 0) = 0.$$

The analytical solution of this equation is

$$u(x, t) = \cos(3\pi t)\sin(\pi x).$$

Here we are going to prove that this solution can be obtained with the decomposition method. As a first term of the series, we can use the function u_0 that verifies the boundary conditions and the equation $L_{tt} = 0$. Then we can

implement the recurrent and invert L_{tt}:

$$u_0 = \sin(\pi x),$$

$$u_{n+1} = L_{tt}^{-1} L_{xx} u_n, \quad n \geq 0,$$

so we calculate:

$$
\begin{aligned}
u_0 &= \sin(\pi x), \\
u_1 &= -\frac{9}{2}(\pi t)^2 \sin(\pi x), \\
u_2 &= \frac{27}{8}(\pi t)^4 \sin(\pi x), \\
u_3 &= -\frac{81}{80}(\pi t)^6 \sin(\pi x), \\
&\vdots
\end{aligned}
$$

$$u_n \approx (-1)^n (\pi t)^{2n} \frac{9^n}{(2n)!} \sin(\pi x).$$

We can notice that we have the first terms of the development as an entire series of a cosinus function. As n increases towards infinity, we obtain the exact solution of our hyperbolic equation:

$$u = \sum_{n=0}^{\infty} u_n \approx \cos(3\pi t) \sin(\pi x).$$

Elliptic Equation

Now consider the elliptic equation

$$L_{xx}u + L_{yy}u = 0. \tag{3.2}$$

Consider also the conditions:

$$u(y = 0) = u(y = 1) = u(x = 0) = 0, \; u(x = 1) = \sin(\pi y).$$

It can be solved by inverting the second order derivation operator concerning the variable x, if the integration constants are calculated.

$$u_0 = L_x^{-2}(0) + a_0(y)x + b_0(y),$$

$$u_{n+1} = -L_x^{-2}L_y^2 u_n + a_{n+1}(y)x + b_{n+1}(y), \quad n \geq 0.$$

So we calculate:

$$u_0 = \sin(\pi y)x,$$

$$u_1 = \left(\frac{1}{6}x^3 - \frac{1}{6}x\right)\pi^2\sin(\pi y),$$

$$u_2 = \left(\frac{1}{120}x^5 - \frac{1}{36}x^3 + \frac{7}{360}x\right)\pi^4\sin(\pi y),$$

$$u_3 = \left(\frac{1}{5040}x^7 - \frac{1}{720}x^5 + \frac{7}{2160}x^3 - \frac{31}{15120}\right)\pi^6\sin(\pi y),$$

$$\vdots$$

In this case, a simple expression of u_n, can't be found. This observation is the result of the calculation of the integration constants and particularly of the term that is proportional to x and which has been created by the condition at $x = 1$. Finally, we can't obtain a general expression of φ_n for all integers n nor consequently an accurate expression of u.

Parabolic Equation

We can try to solve, on the field that is defined by $x \geq 0$, $0 \leq y \leq 1$, the hyperbolic equation

$$L_x u = L_{yy} u \tag{3.3}$$

associated to the boundary conditions

$$u(x = 0) = 0, u(y = 0) = 1 - e^{(-x)}, u(y = 1) = \sin(x)$$

The decomposition method can be implemented by inverting the operator L_y^2 but without calculating the integration constants. We have to initialize the recurrent with u_0 that verifies the conditions and the equation $L_{yy}u_0$:

$$u_0 = (1 - y)(1 - e^{-x}) + y \sin(x),$$

$$u_{n+1} = L_{yy}^{-1} L_x u_n, \quad \forall n \geqslant 0.$$

The calculation is easy and leads to:

$$
\begin{aligned}
u_0 &= -e^{-x}(1 - y) + y \sin(x) + 1 - y, \\
u_1 &= e^{-x} \left(\frac{y^2}{2} - \frac{y^3}{6} \right) + \frac{y^3}{6} \cos(x), \\
u_2 &= -e^{-x}(\frac{y^4}{24} - \frac{y^5}{120}) - \frac{y^5}{120} \sin(x), \\
u_3 &= e^{-x} \left(\frac{y^6}{720} - \frac{y^7}{5040} \right) - \frac{y^7}{5040} \cos(x), \\
&\vdots
\end{aligned}
$$

$$u_{2n} = -e^{-x} \left(\frac{y^{4n}}{(4n)!} - \frac{y^{4n+1}}{(4n+1)!} \right) - \frac{y^{4n+1}}{(4n+1)!} \sin(x),$$

$$u_{2n+1} = e^{-x} \left(\frac{y^{4n+2}}{(4n+2)!} - \frac{y^{4n+3}}{(4n+3)!} \right) - \frac{y^{4n+3}}{(4n+3)!} \cos(x).$$

