Deanship of Graduate Studies and Scientific Research
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# Stability theory of difference equations

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This thesis is submitted in partial fulfillment of the requirement for the degree of Master of Mathematics, Faculty of Graduate Studies, Hebron University, Palestine.

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# Declaration

I declare that the master thesis entitled {Stability theory of difference equations} is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

woroud ishaq shammas

Signature:________________ Date:_____________

# Dedications

*To may husband ,to my parents, my child, my brothers, and sister, also I'll never forget my best friends.*

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# Abstract

This thesis is a survey study of the stability of difference equation. Chapter one introduces the linear first-order one-dimensional difference equation. It includes the study of equilibrium and periodic points, definition and properties of Z-transform. In chapter two, a survey study of system of linear difference equation is presented, in which the basic theory, autonomous systems and periodic systems are discussed. The question of stability for both scalar equations and system of equations is investigated in chapter three. Stability of first order difference equation, two dimensional systems of difference equation and nonlinear system of difference equations is surveyed. The last section of this chapter explores two applications showing the importance of the stability theory.

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# Chapter 1

# Preliminaries

## 1.1 Introduction

Difference equations often used to describe the evaluation of certain phenomena over period of time. For example, consider the general difference equation

$$x(n+1) = f(x(n)). \tag{1.1}$$

By using $x_0$ as an initial point, the following sequence can be generated

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), ...$$

$f(x_0) = x_1$ is called the first iterate of $x_0$ under $f$, $f^2(x_0) = f(f(x_0)) = x_2$ is called the second iterate of $x_0$ under $f$. In general $f^n(x_0) = x_n$ is the $n^{th}$ iterate of $x_0$ under $f$. This iterative procedure is an example of a discrete dynamical system. That is, systems that can be described by a set of $n$ variables $x_1, x_2, \ldots, x_n$ which is specified by the following difference equation,

$$x(n+1) = f(x(n)).$$

The difference equation and discrete dynamical systems are equivalent in the sense that they represent the same physical model. for instance, when we discuss difference equations we often refer to the analytic theory of subject, and when we discuss discrete dynamical system, we refer to the topological and geometrical aspects.
Difference equation (1.1) is called time-invariant or autonomous, if the function $f$

in (1.1) is replaced by a function $g$ of two variables $g : \mathbb{Z}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$, where $\mathbb{R}$ is the set of real number and $\mathbb{Z}^+$ is the set of nonnegative integers, then we have

$$x(n+1) = g(n, (x(n))). \tag{1.2}$$

This equation is called (time-variant) or nonautonomous.

## 1.2 Linear first-order difference equations

Consider a time period $T$ and observe (or measure) the system at times $t = nT$, $n \in N_0$, and let $x(n) = f(nT)$, then a sequence $x(0), x(1), x(2), \ldots$ is generated with values are obtained from a function $f$, which is defined for all $t \geq 0$. This method of obtaining the values is called a recurrence relation or sometimes is refered to as periodic sampling.
First, we consider the simplest case

$$x(n+1) = a(n)x(n), \;\; n \in N_0. \tag{1.3}$$

If we consider the case where $a$ is a given constant, then the solution is given by

$$x(n) = a^n x(0), \tag{1.4}$$

where the value $x(0)$ is called the initial value. We write the equation (1.3) as

$$x(n+1) - ax(n) = 0. \tag{1.5}$$

This equation is called a homogeneous first order difference equation with constant coefficients. The term homogeneous means that the right hand side is zero. A corresponding inhomogeneous equation is given as

$$x(n+1) - ax(n) = g(n), \tag{1.6}$$

where the right hand side to be a constant different from zero. The equation (1.5) is called linear, since it satisfies the superposition principle. i.e. If $y(n)$ and $z(n)$ are two solutions to (1.5) and $\alpha, \beta \in R$ are two real numbers, then $w(n) = \alpha y(n) + \beta z(n)$ is also a solution of (1.5). The following computation clarify that

$$\begin{aligned} w(n+1) - aw(n) &= \alpha y(n+1) + \beta z(n+1) - a(\alpha y(n) + \beta z(n)) \\ &= \alpha(y(n+1) - ay(n)) + \beta(z(n+1) - az(n)) \\ &= \alpha(0) + \beta(0) = 0. \end{aligned}$$

We now turn to solve (1.6). The basic idea is to compute a several terms, then guess the structure of the solution, and then prove that we have indeed found the solution. In the computation of $x(2)$ we give all intermediate steps. These are omitted in the computation of $x(3)$ etc.

$$\begin{aligned}
x(1) &= ax(0) + g(n),\\
x(2) &= ax(1) + g(n) = a(ax(0) + g(n)) + g(n) = a^2x(0) + ag(n) + g(n),\\
x(3) &= ax(2) + g(n) = a^3x(0) + a^2g(n) + ag(n) + g(n),\\
x(4) &= ax(3) + g(n) = a^4x(0) + a^3g(n) + a^2g(n) + ag(n) + g(n),\\
x(5) &= ax(4) + g(n) = a^5x(0) + a^4g(n) + a^3g(n) + a^2g(n) + ag(n) + g(n),\\
&\vdots\\
x(n) &= a^nx(0) + g(n)\sum_{k=0}^{n-1} a^k.
\end{aligned}$$

Thus we have guessed that the solution is given by

$$x(n) = a^nx(0) + g(n)\sum_{k=0}^{n-1} a^k. \tag{1.7}$$

To prove that (1.7) is a solution to (1.6), we plug (1.7) into (1.6) to obtain

$$\begin{aligned}
x(n+1) &= a^{n+1}x(0) + g(n)\sum_{k=0}^{n} a^k\\
&= a^{n+1}x(0) + g(1 + a + a^2 + \ldots + a^{n-1} + a^n)\\
&= a(a^nx(0)) + g(n) + a(g(1 + a + a^2 + \ldots + a^{n-1}))\\
&= a(a^nx(0) + g(n)\sum_{k=0}^{n-1} a^k) + g(n)\\
&= ax(n) + g(n).
\end{aligned}$$

**Theorem 1.1.** [7] *Let $a(n)$ and $g(n)$, $n \in N_0$, be real sequences. Then the linear first order difference equation*

$$x(n+1) = a(n)x(n) + g(n) \quad x(0) = y_0 \tag{1.8}$$

*has the solution*

$$y(n) = \left[\prod_{k=0}^{n-1} a(k)\right] y_0 + \sum_{k=0}^{n-1}\left[\prod_{j=k+1}^{n-1} a(j)\right] g(k). \tag{1.9}$$

*The solution is unique.*

*Proof.* Assume that formula (1.9) holds for $n = m$. Then from $y(m+1) = a(m)y(m)+g(m)$, which by formula (1.9) yields

$$\begin{aligned} y(m+1) &= a(m)\left(\prod_{i=n_0}^{m-1} a(i)\right) y_0 + \sum_{r=n_0}^{m-1}\left[a(m)\prod_{i=r+1}^{m-1} a(i)\right] g(r) + g(m), \\ &= \left(\prod_{i=n_0}^{m} a(i)\right) y_0 + \sum_{r=n_0}^{m-1}\left[\prod_{i=r+1}^{m} a(i)\right] g(r) + \left[\prod_{i=m+1}^{m} a(i)\right] g(m) \\ &= \left(\prod_{i=n_0}^{m} a(i)\right) y_0 + \sum_{r=n_0}^{m-1}\left[\prod_{i=r+1}^{m} a(i)\right] g(r). \end{aligned}$$

□

### 1.2.1 Continuous time

There are two special cases of (1.9) that are important in many applications. The first equation is given by

$$y(n+1) = ay(n) + g(n), \qquad y(0) = y_0.$$

Using formula (1.9)

$$y(n) = a^n y_0 + \sum_{k=0}^{n-1} a^{n-k-1} g(k). \tag{1.10}$$

To show that (1.10) satisfies the initial value problem. First we have

$$\sum_{k=0}^{n-1} a^{n-k-1} g(k) = 0$$

by the usual convention, so $u(0) = u_0$. For $t \geq 1$,

$$\begin{aligned} y(n+1) &= a^{n+1} y_0 + \sum_{k=0}^{n-1} a^{n-k} g(k) \\ &= a^{n+1} y_0 + \sum_{k=0}^{n-1} a^{n-k} g(k) + g(n) \\ &= a[a^{n+1} y_0 + \sum_{k=0}^{n-1} a^{n-k} g(k)] + g(n) \\ &= ay(n) + g(n). \end{aligned}$$

The second equation is given by

$$y(n+1) = ay(n) + b, \qquad y(0) = y_0.$$

Using formula (1.10), we obtain

$$y(n) = \begin{cases} a^n y_0 + b[\frac{a^n-1}{a-1}], & \text{if } a \neq 1, \\ y_0 + bn, & \text{if } a = 1. \end{cases}$$

To establish this

$$\begin{aligned} y_n &= ay_{n-1} + b \\ &= a(ay_{n-2} + b) \\ &= a^2 y_{n-2} + ab + b \\ &= a^2(ay_{n-3} + b) + ab + b \\ &= a^3 y_{n-3} + a^2 b + ab + b \\ &= a^n y_0 + a^{n-1} b + a^{n-2} b + \ldots + ab + b \\ &= a^n y_0 + b[\frac{a^n - 1}{a-1}], \quad a \neq 1. \end{aligned}$$

In the difference equation $y_n = ay_{n-1} + b$, let $a = 1$

$$y_n = y_{n-1} + b,$$

then $y_n = y_0 + b_n$. In following we emphasise the similarity correspondence between the technique used to solve difference equations and that used in solving differential equations. Consider the simple differential equation

$$\frac{dx}{dt} = ax(t), \quad x(0) = x_0,$$

with solution

$$x(t) = x_0 e^{at},$$

and the solution of the non homogeneous differential equation

$$\frac{dy}{dt} = ay(t) + g(t), \qquad y(0) = y_0,$$

is given by

$$y(t) = y_0 e^{at} + \int_0^t e^{a(t-s)} g(s) ds.$$

To prove that for the case where $p(t) = -a$, consider the first ordinary differential equation with

$$\begin{aligned} \frac{dy}{dt} &= ay(t) + g(t), \qquad y(0) = y_0, \\ \frac{dy}{dt} - ay(t) &= g(t), \qquad y(0) = y_0. \end{aligned}$$

This equation has a general solution:

$$y(t) = \frac{1}{\mu(t)} \left[ \int_0^t \mu(s) g(s) dt + c \right]$$

where

$$\mu(t) = e^{\int p(t)dt} = e^{-\int a dt} = e^{-at}.$$

Hence

$$y(t) = e^{at} \left[ \int_0^t e^{-as} gsds + c \right],$$

$$y(t) = ce^{at} + \int_0^t e^{a(t-s)} g(s) ds,$$

using the initial condition to obtain the value of $c$

$$y(0) = c = y_0.$$

Therefore,

$$y(t) = y_0 e^{at} + \int_0^t e^{a(t-s)} g(s) ds.$$

Thus, the exponential $e^{at}$ in differential equations corresponds to the exponential $a^n$ and the integral$\int_0^t e^{a(t-s)} g(s) ds$ corresponds to the summation $\sum_{k=0}^{n-1} a^{n-k-1} g(k)$.

**Example 1.2.** *consider*

$$x(n+1) = 2x(n) + 3^n, \qquad x(0) = 5.$$

*From (1.10), we have*

$$\begin{aligned} x(n) &= 5.2^n + \sum_{k=0}^{n-1} 3^k.2^{n-1-k} \\ &= 5.2^n + 2^{n-1}\sum_{k=0}^{n-1} 2^{-k}.3^k \\ &= 5.2^n + 2^{n-1}\sum_{k=0}^{n-1} (\frac{3}{2})^k \\ &= 5.2^n + 2^{n-1}\frac{3}{2}\left(\frac{(\frac{3}{2})^{n-1} - 1}{\frac{3}{2} - 1}\right) \\ &= 5.2^n + 3^n - 32^{n-1}. \end{aligned}$$

## 1.3 Equilibrium points

The notion of equilibrium points is central in the study of the dynamics of any physical system. In many applications in biology, economics, physics.

**Definition 1.3.** [20] *A point $x^* \in X$ in the domain of $f$ is called an equilibrium point, a steady state, or fixed point if it satisfies the equation*

$$x^* = f(x^*, t); \;\; for \;\; all \;\; t \in \mathbb{Z}. \tag{1.11}$$

**Example 1.4.** *there are two equilibrium points for the equation*

$$x(n+1) = x^2(n) + 2x(n),$$

*where $f(x) = x^2 + 2x$. To find these equilibrium points, we let $x^2 + 2x = x$, and solve for $x$. Hence there are two equilibrium points, $-1, 0$.*

**Definition 1.5.** [1] *Let $x$ be a point in the domain of $f$. If there exists a positive integer $r$ and an equilibrium point $x^*$ of (1.1) such that $f^r(x) = x^*$, $f^{r-1}(x) \neq x^*$, then $x$ is an eventually equilibrium (fixed) point.*

**Example 1.6.** *A popular example of a nonlinear difference equation is the logistic function with $\mu > 1$ :*

$$f(x) = \begin{cases} \mu x(1-x), & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if otherwise.} \end{cases} \tag{1.12}$$

*The equilibrium points are determined by the quadratic equation:* $x^* = \mu x(1-x)$. *The two solutions are* $x^* = 0$ *and* $x^* = \frac{(\mu-1)}{\mu}$. *Hence the corresponding difference equation has two equilibrium points.* $f(x)$ *attains a maximum value of* $\frac{\mu}{4}$ *at* $x = \frac{1}{2}$.

## 1.4 Periodic points and cycles

The notion in the study of dynamical systems is the notion of periodicity. For example, the motion of a pendulum is periodic.

**Definition 1.7.** [4] *A value* $x_0$ *is called a periodic point for* $f$*, and its orbit is called periodic orbit (or an* $n$*-cycle), if there is some value of* $n$ *such that* $f^n(x) = x$*. Any such value of* $n$ *is called a period of* $x_0$*, and the smallest (positive) value of* $n$ *is called exact period (or minimal period).*

A periodic orbit of length $n$ will repeat every $n$ steps: that is

$$x_0, f(x_0), f^2(x_0), \dots, f^{n-1}(x_0), x_0, f(x_0), f^2(x_0), \dots$$

**Example 1.8.** *Take the equation*

$$x(n+1) = -2x(n) + 2.$$

*Then,* $f(x) = -2x + 2$*. Now, to find the* $2$*-periodic points, we have to find* $f^2(x)$

$$f^2(x) = f(f(x)) = 4x - 2 = x.$$

*Solving this equation shows that* $\frac{2}{3}$ *is the 2-periodic point of our difference equation.*

**Proposition 1.9.** [4] *If* $x_0$ *is a periodic point with minimal period* $n$*, then* $f^m(x_0) = f^{(m+n)}(x_0)$ *for any* $m$ *and* $f^k(x_0) = x_0$ *holds if and only if* $n$ *divides* $k$*.*

*Proof.* For the first direction, if $f^n(x_0) = x_0$, then the least period of $x_0$ with respect to $f$ divides $n$. If we let $m$ denote the least period of $x_0$ with respect of $f$ and write $n = km + r$ with $0 \le r < m$, then $x_0 = f^n(x_0) = f^{km+r}(x_0) = f^r(f^{km}(x_0)) = f^r(x_0)$. Since $m$ is the smallest positive integer such that $f^m(x_0) = x_0$, we must have $r = 0$. Therefore, $m$ divides $n$.

For the reverse direction, let $n$ is a minimal period of $f$ such that $f^k(x_0) = x_0$ and $n$ does not divide $k$, then $k = qn + r$ for some $q, r \in \mathbb{Z}$ and $0 \le r \le n$. This implies that $f^r(x_0) = f^{(n+r)}(x_0) = f^{(2n+r)}(x_0) = \dots = f^{(qn+r)}(x_0) = f^k(x_0) = x_0$, but this is a contradiction, since $n$ is the minimal periodic of $f$. $\square$

Finding all the periodic points of order $n$ for $f$ requires solving $f^n(x) = x$. For most functions this is computationally quite difficult: if $f$ is a polynomial of degree $d$, then $f^n(x) - x$ is a polynomial of degree $d^n$.
- For polynomials, we only have a hope of doing this (even with a computer) if $f$ is a polynomial of small degree and the order is small.
- However, if $m \mid n$, then every point of period $m$ will satisfy $f^m(x) = x$ and hence also $f^n(x) = x$. Thus, we can save a small amount of effort by removing the factors of $p^n(x) - x$ that come from terms $p^m(x) - x$ where $m \mid n$.
- This trick is especially helpful if $f$ is a quadratic polynomial and $n = 2$: then $f^2(x) - x$ has degree 4, but it is divisible by the quadratic $f(x) - x$, so we can take the quotient and obtain a quadratic, which is then easy to solve.

**Definition 1.10.** **[18, 4]** *A value $x_0$ is called a preperiodic point for $f$ (or eventually periodic) if there exist positive integers $m$ and $n$ such that $f^m(x_0) = f^{(m+n)}(x_0)$. Equivalently, $x_0$ is preperiodic if there exists some $m$ so that $f^m(x_0)$ is periodic. In the event that $n = 1$ we say $x_0$ is an eventually fixed point.*

**Example 1.11.** *The point $x_0 = -1$ is an eventually fixed point for the function $f(x) = x^4$, since the orbit of $-1$ is $-1, 1, 1, 1, \ldots$*

**Example 1.12.** *The point $x_0 = \frac{1}{3}$ is an preperiodic point for the function $f(x) = 1 - \frac{3}{2}x^2$, since the orbit of $\frac{1}{3}$ is $\frac{5}{6}, \frac{2}{3}, 0, 1, -1, \ldots$*

## 1.5 The Z-transform method

The idea behind the $Z$-transform, which is also called transformation method, is to simplify the study of linear differential equation or difference equation to a study of geometric equation corresponding to the original difference or differential equation. For example, the Laplace transform is used in solving and analysing higher order linear differential equations and linear difference equations, while the $Z$-transform method is most suitable for linear difference equations and discrete systems. Communication, signal processing and digital control are of the main applications that $Z$-transform is most often used for analysing and computing.

### 1.5.1 Definitions and examples

The Z-transform of sequences $x(n)$ is defined by

$$\tilde{x}(z) = Z(x(n)) = \sum_{j=o}^{\infty} x(j)z^{-j}, \tag{1.13}$$

where $\{x(n) = 0; \quad n = -1, -2, ....\}$, and $z$ is a complex number. The domain of convergence of $x(z)$ is defined as the values of $z$ in the complex plane for which the series (1.13) converges. Region of convergence of the series (1.13) is determined usually by the ratio test and root test. Suppose that

$$\lim_{j\to\infty} \mid \frac{x(j+1)}{x(j)} \mid = R.$$

Then by the ratio test, the infinite series (1.13) converges if

$$\lim_{j\to\infty} \mid \frac{x(j+1)z^{-j-1}}{x(j)z^{-j}} \mid < 1,$$

and diverges if

$$\lim_{j\to\infty} \mid \frac{x(j+1)z^{-j-1}}{x(j)z^{-j}} \mid > 1.$$

Hence, the series (1.13) converges in the region $\mid z \mid > R$, and diverges for $\mid z \mid < R$. In addition, the number $R$ is called the radius of converges of series (1.13). If $R = 0$, the Z-transform $\tilde{x}(z)$ converges every where with the possible exception of the origin. On the other hand, if $R = \infty$, the Z-transform diverges every where. We compute the Z-transform of some elementary functions.

**Example 1.13.** *Find the Z-transform of the sequence* $x(n) = cosh(wn)$, *for* $n \geq 0$.

*Solution:*

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} cosh(wn)z^{-n} \\
&= \sum_{n=0}^{\infty} \frac{1}{2}(e^{jwn} + e^{-jwn})z^{-n} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(e^{jw}z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty}(e^{-jw}z^{-1})^n \\
&= \frac{1}{2}(\frac{1}{1-e^{jw}z^{-1}}) + \frac{1}{2}(\frac{1}{1-e^{-jw}z^{-1}}), \quad \mid z \mid > \mid e^{jw} \mid = 1 \\
&= \frac{1}{2}(\frac{z}{z-e^{jw}}) + \frac{1}{2}(\frac{z}{z-e^{-jw}}), \quad \mid z \mid > 1 \\
&= \frac{1}{2}\left(\frac{z(z-e^{-jw})}{z^2 - z(e^{jw}+e^{-jw})+1} + \frac{z(z-e^{jw})}{z^2 - z(e^{jw}+e^{-jw})+1}\right), \quad \mid z \mid > 1 \\
&= \frac{z - zcosh(w)}{z^2 - 2zcosh(w) + 1}, \quad \mid z \mid > 1
\end{aligned}$$

### 1.5.2 Properties of the Z-transform

we will discuss main properties of the Z-transform.