We can't find an analytical expression of the sum of this series but we can assert that there are two functions f and g of the variable y so that this sum can be written

$$u(t, y) = e^{-x}(sin(y) - cos(y)) + sin(x)f(y) + cos(x)g(y),$$

where f and g verify $f'' = -g$ and $g'' = f$, see [7]

3.3   A diffusion of oxygen in absorbing tissue

The successful treatment of cancer by radiotherapy is dictated primarily by the ability to apply a radiation dosage large enough to do substantial damage to the cancerous cells without damaging surrounding healthy cells, and still remain within the tissue tolerance level of radiation. The susceptibility of cancerous cells to radiation has been shown to increase with increasing oxygen concentrations within the tumor. Many experiments have shown that the dependence of tissue radiosensitivity, for bacterial cells, indicates a 2-3-fold increase in the radiation dosage would be required to obtain the degree of destruction for cells in the total absence of oxygen in comparison with oxygenated cells. This effect of oxygen allows the use of smaller radiation doses to achieve the desired percentage of destruction of cancerous cells. It should be noted that the solution of the diffusion of oxygen in absorbing tissues here is not limited to cancerous tumors, but may be used in the diffusion of oxygen in absorbing tissues in general [4].

The solution of the oxygen diffusion problem in a medium, which simultaneously absorbs the oxygen, consists of finding u and s such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 1, \tag{3.4}$$

subject to

$$\frac{\partial u}{\partial x}(t, 0) = 0,$$

$$u(t, s(t)) = 0, \tag{3.5}$$

$$\frac{\partial u}{\partial x}(t, s(t)) = 0, \tag{3.6}$$

and the initial condition

$$u(0, x) = \frac{1}{2}(1 - x)^2, 0 < x < s(0) = 1. \tag{3.7}$$

Method of solution

Consider the general problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - g(x), \quad 0 < x < s(t), \tag{3.8}$$

which is the governing equation, subject to the boundary condition

$$\frac{\partial u}{\partial x}(t, 0) = h(t),$$

the Dirichlet boundary condition

$$u(t, s(t)) = p(t),$$

the Neumann boundary condition

$$\frac{\partial u}{\partial x}(t, s(t)) = q(t),$$

and the initial condition

$$u(0, x) = \varphi(x), \quad 0 < x < s(0).$$

Our problem contains, as a special case, the above system which describes the oxygen diffusion problem. Based on the ADM, we write (3.4) in Adomians operator-theoretic notation as

$$L_{xx}u = \frac{\partial u}{\partial t} + g(x), \tag{3.9}$$

where

$$L_{xx} = \frac{\partial^2}{\partial x^2}. \tag{3.10}$$

Applying the inverse linear operator

$$\int_x^{s(t)} \int_x^{s(t)} (.)dxdx$$

to (3.10) and taking into account that

$$u(t, s(t)) = p(t)$$

and

$$(\partial u/\partial x)(t, s) = q(t),$$

we obtain

$$u(t, x) = p(t) - q(t)(s - x) + \int_x^{s(t)} \int_x^{s(t)} g(x)dxdx + \int_x^{s(t)} \int_x^{s(t)} \frac{\partial u}{\partial t}dxdx.$$

Define the solution $u(t, x)$ by an infinite series of components in the form

$$u(t, x) = \sum_{n=0}^{\infty} u_n(t, x).$$

Consequently, the components u_n can be elegantly determined by setting the recursion scheme:

$$u_0 = p(t) - q(t)(s - x) + \int_x^{s(t)} \int_x^{s(t)} g(x)dxdx,$$

$$u_{n+1}(t, x) = \int_x^{s(t)} \int_x^{s(t)} \frac{\partial u_n}{\partial t}dxdx, \quad n \geq 0,$$

for the complete determination of these components. Replace $p(t) = q(t) = 0$ and $g(x) = 1$ into the recursion scheme to get

$$
\begin{aligned}
u_0 &= \frac{1}{2!}(s-x)^2, \\
u_1 &= \frac{s'}{3!}(s-x)^3, \\
u_2 &= \frac{s'^2}{4!}(s-x)^4 + \frac{s''}{5!}(s-x)^5, \\
&\vdots
\end{aligned}
$$