(i) Linearity, let $\tilde{x}(z)$ be the Z-transform of $x(n)$ with radius of convergence $R_1$ and $\tilde{y}(z)$ be the Z-transform of $y(n)$ with radius convergence $R_2$, then for any complex number $a, b$ we have

$$Z[ax(n) + by(n)] = a\tilde{x}(z) + b\tilde{y}(z),$$

for some $|z| > max(R_1, R_2)$.

(ii) Shifting to the right and left property, consider the sequence $\tilde{x}(z)$ with radius $R$ of convergence, then

(a) shifting $z(\tilde{x}(n))$ to the right is defined as

$$Z[x(n-k)] = z^{-k}\tilde{x}(z),$$

for $\mid z \mid > R$, and $x(-i) = 0$ for $i = 1, 2, \ldots, k$,

(b) shifting $z(\tilde{x}(n))$ to the left is defined as

$$Z[x(n-k)] = z^{-k}\tilde{x}(z) - \sum_{r=0}^{k-1} x(r)z^{k-r}.$$

for $\mid z \mid > R$.

(iii) convolution, $*$, a convolution of two sequences $x(n)$, $y(n)$ is defined by

$$x(n) * y(n) = \sum_{j=0}^{n} x(n-j)y(j) = \sum_{j=0}^{n} x(j)y(n-j)$$

$$Z[x(n) * y(n)] = \tilde{x}(z)\tilde{y}(z)$$

(iv) The property of multiplication, suppose that $\tilde{x}(z)$ is the Z-transform of $x(n)$ with radius of convergence $R$, then

$$Z[b^n x(n)] = \tilde{x}(\frac{z}{b}), \quad |z| > |b|R.$$

(v) The inverse Z-transform, $Z^{-1}[\tilde{x}(z)] = x(n)$, is used to obtain the sequence $x(n)$ for $\tilde{x}(n)$.

**Example 1.14.** *Determine the Z-transform of*

$$x(n) = \frac{1}{5^n}[u(n) - (u-5)], \quad n = 0, 1, 2, ...$$

$$\begin{aligned}
\tilde{x}(n) &= \frac{1}{5^n}u(n) - \frac{1}{5^n}u(n-5) \\
ZT[x(n)] &= ZT[\frac{1}{5^n}u(n)] - ZT[\frac{1}{5^n}u(n-5)] \\
&= \frac{z}{z-\frac{1}{5}} - ZT[\frac{1}{5^n}\frac{1}{5^{n-5}}u(n-5)] \\
x(z) &= \frac{z}{z-\frac{1}{5}} - (0.2)^5 ZT[\frac{1}{5^{n-5}}u(n-5)] \\
&= \frac{z}{z-\frac{1}{5}} - (0.2)^5 z^{-5}\frac{z}{z-\frac{1}{5}} \\
&= \frac{z-(0.2)^{-5}z^{-4}}{z-\frac{1}{5}} \\
&= \frac{z^5-(0.2)^5}{z^4(z-\frac{1}{5})}.
\end{aligned}$$

*Chapter 2*

# System of linear difference equation

## 2.1 Equations of first order with a single variable.

Consider the first-order difference equation

$$x_t + ax_{t-1} = b_t, \tag{2.1}$$

with one variable $x$, $a$ is a time-independent coefficient and $b_t$ is the forcing term. When $b_t = 0$, the difference equation is said to be homogeneous, otherwise it is called non-homogeneous. When the forcing term is a constant ($b_t = b$ for all $t$) the difference equation (2.1) is non-homogeneous and autonomous (or time-invariant). Finally, when $b_t$ is time-dependent the equation is said to be non-autonomous; this more general formulation, allowing to capture seasonality, deterministic shocks or perturbations. The most general form of linear difference equation is one in which also the coefficient a is time-varying.

### 2.1.1 Autonomous equations.

Starting with a non-homogeneous, autonomous difference equation, with $b_t$ being equal to a time-invariant scalar $b$. A first approach is to solve (2.1) by successive iteration. Suppose the initial value $x_0$ is given, then

$$\begin{aligned} x_1 &= b - ax_0 \\ x_2 &= b - a(b - ax_0) \\ &= b - ab + a^2x_0 \\ x_3 &= b - a(b - ab + a^2x_0) \\ &= b - ab + a^2b - a^3x_0 \end{aligned}$$

then

$$\begin{aligned} x_t &= b(1 + (-a) + (-a)^2, \ldots, +(-a)^{t-1}) + (-a)^t x_0 \\ &= \frac{(1 - (-a))^t}{1 - (-a)} b + (-a)^t x_0, \quad if \;\; a \neq -1 \end{aligned}$$

Rearranging and gathering terms in $(-a)^t$ yields

$$x_t = (-a)^t \left[ x_0 - \frac{b}{1+a} \right] + \frac{b}{1+a} \tag{2.2}$$

A general method, analogous to the one used for differential equations, is based on the Superposition Principle (see theorem [SP] below). That is, the solution of a linear difference equation is the sum of the particular solution, any solution to the non-homogeneous difference equation, and the complementary solution, solution of its homogeneous part

$$x_t = x_t^{co} + x_t^{pa}.$$

For autonomous equations, a (very convenient) particular solution is the steady-state solution, $x_t^{pa} = x^*$, which is constant over time. Consider the homogeneous equation, $x_t + ax_{t-1} = 0$. Recalling differential equations, one may guess a solution to this equation to be, $x_t = ck^t$, where $c$ is analogous to a constant of integration and $k = -a$. (Recall that for homogeneous differential equations of the type, $x'(t) + ax(t) = 0$, the complementary solution is $x_t^{co} = ce^{(-a)t}$. This can be derived, first, by integrating $\frac{x'}{x} + a = 0$ over $t$, which yields $\log[x(t)] = B - at$, with $B$ defined by collecting the two constants of integration. Then, by taking the exponential of both sides and letting $c := eB$). In the difference equation, the instantaneous rate of change $e^{(-a)t}$ is replaced by $(-a)^t$. To see the latest, substitute the guessed solution in the equation, $ck^t + ack^{t-1} = 0$; simplifying, $ck^{t-1}(k + a) = 0$, which is satisfied if and only if $k = -a$. To summarize, the complementary solution is,

$$x_t^{co} = c(-a)^t$$

As a particular solution take the steady-state $x^*$; substituting $x_t = x^*$, $x^* + ax^* = b$, hence

$$x_t^{pa} = \frac{b}{1+a},$$

therefore the general solution is

$$x_t = x_t^{co} + x_t^{pa} = c(-a)^t + \frac{b}{1+a}. \tag{2.3}$$

This solution is identical to (2.2), the one found by iterated substitutions, except for $c$ that is still to be determined. Take $t = 0$, then $x_0 = c(-a)^0 + \frac{b}{(1+a)}$, implying $c = x_0 - \frac{b}{(1+a)}$, if $a \neq -1$.
It remains to consider the case in which $a = -1$, implying $x_t = x_{t-1} + b$. Incidentally, the method used to find the general solution, and precisely the complementary solution, is called undetermined coefficients. This is based on guessing a functional form for $x_t$ and verify that this solves the equation (for some coefficients). Guess, $x_t = \alpha t$, a trend line with initial value at $\alpha$ to be determined. Substituting, $\alpha(t + 1) - \alpha t - b = 0$, satisfied for $\alpha = b$. Hence, if $a = -1$,

$$x_t = bt + x_0. \tag{2.4}$$

obviously, any other guess on functional forms would not work. The following theorem establishing the Superposition Principle (SP).

**Theorem 2.1.** *(SP)*[**24**] *Any solution $x_t$ of the equation $x_t + ax_{t-1} = b_t$ can be written as*

$$x_t = x_t^{co} + x_t^{pa}$$

*with $x_t^{co} = c(-a)^t$, for a particular solution $x_t^{pa}$.*

*Proof.* Let $x_t$ is a solution of (2.1).

$$z_t = x_t - x_t^{pa}$$

$$\begin{aligned} x_{t+1} &= x_{t+1} - x_t^{pa} \\ &= -ax_t + b_t - (-ax_t^{pa} + b_t) \\ &= -a(x_t - x_t^{pa}) \\ &= -az_{t-1}. \end{aligned}$$

That is, $z_t = c(-a)^t$ solves the homogeneous equation, so $x_t = z_t + x_t^{pa} = c(-a)^t + x_t^{pa}$. □

The Superposition Principle is a general result that, as you can see from its proof, extends to non-autonomous equations to systems of linear equations.

## 2.2 The Basic Theory

Let $A(n)$ be an $k \times k$ matrix whose elements $a_{ij}(n)$ are real or complex functions and $x(n) \in \mathbb{R}^k$ with entries that are real or complex functions. A linear equation

$$x(n+1) = A(n)x(n) + g(n),$$

$g(n) \in \mathbb{R}^k$, is said to be a nonhomogeneous linear difference equation. If $g(n) = 0$ then the equation is called homogeneous linear difference equation.

$$x(n+1) = A(n)x(n), \tag{2.5}$$

when an initial vector $x_0$ is assigned to (2.5) the solution can be determined uniquely. For example, it follows from (2.5) that the solution takes the form

$$x(n, n_0, x_0) = \left[\prod_{i=n_0}^{n-1} A(i)\right] x_0,$$

$\prod_{i=n_0}^{n-1} A(i)$ where is uniquely determined for all $n$. As the matrices $A(i)$ can not commute, the order in the product became important with the lowest index is always the rightmost. The above product was the following order $A(n-l)A(n-2)\ldots A(n_0)$. Sometimes, in order to avoid confusion, we shall denote the solution of (2.5) having $x_0$ as initial vector by $x(n, n_0, x_0)$. The space $K$ of solutions of (2.5) is a linear space, since any linear combination of two solutions is a solution of the same equation.

**Definition 2.2.** **[1]** *The solutions $x_1(n), x_2(n), ..., x_k(n)$ of (2.5) are said to be linearly independent for $n \geq n_o \geq 0$ if whenever $c_1x_1(n) + c_2x_2(n) + ... + c_kx_k(n) = 0$ for all $n \geq n_o$, then $c_i = 0, \quad 1 \leq i \leq k$.*

Let $\Phi(n)$ be a $k \times k$ matrix whose columns are solutions of (2.5). We write

$$\Phi(n) = [x_1(n), x_2(n), ..., x_k(n)].$$

Now

$$\begin{aligned}\phi(n+1) &= [A(n)x_1(n), A(n)x_2(n), ..., A(n)x_k(n)] \\ &= A(n)[x_1(n), x_2(n), ..., x_k(n)] \\ &= A(n)\Phi(n).\end{aligned}$$

Hence, $\Phi(n)$ satisfies the matrix difference equation:

$$\Phi(n+1) = A(n)\Phi(n). \tag{2.6}$$

Furthermore, the solutions $x_1(n), x_2(n), ..., x_k(n)$ are linearly independent for $n \geq n_o$ if and only if the matrix $\Phi(n)$ is nonsingular ($det\Phi(n) \neq 0$) for all $n \geq n_o$.

**Definition 2.3.** [**1**] *If $\Phi(n)$ is a matrix that is nonsingular for all $n \geq n_o$ and satisfies (2.6), then it is said to be a fundamental matrix for system equation (2.5).*

Note that if $\Phi(n)$ is a fundamental matrix and $C$ is any nonsingular matrix, then $\Phi(n)C$ is also a fundamental matrix.
Thus there are infinitely many fundamental matrices for a given system. However, there is one fundamental matrix that we already know, namely,

$$\Phi(n) = \prod_{i=n_o}^{n-1} A(i), \ \ with \ \ \Phi(n_o) = I.$$

In the autonomous case when $A$ is a constant matrix, $\Phi(n) = A^{n-n_o}$, and if $n_o = 0$, then $\Phi(n) = A^n$. Consequently, it would be more suitable to use algorithm to compute the fundamental matrix for an autonomous system.

**Theorem 2.4.** [**21**] *There is a unique solution $\Psi(n)$ of the matrix (2.6) with $\Psi(n_0) = I$.*

*Proof.* The matrix difference equation (2.6) is a system of $k^2$ first-order difference equations. Define $k^2$-vector solution $\nu$ such that $\nu(n_0) = (1, 0, \ldots, 1, 0, \ldots)^T$, where all entries are $0^{'s}$ except at the first, $(k+2)th$, $(2k+3)th, \ldots$ indices where $1^{'s}$ appear. The vector $v$ is then converted to the $k \times k$ matrix $\Psi(n)$ by grouping the components into sets of $k$ elements in which each set will be a column. Clearly, $\Psi(n_0) = I$. □

The state transition matrix is defined as

$$\Phi(n, n_0) = \Phi^{-1}(n)\Phi(n_0),$$

where $\Phi(n)$ is the fundamental matrix.
In addition, to the two basic properties: $\Phi(n, m) = \Phi(n)\Phi^{-1}(m)$ for any two positive integers $n, m$ with $n \geq m$ and $\Phi(n+1, m) = A(n)\Phi(n, m)$, some of other properties of $\phi(n, m)$ will be listed below that

**Corollary 2.5.** **[10]** *The solution of (2.5) with $x(n, n_0, x_0) = x_o$ is unique and given by*

$$x(n, n_0, x_0) = \Phi(n, n_0)x_0.$$

**Lemma 2.6.** **[1, 18]** *Abel's Formula. For any $n \geq n_0 \geq 0$,*

$$\det \Phi(n) = \left( \prod_{i=n_0}^{n-1} [\det A(i)] \right) \det \Phi(n_0). \tag{2.7}$$

*Proof.* Taking the determinant of both sides of (2.6) we obtain the scalar difference equation

$$\det \Phi(n + 1) = \det A(n) \det \Phi(n)$$

whose solution is given by (2.7). □

**Corollary 2.7.** **[18]** *If $A$ is a constant matrix in (2.5), then*

$$\det \Phi(n) = [\det A]^{n-n_0} \det \Phi(n_0). \tag{2.8}$$

**Corollary 2.8.** **[1]** *The solutions $x_1(n), x_2(n), ..., x_k(n)$ of (2.5) are linearly independent for $n \geq n_o$ if and only if $\Phi(n_o)$ is non singular.*

**Theorem 2.9.** **[18]** *There are $k$ linearly independent solutions of system (2.5) for $n \geq n_0$.*

*Proof.* Define $e_i = (0, 0, \ldots, 1, \ldots, 0)^T \in R^k$ to be the standard unit vector for all $i \in [1, k]$. for each $e_i$, there exist a solution of (2.5) with $x(n_0, n_0, e_i) = e_i$. The set of solutions are linearly independent since $|\phi(n_0)| = |I| \neq 0$. □

The solutions of system (2.5) has two main features that they are closed under addition and scalar multiplication.

(1) if $x_1(n)$ and $x_2(n)$ are a solution of (2.5) then $x_1(n) + x_2(n)$ is also a solution.

(2) if $x_1(n)$ is a solution, then $cx_1(n)$ is also a solution of (2.5).

*Proof.* (1) Let $x(n) = x_1(n) + x_2(n)$, then

$$\begin{aligned} x(n+1) &= x_1(n+1) + x_2(n+1) \\ &= Ax_1(n) + Ax_2(n) \\ &= A[x_1(n) + x_2(n)] \\ &= Ax(n). \end{aligned}$$

(2) Let $x(n) = cx_1(n)$, then

$$\begin{aligned} x(n+1) &= cx_1(n+1) \\ &= cAx_1(n) \\ &= Ax(n). \end{aligned}$$

□

One can generalize the linearity principle as follows if $x_1(n), x_2(n), ..., x_k(n)$ are solutions of system (2.5), then $x(n) = c_1x_1(n) + c_2x_2(n) + ... + c_kx_k(n)$ is also a solution. This leads to the following definition.

**Definition 2.10.** [**1**] *Assuming that $\{x_i(n)|1 \leq i \leq k\}$ is any linearly independent set of solutions of (2.5), the general solution of (2.5) is defined to be*

$$x(n) = \sum_{i=1}^{k} c_i x_i(n), \tag{2.9}$$

*where $c_i \in \mathbb{R}$ and at least one $c_i \neq 0$. Formula (2.9) may be written as*

$$x(n) = \Phi(n)c, \tag{2.10}$$

*where $\Phi(n) = (x_1(n), x_2(n), ..., x_k(n))$ is a fundamental matrix, and $c = (c_1, c_2, ..., c_k)^T \in \mathbb{R}^k$.*

**Remark 2.11.** [**1**] *The set $S$ of all solutions of system (2.5) forms a linear (vector) space under addition and scalar multiplication. Its basis is any fundamental set of solutions and hence its dimension is $k$. The basis $x_1(n), x_2(n), ..., x_k(n)$ spans all solutions of equation (2.5). Hence any solution $x(n)$ of equation (2.5) can be written in the form (2.9) equivalently (2.10) This is may we call $x(n)$ in (2.9) a general solution.*

Now, consider the nonhomogeneous system (2.5). We define a particular solution $y_p(n)$ of (2.5) as any $k-vector$ function that satisfies the non homogeneous difference system. The following result gives us a mechanism to find the general solution of system (2.5).

**Theorem 2.12.** [**18**] *Any solution $y(n)$ of (2.5) can be written as*

$$y(n) = \Phi(n)c + y_p(n),$$

*for an appropriate choice of the constant vector $c$, and a particular solution $y_p(n)$.*

*Proof.* Let $y(n)$ be a solution of (2.5) and $y_p(n)$ be a particular solution of (2.5). Let $x(n) = y(n) - y_p(n)$, then

$$x(n+1) = y(n+1) - y_p(n+1)$$

by applying (2.2)

$$\begin{aligned} x(n+1) &= A(n)y(n) - A(n)y_p(n) \\ &= A(n)[y(n) - y_p(n)] \end{aligned}$$

this implies that

$$= A(n)x(n).$$

So $x(n)$ is a solution of (2.5). Hence $x(n) = \Phi(n)c$ for some vector constant $c$. So

$$y(n) - y_p(n) = \Phi(n)c.$$

□

**Lemma 2.13.** **[1, 18]** *A particular solution of (2.5) may be given by*

$$y_p(n) = \sum_{r=n_o}^{n-1} \Phi(n, r+1)g(r)$$

*with* $y_p(n_o) = 0$

*Proof.*

$$\begin{aligned} y_p(n+1) &= \sum_{r=n_0}^{n} \Phi(n+1, r+1)g(r) \\ &= \sum_{r=n_0}^{n-1} A(n)\Phi(n, r+1) + \Phi(n+1, n+1)g(n) \\ &= A(n)y_p(n) + g(n) \end{aligned}$$

Hence, $y_p(n)$ is a solution of (2.2). Furthermore, $y_p(n_0) = 0$. □

**Theorem 2.14.** ***(Variation of Constant Formula)*****[1, 18]** *The unique solution of the initial value problem*

$$y(n+1) = A(n)y(n) + g(n), \qquad y(n_0) = y_0, \tag{2.11}$$

*is given by*

$$y(n, n_0, y_0) = \Phi(n, n_0)y_0 + \sum_{r=n_0}^{n-1} \Phi(n, n_0)\Phi(n_0, r+1)g(r),$$

$$= \Phi(n, n_0)y_0 + \sum_{r=n_0}^{n-1} \Phi(n, r+1)g(r), \tag{2.12}$$

it follows that $\Phi(n, n_0) = \prod_{i=n_0}^{n-1} A(i)$ where $\prod_{i=n_0}^{n_0-1} A(i) = I$. We can rewrite (2.12) in the form

$$y(n, n_0, y_0) = \left(\prod_{i=n_0}^{n-1} A(i)\right) y_0 + \sum_{r=n_0}^{n-1} \left(\prod_{i=r+1}^{n-1} A(i)\right) g(r), \tag{2.13}$$

in the case where $A$ is a constant matrix, $\Phi(n, n_0) = A^{n-n_0}$, and of course $\Phi(n, n_0) = \Phi(n - n_0, 0)$. The equation (2.13) reduces to

$$y(n, n_0, y_0) = A^{n-n_0}y_0 + \sum_{r=n_0}^{n-1} A^{n-r-1}g(r).$$

**Example 2.15.** *Solve the system $y(n+1) = Ay(n) + g(n)$, where*

$$A = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \quad g(n) = \begin{pmatrix} n \\ 2 \end{pmatrix}, \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