A polynomial profile of fifth degree is now obtained by the ADM, which is the truncated decomposition series

$$
u(t,x) = u_0(t,x) + u_1(t,x) + u_2(t,x),
$$

so that

$$
u(t,x) = \frac{1}{2!}(s-x)^2 + \frac{s'}{3!}(s-x)^3 + \frac{s'^2}{4!}(s-x)^4 + \frac{s''}{5!}(s-x)^5, \tag{3.11}
$$

and which automatically satisfies the boundary conditions (3.5) and (3.6). We can now obtain an expression for the location of the moving boundary, $s(t)$. This is derived from integrating (3.8) with respect to x from 0 to x and taking into account that $(\frac{\partial u}{\partial x})(t,0) = h(t)$; we obtain

$$
\frac{\partial u}{\partial x} = h(t) + \int_0^x g(x)dx + \int_0^x \frac{\partial u}{\partial t}dx. \tag{3.12}
$$

Substitute $x = s$ into (3.12) and using the fact that $(\partial u/\partial x)(t,s) = q(t)$. Thus

$$
\int_0^{s(t)} g(x)dx + \int_0^{s(t)} \frac{\partial u}{\partial t}dx = q(t) - h(t), \tag{3.13}
$$

$$
s(0) = 1.
$$

Using the following Leibniz's rule for differentiation under the integral sign:

$$\frac{d}{dt}\int_0^{s(t)} u(t,x)dx = \int_0^{s(t)} \frac{\partial u}{\partial t}dx + u(t,s(t))\frac{ds}{dt},$$

and taking into account that $u(t,s(t)) = p(t)$, we obtain

$$\int_0^{s(t)} \frac{\partial u}{\partial t}dx = \frac{d}{dt}\int_0^{s(t)} u(t,x)dx - p(t)\frac{ds}{dt}. \qquad (3.14)$$

substituting (3.14) into (3.13), we get

$$\int_0^{s(t)} g(x)dx + \frac{d}{dt}\int_0^{s(t)} u(t,x)dx - p(t)\frac{ds}{dt} = q(t) - h(t),$$

where $s(0) = 1$. If we consider $p(t) = q(t) = h(t) = 0$ and $g(x) = 1$, then (1.8)

becomes

$$\frac{d}{dt}\int_0^{s(t)} u(t,x)dx = -s, \qquad (3.15)$$

substitute the profile equation (3.11) into (3.15) gives an ODE to solve for s(t),

namely,

$$\frac{s^2 s'}{2!} + \frac{s^3 s'^2}{3!} + \frac{s^4 s''}{4!} + \frac{s^4 s'^3}{4!} + \frac{s^5 s' s''}{5!} + \frac{s^6 s'''}{6!} = -s,$$

with $s(0) = 1$. So that

$$\frac{ss'}{2!} + \frac{s^2 s'^2}{3!} + \frac{s^3 s''}{4!} + \frac{s^3 s'^3}{4!} + 3\frac{s^4 s' s''}{5!} + \frac{s^5 s'''}{6!} + 1 = 0, \qquad (3.16)$$

we now can determine the location of the moving boundary $s(t)$ as a function of

time by solving the nonlinear equation. Indeed, the solution $s(t)$ follows

immediately by setting the following form:

$$\sqrt{1 + 2\lambda t}, \qquad (3.17)$$

where $\lambda \in \mathbb{R}$ is a parameter to be determined. Simple computations lead to

$$ss' = \lambda,$$

$$s''s^3 = -\lambda^2,$$

$$s'''s^5 = 3\lambda^3,$$

$$s''s's^4 = -\lambda^3.$$

Substituting these expressions into (3.16), we obtain $\lambda^3 + 6\lambda^2 + 24\lambda + 48 = 0$. Consequently, we find $\lambda \approx -3.19$ which is a real root of this equation. Hence, the concentration and the location of the moving boundary for $0 \le t \le 1/6.4$ can be represented fairly accurately by the approximate expression equation (3.11) and $\sqrt{1 - 6.4t}$, respectively.

It should be noted that this solution is applicable for the time $0 \le t \le 1/6.4$ only. An important note can be made here that the t-solution can be obtained by using the initial condition equation (3.7) only. To do this, we apply the inverse linear operator $L_t^{-1}(.) = \int_0^t (.)dt$ to both sides of (3.8) and use the initial condition equation (3.7) to obtain

$$u(t, x) = \varphi(x) - g(x)t + \int_0^t \frac{\partial^2 u}{\partial x^2}dt,$$

where

$$\varphi(x) = (1/2)(1 - x)^2 \text{ and } g(x) = 1.$$

So that the decomposition method consists of decomposing the unknown function $u(x, t)$ into a sum of components defined by the series $u(t, x) = \sum_{n=0}^{\infty} u_n(t, x)$. Thus the components can be elegantly determined in a recursive manner as will be

discussed later; we therefore set the recurrence scheme:

$$u_0 = (1/2)(1-x)^2 - t,$$

$$u_{n+1} = \int_0^t \frac{\partial^2 u_n}{\partial x^2} dt, n \geq 0.$$

3.4   ADM for nonlinear wave equation

In this section, we will again make use of the ADM in order to obtain analytic nonhomogeneous solutions of the nonlinear partial differential equation

$$u_{xx} - uu_{tt} = \varphi(x, t), \tag{3.18}$$

with initial conditions

$$u(0, t) = f(t), u_x(0, t) = g(t). \tag{3.19}$$

To apply the decomposition method, we write equation (3.18) in an operator form

$$L_{xx}(u(x, t)) = \varphi(x, t) + Nu, \tag{3.20}$$

with nonlinear term $Nu = uu_{tt}$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$ are the differential operators. It is clear that L_{xx}^{-1} is the two fold integration from 0 to x, Applying the inverse operator to (3.20) yields

$$L_{xx}^{-1} L_{xx}(u(x, t)) = L_{xx}^{-1}(\varphi(x, t)) + L_{xx}^{-1}(Nu),$$

from which it follows that

$$u(x, t) = f(t) + xg(t) + L_{xx}^{-1}(\varphi(x, t)) + L_{xx}^{-1}(Nu). \tag{3.21}$$

The decomposition method consists of decomposing the unknown function $u(x, t)$ into a sum of components defined by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.22}$$

and the nonlinear term $Nu = uu_{tt}$ can be expressed in the A_n Adomian's polynomials, thus;

$$Nu = \sum_{n=0}^{\infty} A_n,$$

where

$$
\begin{aligned}
A_0 &= u_0 \frac{\partial^2}{\partial t^2} u_0, \\
A_1 &= u_0 \frac{\partial^2}{\partial t^2} u_1 + u_1 \frac{\partial^2}{\partial t^2} u_0, \\
A_2 &= u_0 \frac{\partial^2}{\partial t^2} u_2 + u_1 \frac{\partial^2}{\partial t^2} u_1 + u_2 \frac{\partial^2}{\partial t^2} u_0, \\
&\vdots
\end{aligned}
$$

which leads to the recursive relationship

$$
\begin{aligned}
u_0 &= f(t) + xg(t) + L_x^{-1}\phi(x,t), \\
u_1 &= L_{xx}^{-1}(A_0), \\
u_2 &= L_{xx}^{-1}(A_1), \\
&\vdots \\
u_{n+1} &= L_x^{-1}(A_n), \quad n \geq 0.
\end{aligned}
\tag{3.23}
$$

Example 3.1. *Let us consider a nonhomogeneous nonlinear wave equation. The equation of the form*

$$u_{xx} - uu_{tt} = 2 - 2(t^2 + x^2), \tag{3.24}$$

the initial conditions posed are

$$u(x,0) = x^2, \tag{3.25}$$

$$u(0,t) = t^2, \tag{3.26}$$

$$u_x(0, t) = 0. \tag{3.27}$$

Using (3.23) to determine the individual terms of the decomposition, we find

$$u_0 = x^2 + t^2 - x^2 t^2 - \frac{1}{6} x^4, \tag{3.28}$$

and

$$
\begin{aligned}
u_1 &= L_x^{-1}(A_0) = x^2 t^2 + \frac{1}{6} x^4 - \frac{1}{3} x^4 t^2 - \frac{7}{90} x^6 + \frac{2}{15} x^6 t^2 + \frac{1}{16} x^8, \\
u_2 &= L_x^{-1}(A_1) = \frac{1}{3} x^4 t^2 + \frac{7}{90} x^6 - \frac{2}{15} x^6 t^2 - \frac{1}{16} x^8 - \cdots,
\end{aligned}
$$

and so on for other components. It can be easily observed that the self canceling noise terms appear between various components. Canceling the third term in u_0 and the first term in u_1 , the fourth term in u_0 and the first term in u_1 , in keeping the non canceled terms in u_0 yields the exact solution of (3.24) given by

$$u(x, t) = x^2 + t^2. \tag{3.29}$$

This can be verified through substitution, see [12]

3.5   One dimensional nonlinear Burgers' equation

The study of Burgers' equation is important since it arises in the approximate theory of flow through a shock wave propagating in a viscous fluid and in the modeling of turbulence [10]. The exact solutions of Burgers' equation have been surveyed by Benton and Platzman [20]. In many cases these solutions involve infinite series which may converge very slowly or for small values of the viscosity coefficients.