*Solution*

$$A^n = \begin{pmatrix} 4^n & n4^{n-1} \\ 0 & 4^n \end{pmatrix}.$$

*Hence,*

$$y(n) = \begin{pmatrix} 4^n & n4^{n-1} \\ 0 & 4^n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{r=0}^{n-1} \begin{pmatrix} 4^{n-r-1} & (n-r-1)4^{n-r-2} \\ 0 & 4^{n-r-1} \end{pmatrix} \begin{pmatrix} r \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4^n \\ 0 \end{pmatrix} + \sum_{r=0}^{n-1} \begin{pmatrix} r4^{n-r-1} + (n-r-1)4^{n-r-2} \\ 4^{n-r-1} \end{pmatrix}$$

$$= \begin{pmatrix} 4^n \\ 0 \end{pmatrix} + 4^n \begin{pmatrix} \frac{1}{4} \sum_{r=1}^{n-1} r \left(\frac{1}{4}\right)^r + \frac{n-1}{4} \sum_{r=1}^{n-1} \left(\frac{1}{4}\right)^r \\ \frac{1}{2} \sum_{r=0}^{n-1} \left(\frac{1}{4}\right)^r \end{pmatrix}$$

$$= \begin{pmatrix} 4^n \\ 0 \end{pmatrix} + 4^n \begin{pmatrix} \frac{1}{4} \left[1 - (\frac{1}{4})^n\right] - n(\frac{1}{4})^{n+2} + \frac{n-1}{2} \left[1 + (\frac{1}{4})^n\right] \\ 1 - (\frac{1}{4})^n \end{pmatrix}$$

$$= \begin{pmatrix} 4^n \\ 0 \end{pmatrix} + 4^n \begin{pmatrix} -\frac{n}{4} \left(\frac{1}{4}\right)^n + \frac{n}{4} + \frac{n}{4} \left(\frac{1}{4}\right) \\ 1 - (\frac{1}{4})^n \end{pmatrix}$$

$$= \begin{pmatrix} 4^n \\ 0 \end{pmatrix} + \begin{pmatrix} n4^{n-1} - \frac{3}{4}n \\ 4^{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} 4^n + n4^{n-1} + \frac{3}{4}n \\ 4^{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} 2^{2n} + n2^{2(n-1)} + \frac{3}{4}n \\ 2^{2(n-1)} \end{pmatrix}.$$

## 2.3 The Jordan form: Autonomous (time-invariant) systems revisited

In this section we will present the Jordan form of a matrix. The importance of Jordan form appears in both sides theoretical and computational aspects of autonomous systems.

### 2.3.1 Diagonalizable matrices

let $A$ and $B$ be two $m \times m$ matrices, $A$ is said to be similar to $B$ iff $P^{-1}AP = B$, for a nonsingular matrix $P$. For special case, when $B$ is a diagonal matrix, say $B = (\lambda_1, \lambda_2, ..., \lambda_m)$, then $A$ is said to be diagonalizable. That is

$$P^{-1}AP = B = diag[\lambda_1, \lambda_2, ..., \lambda_m],$$

and

$$A = P^{-1}BP,$$
$$A^k = (PDP^{-1})^k = PD^kP^{-1}.$$

Explicitly,

$$A^n = P \begin{bmatrix} \lambda_1^k & & 0 & \\ & \lambda_2^k & & \\ & & \ddots & \\ & 0 & & \lambda_m^k \end{bmatrix} P^{-1}.$$

Now, for the equation

$$x(n+1) = Ax(n), \tag{2.14}$$

the fundamental matrix $\Phi(n)$ is

$$\Phi(n) = A^k P = P \begin{bmatrix} \lambda_1^k & & 0 & \\ & \lambda_2^k & & \\ & & \ddots & \\ & 0 & & \lambda_m^k \end{bmatrix}, \tag{2.15}$$

using the condition $\Phi(0) = P$, it gives that

$$A^k = \Phi(k)\Phi^{-1}(0).$$

Now, formula (2.15) is useful only once the matrix $P$ is determined. Let $\xi_i$ be the eigenvector of $A$ corresponding to eigenvalues $\lambda$, $i = 1, \ldots, m$, then $P = (\xi_1, \xi_2, ..., \xi_k)$.

**Example 2.16.** *Find the general solution of $x(n+1) = Ax(n)$, where*

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & -1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

*Solution: The eigenvalues of $A$ can be obtained by solving the characteristic equation*

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} = 0$$

*This determinant produces $(\lambda+1)^2(\lambda-8) = 0$. Thus, $\lambda_1 = 8$, and $\lambda_2 = \lambda_3 = -1$. To find the corresponding eigenvectors, we solve the equation $(A - \lambda I)x = 0$. Hence, for $\lambda_1 = 8$, we have*

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

*Then $x_1 = x_3$ , $x_2 = \frac{1}{2}x_3$ choose $x_3$.*
*Solving this system gives us the first eigenvector*

$$\xi_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

*Now, for* $\lambda_2 = \lambda_3 = -1$*, we have*

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

*Consequently,* $x_1 + \frac{1}{2}x_2 + x_3 = 0$ *is the only equation obtained from this algebraic system. To solve the system, two of the three unknown terms* $x_1$*,* $x_2$*, and* $x_3$ *must be arbitrarily chosen. So, if we let* $x_1 = 0$ *and* $x_2 = 2$*, then* $x_3 = -1$*. Hence, we get eigenvector*

$$\xi_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

*On the other hand, if we let* $x_1 = 1$ *and* $x_2 = 0$*, then* $x_3 = -1$*. This gives the third eigenvector*

$$\xi_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

*Obviously, there are infinitely many choices for* $\xi_2$*,* $\xi_3$*. We see that the general solution is*

$$x(n) = c_1 8^n \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - c_2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} - c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

*or*

$$x(n) = \begin{pmatrix} 2c_1 8^n + c_3 \\ c_1 8^n + 2c_2 \\ 2c_1 8^n - c_2 - c_3 \end{pmatrix} \tag{2.16}$$

*Suppose that in the above problem we are given an initial value*

$$x(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

*and must find the solution* $x(n)$ *with this initial value. One way of doing this is by letting* $n = 0$ *in the solution given by formula (2.16) and evaluating the constants*

*$c_1$, $c_2$, and $c_3$. This yields*

$$2c_1 + c_3 = 0,$$

$$c_1 + 2c_2 = 1,$$

$$2c_1 - c_2 - c_3 = 0.$$

*Solving this system gives $c_1 = \frac{1}{9}$, $c_2 = \frac{4}{9}$, and $c_3 = -\frac{2}{9}$, leading us to the solution*

$$x_(n) \begin{pmatrix} \frac{2}{9}8^n - \frac{2}{9} \\ \\ \frac{1}{9}8^n + \frac{8}{9} \\ \\ \frac{2}{9}8^n - \frac{2}{3} \end{pmatrix}.$$

### 2.3.2 The Jordan form

Here we focus on the general case where the matrix $A$ is not diagonalizable. That is, the case where $A$ has repeated eigenvalues, and $\ell$-linearly independent eigenvectors can′t be obtained. For example, the following matrices are not diagonalizable:

$$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

If a $m \times m$ matrix $A$ is not diagonalizable, then it can be reduced to diagonal matrix, that is, $P^{-1}AP = D$, using Jordan form

$$D = diag(J_1, J_2, ..., J_r), \qquad 1 \le r \le m, \tag{2.17}$$

and

$$D_i = \begin{pmatrix} \lambda_i & & 0 & \dots & 0 \\ 0 & \lambda_i & & & 0 \\ 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & \lambda_i \end{pmatrix} \tag{2.18}$$

The matrix $D_i$ is called a Jordan block. The following theorem clarify these notes

**Theorem 2.17.** **[1, 20]***(The Jordan Canonical Form).*
*Any $m \times m$ matrix $A$ is similar to a Jordan form given by the formula (2.17), where each $D_i$ is a $\ell_i \times \ell_i$ matrix of the form (2.18), and $\sum_{i=1}^{r} \ell_i = m$.*

The number of linearly independent eigenvectors corresponding to $\lambda$ is called geometric multiplicity of $\lambda$. The number of times that $\lambda$ is repeated is called the algebraic multiplicity of $\lambda$ and is denoted by $a_\lambda$.

(i) If $a_\lambda = 1$, then $\lambda$ is not repeated and $\lambda$ is refered to as simple.

(ii) If $a_\lambda$ equals to geometric multiplicity of $\lambda$ then it is called semisimple.

For example, the matrix

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

has one simple eigenvalue 2, one semisimple eigenvalue 1, and one eigenvalue 4, which is neither simple nor semisimple.
We list below the possible Jordan forms of a $3 \times 3$ matrix with an eigenvalue $\lambda = 4$, of multiplicity 3. In the matrix, different Jordan blocks figured by squares.

$$\underbrace{\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}}_{(a)} \underbrace{\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}}_{(b)} \underbrace{\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}}_{(c)} \underbrace{\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}}_{(d)}$$

In $(d)$ the matrix is diagonalizable, and we have three Jordan blocks of order 1. Thus, $\ell_1 = \ell_2 = \ell_3 = 1$, $r = 3$, and the geometric multiplicity of $\lambda$ is 3.
In $(c)$ there are two Jordan blocks with $\ell_1 = 2$, $\ell_2 = 1$, $r = 2$, and the geometric multiplicity of $\lambda$ is 2.
In $(b)$ there are also two Jordan blocks with $\ell_1 = 1$, $\ell_2 = 2$, $r = 2$, and the geometric multiplicity of $\lambda$ is 2.
In $(a)$ there is only one Jordan block with $\ell_1 = 3$, $r = 1$, and the geometric multiplicity of $\lambda$ is 1. The linearly independent eigenvectors corresponding to $\lambda = 4$ in

$(d)$, $(c)$, $(b)$, $(a)$ are, respectively,

$$\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}}_{d}\quad\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}}_{c}\quad\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}}_{b}\quad\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{a}$$

Note that a matrix of the form

$$\begin{pmatrix}\lambda & & & \dots & 0\\ 0 & \lambda & & \dots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \dots & \\ 0 & 0 & 0 & \dots & \lambda\end{pmatrix}$$

has only one eigenvector, which is, the unit vector $e_1 = (1, 0, ..., 0)^T$. This verifies that the linearly independent eigenvectors of the Jordan form $J$ given by formula (2.17)

$$e_1, e_{\ell_1+1}, e_{\ell_1+\ell_2+1}, ..., e_{\ell_1+\ell_2+...+\ell_{r-1}} + 1.$$

Now, since $P^{-1}AP = D$, then

$$AP = PD. \tag{2.19}$$

Let $P = (\xi_1, \xi_2, ..., \xi_k)$. Equating the first $\ell_1$ columns of both sides in formula (2.19), we obtain

$$A\xi_1 = \lambda_1\xi_1, ..., A\xi_i = \lambda_1\xi_i + \xi_{i-1}, \quad i = 2, 3, ..., \ell_1. \tag{2.20}$$

It is clear that, $\xi_1$ is the only eigenvector of $A$ in the Jordan chain $\xi_1, \xi_2, ..., \xi_{\ell_1}$. The other vectors $\xi_2, \xi_3, ..., \xi_{\ell_1}$ are called generalized eigenvectors of $A$, and they obtained using the difference equation

$$(A - \lambda_1 I)\xi_i = \xi_{i-1}, \quad i = 2, 3, ..., \ell_1.$$

As the same as for the remainder of the Jordan blocks, it can be found the generalized eigenvectors corresponding to the $v^{th}$ Jordan block using the difference equation

$$(A - \lambda_v I)\xi_v = \xi_{v_{i-1}}, \quad i = 2, 3, ...., \ell_m. \tag{2.21}$$

Now, we know that $A^k = (PDP^{-1})^k = PD^kP^{-1}$, where

$$D^n = \begin{bmatrix}D_1^k & & & 0\\ & D_2^k & & \\ & & \ddots & \\ & 0 & & D_m^k\end{bmatrix}.$$

Notice that for any $D_i$, $i = 1, 2, ..., r$, we have $D_i = \lambda_i I + N_i$, where

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & & 1 \\ 0 & 0 & & \dots & 0 \end{pmatrix}$$

is an $\ell_i \times \ell_i$ nilpotent matrix (i.e., $N_i^r = 0$ for all $r \geq \ell_i$). Hence,

$$\begin{aligned} D_i^n &= (\lambda_i I + N_i)^k = \lambda_i^k I + \binom{n}{1}\lambda_i^{k-1} N_i + \binom{n}{2}\lambda_i^{k-2} N_i^2 + \dots + \binom{n}{\ell_i - 1}\lambda_i^{n-\ell_i+1} N_i^{\ell_i - 1} \\ &= \begin{pmatrix} \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \binom{n}{2}\lambda_i^{n-2} & \dots & \binom{n}{s_i-1}\lambda_i^{n-s_i+1} \\ 0 & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \dots & \binom{n}{s_i-2}\lambda_i^{n-s_i+2} \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & \binom{n}{1}\lambda_i^{n-1} \\ 0 & 0 & & \dots & \lambda_i^n \end{pmatrix} \end{aligned} \tag{2.22}$$

**Example 2.18.** *Find the general solution of $x(n+1) = Ax(n)$ with*

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 0 & 2 & -5 \\ 0 & 1 & 2 \end{pmatrix}.$$

*Solution: The eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 5$. To find the eigenvectors, we solve the equation $(A - \lambda I)\xi = 0$,*
*or*

$$\begin{pmatrix} 5 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

*Hence,*

$$5d_1 + d_2 + 2d_3 = 0,$$

$$-2d_2 - 5d_3 = 0,$$

$$d_2 + 2d_3 = 0,$$

*which generating two eigenvectors,*

$$\xi_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

*Now, to find one generalized eigenvector* $\xi_3$, *we apply the formula (2.20) and solve* $(A - 5I)\xi_3 = \xi_1$ :

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & -3 & -5 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

*gives a system that has no solution. The second attempt will b*

$$(A - 5I)\xi_3 = \xi_2,$$

*or*

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & -3 & -4 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$

*which result*

$$\xi_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

*Thus,*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -2 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

*and*

$$D^n = \begin{pmatrix} 5^n & 0 & 0 \\ 0 & 5^n & n5^{n-1} \\ 0 & 0 & 5^n \end{pmatrix}.$$

*Hence,*

$$x(n) = PD^n\hat{c} \begin{pmatrix} 0 & 5^n & n5^{n-1} \\ -2\cdot 5^n & -2\cdot 5^n & -2n5^{n-1}-5^n \\ -5^n & 5^n & n5^{n-1}+5^n \end{pmatrix} \begin{pmatrix} \hat{c_1} \\ \hat{c_2} \\ \hat{c_3} \end{pmatrix}.$$

*where $\hat{c}$ is a vector constant can be found if initial condition is given.*

### 2.3.3 Block-Diagonal Matrices

In general, the generalized eigenvectors corresponding to an eigenvalue $\lambda$ of algebraic multiplicity m are the solutions of the equation

$$(A-\lambda I)^m\xi = 0.$$

The first eigenvector $\xi_1$ corresponding to $\lambda$ is obtained by solving the equation

$$(A-\lambda I)\xi = 0.$$

The second eigenvector or generalized eigenvector $\xi_2$ is obtained by solving the equation

$$(A-\lambda I)^2\xi = 0.$$

And so on.
Now if $J$ is the Jordan form of $A$, that is, $P^{-1}AP = J$ or $A = PJP^{-1}$, then $\lambda$ is an eigenvalue of $A$ if and only if it is an eigenvalue of $J$. Moreover, if $\xi$ is an eigenvector of $A$, then $\tilde{\xi} = P^{-1}\xi$ is an eigenvector of $J$.
We would like to know the structure of the eigenvectors $\tilde{\xi}$ of $J$. For this we appeal to the following lemma from Linear Algebra.

**Lemma 2.19.** [1] *let* $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ *be a $k\times k$ block-diagonal matrix such that $A$ is an $r\times r$ matrix and $B$ is an $s\times s$ matrix, with $r+s=k$. Then the following statements hold true:*

- *i: if $\lambda$ is an eigenvalue of $A$, then it is an eigenvalue of $C$. Moreover, the eigenvector and the generalized eigenvectors corresponding to $\lambda$ are of the form $\xi = (a_1, a_2, \ldots, a_r, 0, \ldots, 0)^T$ for some $a_i \in \mathbb{R}$.*

- *ii: If $\lambda$ is an eigenvalue of $B$, then it is an eigenvalue of $C$. Moreover, the eigenvector and the generalized eigenvectors corresponding to $\lambda$ are of the form $\xi = (0, \ldots, 0, a_{r+1}, a_{r+2}, \ldots, a_s)$ for some $a_{r+i} \in \mathbb{R}$.*

*Proof.* i: Suppose that $\lambda$ is an eigenvalue of $A$, and $V = (a_1, a_2, \ldots, a_r)^T$ is the corresponding eigenvector. Define $\xi = (a_1, \ldots, a_r, 0, \ldots, 0) \in \mathbb{R}^k$. Then clearly $C\xi = \lambda\xi$, and thus $\lambda$ is an eigenvalue of $C$. Let the $k \times k$ identity matrix $I$ be written in the form $I = \begin{pmatrix} I_r & 0 \\ 0 & I_s \end{pmatrix}$, where $I_r$ and $I_s$ are, respectively, the $r \times r$ and $s \times s$ identity matrices. Let $\lambda$ be an eigenvalue of $A$ with algebraic multiplicity $m$. Then

$$(C - \lambda I)\xi = \begin{pmatrix} A - \lambda I_r & 0 \\ 0 & B - \lambda I_s \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_2 \\ \vdots \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence,

$$(A - \lambda I_r) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a nontrivial solution

$$\tilde{\xi} = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}.$$

However

$$(B - \lambda I_s) \begin{pmatrix} \xi_{r+1} \\ \vdots \\ \xi_s \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has only the trivial solution

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then $\xi = (a_1, \ldots, a_r, 0, \ldots, 0)^T$ is an eigenvector of $C$ corresponding to $\lambda$. The same analysis can be done for generalized eigenvectors by solving $(C - \lambda I)^i \xi = 0, \quad 1 \leq i \leq m$.

□

## 2.4 Linear periodic systems

The linear system

$$x(n+1) = A(n)x(n), \tag{2.23}$$

where for all $n \in \mathbb{Z}$, is called periodic if $A(\alpha + N) = A(\alpha)$, for some positive integer $N$. The periodic system (2.23) share basic properties of the corresponding autonomous system.

**Lemma 2.20.** **[10]** *Let $B$ be a $k \times k$ nonsingular matrix and let $m$ be any positive integer. Then there exists some $k \times k$ matrix $C$ such that $C^m = B$.*

*Proof.* Let

$$P^{-1}BP = \begin{pmatrix} j_1 & & & \\ & j_2 & & \\ & & \ddots & \\ & & & j_r \end{pmatrix}$$

be the Jordan form of $B$. Define

$$J_i = \lambda_i \left( I_i + \frac{1}{\lambda_i} N_i \right).$$

where $I_i$ is the $\ell_i \times \ell_i$ identity matrix and

$$N_i = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}.$$

$$N_i^2 = 0, \;\; N_i^3 = 0, \ldots, \;\; N_i^{\ell_i} = 0. \;\; i = 1, \ldots, r \tag{2.24}$$

Define

$$\begin{aligned} G_i &= \exp\left[\frac{1}{m} \ln J_i\right], \\ &= \exp\left(\frac{1}{m} \ln \left(\lambda_i I_i + \left(I_i + \frac{1}{\lambda_i} N_i\right)\right)\right) \\ &= \exp\left(\frac{1}{m}\left(\ln \lambda_i I_i + \ln\left(I_i + \frac{1}{\lambda_i} N_i\right)\right)\right) \end{aligned}$$

$$= \exp\left(\frac{1}{m}\left(\ln \lambda_i I_i + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+t}}{\ell}\left(\frac{N_i}{\lambda_i}\right)^{\ell}\right)\right) \tag{2.25}$$

Hence, $G_i$ is a well-defined matrix. As well as , $G_i^m = J_i$ □

Now, if we let

$$G = \begin{pmatrix} G_1 & & 0 & \\ & G_2 & & \\ & & \ddots & \\ 0 & & & G_r \end{pmatrix},$$

where $G_i$ is defined in formula (2.25) , then

$$G^m = \begin{bmatrix} G_1^m & & & 0 \\ & G_2^m & & \\ & & \ddots & \\ 0 & & & G_r^m \end{bmatrix} = J.$$

Finally if $E = PGP^{-1}$. Then $E^m = PG^mP^{-1} = PJP^{-1} = B$.