Consider the one-dimensional nonlinear Burgers' equation for a given field $u(x, t)$ and diffusion coefficient (or viscosity, as in the original fluid mechanical context) v,

see [6].

$$u_t + \varepsilon\, uu_x - vu_{xx} = 0, \quad a \le x \le b, \tag{3.30}$$

with initial conditions:

$$u(x,0) \;=\; f(x) \tag{3.31}$$

$$u(a,t) \;=\; \beta_1,\; u(b,t) = \beta_2 \quad \forall\, t > 0, \tag{3.32}$$

In this section, the use of tanh function method is demonstrated to get an analytical solution of eq. (3.30) which is not of series form. Secondly, an approximate solution is obtained by applying ADM using the initial condition $u(x,0) = f(x)$ only. Then, a test example is given to demonstrate the accuracy of the method and to illustrate its pertinent feature, another approach for using ADM with the boundary conditions is proposed to get a numerical solution of eq. (3.30).

Analytical solution using the tanh function method for Burgers' equation

we find particular solutions for Burgers' eq. (3.30) using the recent tanh function method. For this, consider the transformations:

$$u(x,t) = f(\xi), \tag{3.33}$$

where $\xi = c(x - \lambda t)$, where c and λ are arbitrary (real) constants. Based on this we use the following change of variables

$$\frac{\partial}{\partial t}(.) = -c\lambda\frac{d}{d\xi}(.),\;\; \frac{\partial}{\partial x}(.) = c\frac{d}{d\xi}(.),\;\; \frac{\partial^2}{\partial x^2}(.) = c^2\frac{d^2}{d\xi^2}. \tag{3.34}$$

Applying the change of variable to Burgers' eq.(3.30), the following ordinary differential equation is obtained.

$$-c\lambda\frac{\mathrm{d}f\left(\xi\right)}{\mathrm{d}\xi} + \varepsilon c f\left(\xi\right)\frac{\mathrm{d}f\left(\xi\right)}{\mathrm{d}\xi} - c^2 v\frac{\mathrm{d}^2 f\left(\xi\right)}{\mathrm{d}\xi^2} = 0, \qquad (3.35)$$

Integrating eq. (3.35), we get

$$-c\lambda f\left(\xi\right) + \frac{\varepsilon c}{2}f^2\left(\xi\right) - c^2 v\frac{\mathrm{d}f\left(\xi\right)}{\mathrm{d}\xi} = B, \qquad (3.36)$$

where B is the constant of integration. Now we introduce a new independent variable:

$$y = \tanh\left(\xi\right),$$

that leads to the change of derivative

$$\frac{d}{d\xi}\left(.\right) = (1 - y^2)\frac{d}{dy}(.). \qquad (3.37)$$

We introduce the following tanh series

$$f\left(\xi\right) = s\left(y\right) = \sum_{i=0}^{m} a_t y^t, \qquad (3.38)$$

where m is a positive integer. From eqns. (3.37) and (3.38) we get

$$-c\lambda s + \frac{c\varepsilon}{2}s^2 - c^2 v\left(1 - y^2\right)\frac{ds}{dy} = 0. \qquad (3.39)$$

To determine the parameter m we balance the linear term of highest order in eq. (3.39) with the highest order nonlinear term. This in turn gives m=1, so we get

$$s\left(y\right) = a_0 + a_1 y. \qquad (3.40)$$

Substituting $s(y)$, and $s'(y)$, from eq. (3.40) into eq. (3.39) yields the system of algebraic equations for a_0 ,a_1, c, and λ :

$$y^0 : -c^2 v a_1 + 0.5 c \varepsilon a_0^2 - c \lambda a_0 \; = \; 0,$$

$$y^1 : a_0 a_1 c \varepsilon - a_1 c \lambda = 0, \tag{3.41}$$

$$y^2 : 0.5 c \varepsilon a^2 + c^2 v a_1 = 0,$$

with the aid of Mathematica we find two solutions:

$$a_0 = \frac{-2cv}{\varepsilon}, a_1 = \frac{-2cv}{\varepsilon}, \lambda = -2cv,$$
$$a_0 = \frac{2cv}{\varepsilon}, a_1 = \frac{-2cv}{\varepsilon}, \lambda = 2cv.$$

So we obtain the solutions

$$u(x,t) = \frac{2cv}{\varepsilon} \left(-1 - \tanh\left[c \left(x + 2cvt \right) \right] \right),$$
$$u(x,t) = \frac{2cv}{\varepsilon} \left(1 - \tanh\left[c \left(x - 2cvt \right) \right] \right), \tag{3.42}$$

The ADM for Burgers' equation using the initial condition

Let $L \left(L(.) = \frac{\partial(.)}{\partial t} \right)$ is a linear operator. Then the approximate solution of the nonlinear Burgers' equation (3.30) is rewritten in the operator form with the initial condition $u(x,0) = u_0 = f(x)$, can be determined by Adomian's polynomials with the iterative process:

$$
\begin{aligned}
u_0 \left(x, t \right) &= f \left(x \right), \\
u_{n+1} \left(x, t \right) &= L^{-1} \left(g \left(t \right) - R \left(u_n \right) - A_n \right), \quad n \geq 0.
\end{aligned} \tag{3.43}
$$