**Lemma 2.21.** [**1**] *The periodic system $x(n+1) = A(n)x(n)$ has two main properties*

(1) If $\Phi(n)$ is a fundamental matrix of the system, then $\Phi(n+N)$ is also fundamental.

(2) $\Phi(n+N) = \Phi(n)E$, for some nonsingular matrix $E$.

*Proof.* (1) Let $\Phi(n)$ be a fundamental matrix of the given system then $\Phi(n+1) = A(n)\Phi(n)$. Now

$$\Phi(n+N+1) = A(n+N)\Phi(n+N) = A(n)\Phi(n+N),$$

since the system is periodic. Hence $\Phi(n+N)$ is also a fundamental matrix of system.

(2) Suppose that $\psi_1(n, n_0) = \Phi(n+N)\Phi^{-1}(n_0+N)$ and $\psi_2(n, n_0) = \Phi(n)\Phi^{-1}(n_0)$ are two fundamental matrices of system (2.23) that has the same initial condition $\psi_1(n_0, n_0) = \psi_2(n_0, n_0) = I$. By the uniqueness of fundamental matrices $\psi_1(n, n_0) = \psi_2(n, n_0)$, we have

$$\Phi(n+N) = \Phi(n)\Phi^{-1}(n_0)\Phi(n_0+N) = \Phi(n)E,$$

for nonsingular matrix $E$.

□

The following theorem, is a consequences of the lemma

**Theorem 2.22.** **[18]** *For every fundamental matrix $\Phi(n)$ of system (2.24), there exists a nonsingular periodic matrix $P(n)$ of period $N$ such that*

$$\phi(n) = P(n)B^n. \tag{2.26}$$

*Proof.* By Lemma (2.21), there exists some matrix $B$ such that $B^N = E$, where $E$ is the matrix specified in Lemma (2.21)(2). Define $P(n) = \Phi(n)B^{-n}$, where $B^{-n} = (B^n)^{-1}$, then

$$\begin{aligned} P(n+N) &= \Phi(n+N)B^{-N}B^{-n} \\ &= \Phi(n)EB^{-N}B^{-n} \\ &= \Phi(n)B^{-n} \\ &= P(n). \end{aligned}$$

Since $P(n)$ has period $N$ and nonsingular. From the definition of $P(n)$ it thus follows that $\Phi(n) = P(n)B^n$. □

**Lemma 2.23.** **[1]** *If $\Phi(n)$ and $\psi(n)$ are two fundamental matrices of (2.24) such that*

$$\Phi(n+N) = \Phi(n)C,$$

$$\psi(n+N) = \psi(n)E,$$

*then $C$ and $E$ are similar (and thus they have the same eigenvalues).*

**Lemma 2.24.** **[10]** *A complex number $\lambda$ is a Floquet exponent of (2.24) if and only if there is a nontrivial solution of (2.24) of the form $\lambda^n q(n)$, where $q(n)$ is a vector function with $q(n+N) = q(n)$ for all $n$.*

*Proof.* First, we assume that $\lambda$ is a Floquet exponent of (2.24). Then,we also know that $\det(B^n - \lambda^n I) = 0$. Now choose $x_0 \in R^k$, $x_0 \neq 0$, such that $(B^n - \lambda^n I)x_0 = 0$ for all $n$. Hence, we have the equation $B^n x_0 = \lambda^n x_0$. Thus, $P(n)B^n x_0 = \lambda^n P(n)x_0$, where $P(n)$ is the periodic matrix defined in formula (2.26). By formula (2.26) now,

$$x(n, n_0, y_0) = \Phi(n, n_0)x_0 = P(n)B^n x_0 = \lambda^n P(n)x_0 = \lambda^n q(n),$$

and we have the desired periodic solution of (2.24), where $q(n) = P(n)x_o$. Conversely, if $\lambda^n q(n)$, $q(n+N) = q(n) \neq 0$ is a solution of (2.24), Theorem (2.22) then implies that

$$\lambda^n q(n) = P(n)B^n x_0 \tag{2.27}$$

for some nonzero vector $x_o$. This implies that

$$\lambda^{n+N} q(n) = P(n) B^{n+N} x_o. \tag{2.28}$$

But, from (2.27),

$$\lambda^{n+N} q(n) = \lambda^N P(n) B^n x_o. \tag{2.29}$$

Equating the right-hand sides of formulas (2.28) and (2.29), we obtain

$$P(n) B^n [B^N - \lambda^N I] x_o = 0,$$

and thus

$$\det [B^N - \lambda^N I] = 0.$$

This manipulation shows that $\lambda$ is a Floquet exponent of (2.24). □

**Corollary 2.25.** **[18]** *The following statements hold:*

*i) System (2.24) has a periodic solution of period $N$ if and only if it has a Floquet multiplier equal to 1.*

*ii) There is a Floquet multiplier equal to $-1$ if and only if system (2.24) has a periodic solution of period $2N$.*

**Remark 2.26.** *corollary (2.25), part (ii), gives us a formula to find the matrix $C = B^N$, whose eigenvalues happen to be the Floquet multipliers of (2.24).*

$$C = \Phi^{-1}(n)\Phi(n+N).$$

*By letting $n = 0$, we have*

$$C = \phi^{-1}(0)\Phi(N). \tag{2.30}$$

*If we take $\Phi(N) = A(N-1)A(N-2)...A(0)$, then $\Phi(0) = I$. Thus, formula (2.30) becomes*

$$C = \Phi(N),$$

*or*

$$C = A(N-1)A(N-2)...A(0). \tag{2.31}$$

*We now give an example to illustrate the above results.*

**Example 2.27.** *Consider the planar system*

$$x(n+1) = A(n)x(n),$$

$$A(n) = \begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{pmatrix}.$$

*Clearly,* $A(n+2) = A(n)$ *for all* $n \in \mathbb{Z}$*. Applying formula (2.30),*

$$B^2 = C = A(1)A(0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

*Thus the Floquet multipliers are* $-1, -1$*. By virtue of Corollary (2.25), the system has a 4-periodic solution. Note that since* $A(n)$ *has the constant eigenvalues* $-1, 1$*,* $\rho(A(n)) = 1$*.*

*Chapter 3*

# Stability Theory

## 3.1 Stability theory of first order difference equation

Consider the nonautonomous first order difference equation

$$x(n+1) = f(x(n)) \tag{3.1}$$

With $f : \mathbb{R} \to \mathbb{R}$, and $x^*$ is the equilibrium point.

**Definition 3.1. [20]**

*(1)* *stable if for all* $\varepsilon > 0$*, there exists* $\delta_\varepsilon > 0$ *such that* $\|x_0 - x^*\| < \delta_\varepsilon$ *implies* $\|f^n(x_0) - x^*\| < \varepsilon$ *all* $n > 0$*. Hence, if the system is initiated within a distance* $\delta_\varepsilon$ *from the equilibrium point, all subsequent values will remain within a distance* $\varepsilon$ *from the equilibrium point.*
*If* $x^*$ *is not stable, it is called unstable.*

*(2)* *attracting if there exists* $\eta > 0$ *such that* $\|x(0)-x^*\| < \eta$ *implies* $\lim_{n\to\infty} x(n) = x^*$*. If* $\eta = \infty$*,* $x^*$ *is called globally attracting. Hence, if the system is initiated within a distance* $\eta$ *from the equilibrium point.*

*(3)* *asymptotically stable or is an asymptotically stable equilibrium point if it is stable and attracting. If* $\eta = \infty$*,* $x^*$ *is called globally asymptotically stable.*

**Definition 3.2. [10]** *The equilibrium point* $x^*$ *is called the hyperbolic if* $| f'(x^*) | \neq 1$ *and non hyperbolic if* $| f'(x^*) | = 1$

**Theorem 3.3.** **[11, 19]** *$x^*$ is an equilibrium point of the nonlinear autonomous difference equation*

$$x(n+1) = f(x(n)).$$

*Where $f$ is continuously differentiable at $x^*$. the following statements then hold true:*

*(i) The $x^*$ is asymptotically stable, if $\mid f'(x^*) \mid < 1$.*

*(ii) The $x^*$ is unstable, if $\mid f'(x^*) \mid > 1$.*

*Proof.* (i) Suppose that $\mid f'(x^*) \mid < V$ for some $V < 1$. Then, because of the continuity of the derivative, there exists an interval $I = (x^* - \gamma, x^* + \gamma)$, $\gamma > 0$, such that $\mid f'(x^*) \mid < V < 1$ for all $x \in I$. For $x_0 \in I$,

$$|x_1 - x^*| = |f(x_0) - f(x^*)|.$$

The mean value theorem then implies that there exists $\xi$, $x_0 < \xi < x^*$, such that

$$|f(x_0) - f(x^*)| = |f'(\xi)||x_0 - x^*|.$$

Hence, we have

$$|x_1 - x^*| \leq M|x_0 - x^*|$$

This shows that $x_1$ is closer to $x^*$ than $x_0$ and is thus also in $I$ because $V < 1$. By induction we therefore conclude that

$$|x(n) - x^*| \leq M^n|x_0 - x^*|$$

For any $\varepsilon > 0$, let $\delta_\varepsilon = min(\gamma, \varepsilon)$ then $|x_0 - x^*| < \delta_\varepsilon|$ implies $|x(n) - x^*| < \varepsilon$ for all $t \geq 0$. $x^*$ is therefore a stable equilibrium point. In addition, $x^*$ is attractive because $\lim_{n\to\infty} |X(n) - X^*| = 0$. Thus, $X^*$ is asymptotically stable.

(ii) Assume that $\mid f'(x^*) \mid > 1$, Choose $\lambda > 1$ and $I = (x^* - \epsilon, x^* + \epsilon)$ for some $\epsilon > 0$ so that

$$|f(x) - f(w)| = |f'(c)||x - w| \geq \lambda|x - w|$$

for all $x, w$ in $I$. By induction,

$$|f^n(x) - x^*| \geq \lambda^n|x - x^*|$$

as long as $f^n(x)$ is in $I$. Since $\lambda > 1$, it follows that all solutions of Eq.(3.1) that originate in $I$, except for the constant solution $x(n) = x^*$, must leave $I$ for sufficiently large $n$. Then $x^*$ is unstable.