Applying the inverse operator L^{-1} on both sides of eq. (3.30) we get:

$$u \left(x, t \right) = f \left(x \right) - L^{-1} \left(u u_x - v u_{xx} \right). \tag{3.44}$$

Now, we get

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) - L^{-1} \left(\sum_{n=0}^{\infty} A_n - v \left(\sum_{n=0}^{\infty} u_n \right) \bigg|_{xx} \right) \tag{3.45}$$

Identifying the zeroth component $u_0(x,t)$ as $f(x)$, the remaining components

$u_n(x,t), n > 1$ can be determined by using the recurrence relation (3.45). That is,

$$
\begin{aligned}
u_0(x,t) &= f(x), \\
u_{n+1}(x,t) &= L^{-1}(A_n - v(u_n)_{xx}), n \geq 0.
\end{aligned}
\tag{3.46}
$$

where A_n are adomian's polynomials that represent the nonlinear term (uu_x). One

can see that the first few terms of A_n are given by:

$$
\begin{aligned}
A_0 &= u_{0x}u_0, \\
A_1 &= u_{0x}u_1 + u_{1x}u_0, \\
A_2 &= u_{0x}u_2 + u_{1x}u_1 + u_{2x}u_0, \\
A_3 &= u_{0x}u_3 + u_{1x}u_2 + u_{2x}u_1 + u_{3x}u_0.
\end{aligned}
$$

The rest of polynomials can be generated in a similar way. The scheme in (3.46)

can easily determine the components $u_n(x,t)$, $n > 0$ and the first few components

of $u_n(x,t)$ take the following form

$$
\begin{aligned}
u_0(x,t) &= f(x), \\
u_1(x,t) &= L^{-1}(A_0 - v(u_0)_{xx}), \\
u_2(x,t) &= L^{-1}(A_1 - v(u_1)_{xx}), \\
u_3(x,t) &= L^{-1}(A_2 - v(u_2)_{xx}), \\
u_4(x,t) &= L^{-1}(A_3 - v(u_3)_{xx}).
\end{aligned}
\tag{3.47}
$$

Calculating more components in the solution series can enhance the numerical solution obtained by decomposition series. Consequently, one can recursively determine each individual term of the series $\sum_{n=0}^{\infty} u_n(x,t)$, and hence the solution $u(x,t)$ is readily obtained in a series form. For numerical purposes to test the accuracy of the proposed method, based on ADM, we consider two test cases for the Burgers' equation. The obtained numerical approximate solution for each case, $u_{appr.}(x,t)$, is compared with the exact solution where

$$
\begin{aligned}
u_{appr.}(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) \\
+ u_3(x,t) + u_4(x,t) + ...
\end{aligned}
\tag{3.48}
$$

Test Case :

Consider the following analytic solution of Burgers' eq. (3.30):

$$
u(x,t) = \frac{1}{2} \left[1 - \tanh \left\{ \frac{1}{4v} \left(x - 15 - \frac{1}{2}t \right) \right\} \right], \quad t \geq 0,
$$

and the initial condition

$$
u(x,0) = \frac{1}{2} \left[1 - \tanh \left\{ \frac{1}{4v} \left(x - 15 \right) \right\} \right], \quad t \geq 0,
$$

where $x \in [0, 28]$

This test problem has known initial conditions and applying ADM one needs initial conditions only the According to this example and the scheme in (3.47), we get:

$$
\begin{aligned}
u_1 &= \frac{0.0625t}{v} \left[1 - \tanh^2 \left\{ \frac{1}{4v} \left(x - 15 \right) \right\} \right], \\
u_2 &= \frac{0.0078125t^2}{v^2} \sec h^2 \left\{ \frac{1}{4v} \left(x - 15 \right) \right\} \tanh \left\{ \frac{1}{4v} \left(x - 15 \right) \right\}, \\
u_3 &= -\frac{t^3}{3072v^3} \sec h^2 \left\{ \frac{1}{4v} \left(x - 15 \right) \right\} \left[1 - 3\tanh^2 \left\{ \frac{1}{4v} \left(x - 15 \right) \right\} \right].
\end{aligned}
$$

We obtain a numerical approximate solution for Burgers' equation. The obtained

numerical results are summarized in Tables **1-4**. From these results, we conclude that the proposed method, to calculate the approximate numerical solution of the Burgers' equation, gives remarkable accuracy in comparison with the exact solution for some values of time t.

x	U Approximate	U Exact	Absolute error
4	0.99998986	0.99998987	$4.00932 * 10^{-9}$
8	0.99944700	0.99944722	$2.13127 * 10^{-7}$
12	0.97068831	0.97068776	$5.45743 * 10^{-7}$
16	0.37754702	0.37754066	$6.35561 * 10^{-6}$
20	0.01098545	0.01098694	$1.48505 * 10^{-6}$
24	0.00020339	0.00020342	$3.48687 * 10^{-8}$
28	$3.726 * 10^{-6}$	$3.7266 * 10^{-6}$	$6.4136 * 10^{-10}$

Table 3.1: Absolute errors at $t = 1$ and $v = 0.5$.

x	U Approximate	U Exact	Absolute error
4	0.99999543	0.99999627	$8.39849 * 10^{-7}$
8	0.99975179	0.99979657	$4.47788 * 10^{-5}$
12	0.98891035	0.98901305	$1.02707 * 10^{-4}$
16	0.62015142	0.62245933	$2.30791 * 10^{-3}$
20	0.02890743	0.02931223	$4.04797 * 10^{-4}$
24	0.00054256	0.00055277	$1.02176 * 10^{-5}$
28	$9.9418 * 10^{-6}$	$1.013 * 10^{-5}$	$1.88157 * 10^{-7}$

Table 3.2: Absolute errors at $t = 3$ and $v = 0.5$.

x	U Approximate	U Exact	Absolute error
4	0.99999942	0.99999942	$7.6743 * 10^{-10}$
8	0.99991507	0.99991518	$1.1337 * 10^{-7}$
12	0.98756129	0.98756834	$7.0518 * 10^{-6}$
16	0.34871176	0.34864513	$6.66311 * 10^{-5}$
20	0.00359200	0.00359360	$1.5927 * 10^{-6}$
24	0.00002428	0.00002430	$1.1512 * 10^{-8}$
28	$1.6366 * 10^{-7}$	$1.6373 * 10^{-7}$	$7.7609 * 10^{-11}$

Table 3.3: Absolute errors at $t = 1$ and $v = 0.4$.

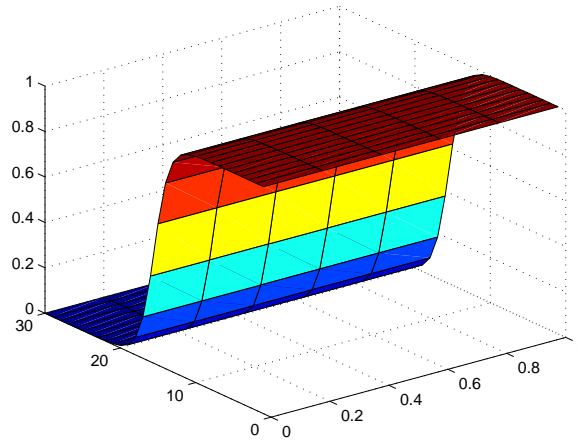The following figures (3-1-3-3) show the behavior of the approximation solutions for the first test case.

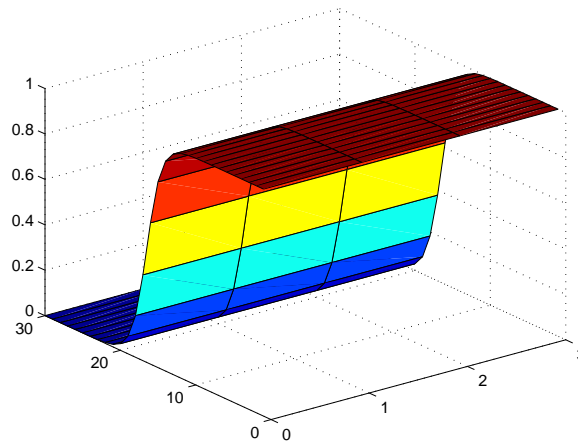Figure 3-1: The numerical solution (v=0.5 from t=0 to t=1).

Figure 3-2: The numerical solution (v=0.5 from t=0 to t=3).
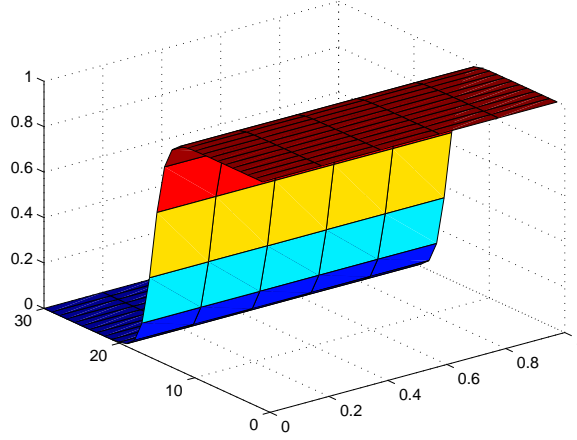
Figure 3-3: The numerical solution (v=0.4 from t=0 to t=1).

The ADM for Burgers' equation (considering the boundary conditions)

Let $R = \frac{\partial^2}{\partial x^2}$. Then eq. (3.30) can be expressed as

$$Ru = \frac{1}{v}[u_t + uu_x], \quad x \in [a, b]. \tag{3.49}$$

Applying the inverse operator R^{-1} on both sides to eq. (3.49) yields

$$u(x, t) = \mu + \frac{1}{v}[u_t + uu_x], \tag{3.50}$$

where $R^{-1} = \int \int (.) dx dx$ and $\mu = C(t) + xB(t)$. Using eq.(3.50) becomes

$$\sum_{n=0}^{\infty} u_n(x, t) = \mu + \frac{1}{v}R^{-1}\left[\sum_{n=0}^{\infty} u_{nt} + \sum_{n=0}^{\infty} A_n\right],$$

where $A_n = \sum_{m=0}^{n} u_{n-m}u_{mx}$. Now we decompose μ into $\mu = \sum_{n=0}^{\infty} \mu_n$.
We have

$$\sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} \mu_n + \frac{1}{v}R^{-1}\left[\sum_{n=0}^{\infty} u_{nt} + \sum_{n=0}^{\infty} A_n\right],$$

Identify $u_0 = \mu_0 = C_0(t) + xB_0(t)$, all other components are determined by

$$u_{n+1} = \mu_{n+1} + \frac{1}{v}R^{-1}\left[u_{nt} + A_m\right],$$

where $\mu_{n+1} = C_{n+1} + xB_{n+1}$, $n \geq 0$. The integration constants C's and B's are determined by satisfying the boundary conditions with the approximate solution$\phi_{n+1} = \sum_{k=0}^{n} u_k$, $n \geq 0$; Thus,

$$\phi_{n+1}(a,t) = u(a,t) = \beta_1,$$

$$\phi_{n+1}(b,t) = u(b,t) = \beta_2.$$

Our first approximation is $\phi_1 = u_0$, or$\phi_1 = C_0(t) + xB_0(t)$. Since $\phi_1(a,t) = u(a,t) = \beta_1$, $\phi_1(b,t) = u(b,t) = \beta_2$.

Therefor,

$$C_0 + aB_0 = \beta_1, \tag{3.51}$$

$$C_0 + bB_0 = \beta_2. \tag{3.52}$$

Solving (3.51) and (3.52), we get

$$B_0 = \frac{\beta_2 - \beta_1}{b - a},$$

$$C_0 = \frac{b\beta_1 - a\beta_2}{b - a}.$$

Hence,

$$u_0 = \frac{(x - a)b\beta_2 - (b - x)\beta_1}{b - a}.$$

To calculate u_1, we have

$$u_1 = C_1 + xB_1 + \frac{1}{v}R^{-1}\left[u_{0t} + A_0\right].$$

A two term approximation is given by

$$\phi_2 = \phi_0 + u_1 = u_0 + u_1.$$

Hence,

$$\phi_2 = \frac{(x-a)b\beta_2 - (b-x)\beta_1}{b-a} + C_1 + xB_1 + \frac{1}{v}R^{-1}\left[u_{0t} + A_0\right]. \qquad (3.53)$$

Since $\phi_2(a,t) = \beta_1$ and $\phi_2(b,t) = \beta_2$, we have

$$\xi_a + C_1 + aB_1 = 0, \qquad (3.54)$$

$$\xi_b + C_1 + bB_1 = 0, \qquad (3.55)$$

where,

$$\xi_a = \frac{1}{v}\left[R^{-1}\left(u_{0t} + A_0\right)\right]\big|_{x=a},$$

and

$$\xi_b = \frac{1}{v}\left[R^{-1}\left(u_{0t} + A_0\right)\right]\big|_{x=b}.$$

Eqns. (3.54) and (3.55) give

$$B_1 = \frac{\xi_a - \xi_b}{b-a}, \qquad (3.56)$$

$$C_1 = \frac{a\xi_a - b\xi_b}{b-a}. \qquad (3.57)$$

Using(3.56), (3.57) and (3.53), we get

$$u_1 = \frac{a\xi_a - b\xi_b}{b-a} + x\frac{\xi_a - \xi_b}{b-a} + \frac{1}{v}\left[R^{-1}\left(u_{0t} + A_0\right)\right],$$

we can continue in this manner to calculate $u_2, u_3, \ldots$

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