□

For non hyperbolic case where $| f'(x^*) |= 1$, we have two cases

(1) If $f'(x^*) = 1$

(2) If $f'(x^*) = -1$.

For the second case $f'(x^*) = -1$, then the schwarzian derivative of a function $f$.

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$$

Because $f'(x^*) = -1$, then

$$Sf(x^*) = -f'''(x^*) - \frac{3}{2}(f''(x^*))^2$$

**Theorem 3.4.** **[1]** *Suppose that for an equilibrium point $x^*$ of eq.(3.1), $f'(x^*) = 1$. The following statements then hold:*

*(1) The $x^*$ is unstable, if $f''(x^*) \neq 0$.*

*(2) The $x^*$ is asymptotically stable, if $f'''(x^*) < 0$ and $f''(x^*) = 0$.*

*(3) The $x^*$ is unstable, if $f'''(x^*) > 0$ and $f''(x^*) = 0$.*

**Theorem 3.5.** **[11]** *Suppose that for the equilibrium point $x^*$ of (3.1) $f'(x^*) = -1$ The following statements then hold:*

*(1) The $x^*$ is asymptotically stable, if $Sf(x^*) < 0$.*

*(2) The $x^*$ is unstable, if $Sf(x^*) > 0$.*

**Example 3.6.** *Consider the difference equation*

$$x(n+1) = 3x^3(n) - 2x(n)$$

The equilibrium point are $0, 1$ and $-1$ $f'(x) = 9x^2 - 2$
$f'(0) = -2$, so theorem (3.5) applies $f''(x) = 18x$, $f'''(x) = 18$

$$Sf(x^*) = -18 - \frac{3}{2}(18x)^2$$

$Sf(0) = -18 < 0$, then $x^*$ is asymptotically stable, it follows from theorem (3.3) that 1 is unstable , so $-1$ is unstable by theorem (3.3).

**Example 3.7.** *Consider the difference equation*

$$x(n+1) = \tan^{-1} x(n)$$

The equilibrium point is 0, $f'(0) = 1$ so theorem (3.4) applies $f''(0) = 0$, and $f'''(0) = -2$ , then 0 is asymptotically stable.

**Definition 3.8.** [**1**] *Consider $f : \mathbb{R} \to \mathbb{R}$ be continuous map, $f(x^*) = x^*$ $x^*$ is called the semi-asymptotically stable from the left (resp.right)if it is both,*

*(1) stable from the left (resp.right), i.e for every $\varepsilon > 0$, there exist $\delta$ such that $x \in (x^* - \delta, x^*)$, (resp.$(x^*, x^* + \delta)$) implies that $| f^n(x_o) - x^* | < \varepsilon$.*

*(2) attracting from the left (resp.right) i.e there exist $\eta > 0$ such that $x \in (x^* - \eta, x^*)$, (resp.$(x^*, x^* + \eta)$) implies that $\lim_{n\to\infty} f^n(x_o) = x^*$.*

If we divide the plane into regions $A_i$,$B_i$, $1 \le i \le 4$
$A_1 = \{(x, y) : y \ge x, x \ge x^*\}$
$A_2 = \{(x, y) : x^* \le y < x\}$
$A_3 = \{(x, y) : -x + 2x^* \le y \le x^*\}$
$A_4 = \{(x, y) : y \le -x + 2x^*, x \ge x^*\}$
$B_1 = \{(x, y) : -x + 2x^* \le y \le x^*\}$
$B_2 = \{(x, y) : x^* \le y < -x + 2x^*\}$
$B_3 = \{(x, y) : x \le y < x^*\}$
$B_4 = \{(x, y) : y \le x \le x^*\}$
Let $f : R \to R$ and $x^*$ be the only fixed point of $f$. write if as

$$f(x) = \begin{cases} \psi(x), & \text{if } x \le x^*, \\ \varphi(x), & \text{if } x \ge x^*, \end{cases}$$

Where $\psi(x^*) = \varphi(x^*) = x^*$ and both $\psi(x)$ and $\varphi(x)$ are continuous maps. The lines $y_1 = x_1$,$y_2 = 2x^* - x$, $y_3 = x^*$, and, $x = x^*$ divide the plane into the above -mentioned eight regions.

**Definition 3.9.** [**1, 10**] *If the set $\{(x, \psi(x) : x \le x^*\} \subset B_i$, for some $i$, then we say that $\psi(x)$ stays in $B_i$, on the other hand, if the set $\{(x, \varphi(x)) : x \ge \bar{x}\} \subset A_i$, for some $i$, then we say that $\varphi(x)$ stays in $A_i$. The following proposition regarding the behavior of the map $f$ in the regions $A_i$ and $B_i$, $1 \le i \le 4$.*

**Proposition 3.10.** **[11]** *if $\varphi(x)$ stays in $A_i$ for all $x \in R$, then*

*(1) If $\psi(x)$ stays in one of the regions $B_1, B_2$, or $B_3$, then $x^*$ is globally asymptotically stable.*

*(2) If $\psi(x)$ stays in $B_4$, then $x^*$ is semi-asymptotically stable from the right.*

**Proposition 3.11.** **[11]** *The following statements hold true.*

*(1) if $\varphi(x)$ stays in $A_3$ for all $x \in R$, then $x^*$ is globally asymptotically stable if $\psi(x)$ stays in $B_3$, it is unstable if $\psi(x)$ stays in $B_4$.*

*(2) if $\varphi(x)$ stays in $A_4$ for all $x \in R$, then $x^*$ is unstable if $\psi(x)$ stays in $B_3$, it is globally asymptotically stable if $\psi(x)$ stays in $B_4$.*

*(3) if $\varphi(x)$ stays in $A_1$ for all $x \in R$, then $x^*$ is unstable if $\psi(x)$ stays in one of the regions $B_1, B_2$ or $B_4$ it is semi-asymptotically stable from the left if $\psi(x)$ stays in $B_3$.*

**Theorem 3.12.** **[11]** *Let $x^*$ be a fixed point of $f$, then the following statement hold true*

*(1) suppose that $f \in C^{2k}$, if $f''(x^*) = \ldots = f^{(2k-1)}(x^*) = 0$ and $f'(x^*) = 1$, but $f^{2k}(x^*) \neq 0$ then $\bar{x}$ is semi-asymptotically stable*

  *(a) from the left if $f^{2k}(x^*) > 0$*

  *(b) from the right if $f^{2k}(x^*) < 0$*

*(2) suppose that $f \in C^{2k+1}$, if $f''(x^*) = \ldots = f^{2k}(x^*) = 0$ and $f'(x^*) = 1$, but $f^{(2k+1)}(x^*) \neq 0$ then*

*(a) $x^*$ is (locally) asymptotically stable if $f^{(2k+1)} < 0$*

*(b) $x^*$ is unstable if $f^{(2k+1)} > 0$*

*Proof.* (1) Let $q(x) = f(x) - x$ and let $\delta$ be a sufficiently small number. then, by Taylor Theorem.

$$q(x^* + \delta) = q(x^*) + q'(x^*)\delta + \ldots + \frac{q^{(2k-1)}(x^*)}{(2k-1)!}\delta^{(2k-1)} + \frac{q^{(2k)}(\xi)}{(2k)!}\delta^{2k}$$

Holds for some $x^* < \xi < x^* + \delta$, and hence

$$q(x^* + \delta) = \frac{f^{2k}(\xi)}{(2k)!}\delta^{2k}, \quad x^* < \xi < x^* + \delta \tag{3.2}$$

It is clear from equation (3.2) that whenever $f^{2k}(x^*) > 0$ and $f^{2k}(x^*) < 0$, it follows that $q(x) > 0$ and $q(x) < 0$ in $(x^* - \delta, x^* + \delta)$ for an sufficiently small $\delta$. Applying proposition (3.10) and (3.11) now yeilds the desired result.

(2) Under the given assumption in part 1, and by using Taylors theorem, we obtain

$$q(x^* + \delta) = \frac{f^{2k+1}(\xi)}{(2n+1)!}\delta^{2n+1}, \quad x^* < \xi < x^* + \delta \tag{3.3}$$

Now, from equation (3.3) if $f^{2k+1}(x^*) > 0$, then $q(x) > 0$ in $(x^*, x^* + \delta)$ and $q(x) < 0$ in $(x^* - \delta, x^*)$, for a sufficiently small $\delta$, this implies, by proposition (3.11), that $x^*$ is unstable. on the other hand, if $f^{2k+1}(x^*) < 0$, then $q(x) < 0$ in $(x^*, x^* + \delta)$ and $q(x) > 0$ in $(x^* - \delta, x^*)$, for a sufficiently small $\delta$, this implies, by proposition (3.10) that $x^*$ is asymptotically stable. □

**Example 3.13.** *Consider the map $f(x) = x + x^4$*

$x^* = 0$ is a fixed point of $f$
$f'(x^*) = 1$, $f''(x^*) = f'''(x^*) = 0$, but $f^{(4)}(x^*) = 24 > 0$ By theorem (3.12) $x^*$ is semi-asymptotically stable from the left.
Note that: If $x^*$ is a equilibrium point of $f$, it must be a equilibrium point of $h$ and $h'(x^*) = 1$.

**Theorem 3.14.** **[12]** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $f(x^*) = \bar{x}$ and $f'(x^*) = -1$, set $h(x) = f(f(x))$. Then $x^*$ is classified under $f$ in the same way as under $h$.*

Note that $| f^n(x^*) | < 1$ if and only if $| h^n(x^*) | < 1$.
Hence, $x^*$ is asymptotically stable ( unstable) under $f$ if and only if it is asymptotically stable ( unstable) under $h$.

**Theorem 3.15.** **[3, 11]** *Assume that $f \in C^{(2k+1)}$ and $x^*$ is a fixed point of $f$ such that $f'(x^*) = -1$,*
*If $h^{(2k+1)}(x^*) \neq 0$ and $h''(x^*) = \ldots = h^{(2k)}(x^*) = 0$, then*

*(1) If $g^{(2k+1)}(x^*) < 0$, then $x^*$ is a asymptotically stable.*

*(2) If $g^{(2k+1)}(x^*) > 0$, then $x^*$ is unstable.*

**Theorem 3.16.** **[11]** *Let $f$ be analytic with $f'(x^*) = -1$, then for some $m > 1$,*

*(1) if $h''(x^*) = \ldots = h^{(2m-1)}(x^*) = 0$ then $h^{(2m)}(x^*) = 0$.*

*(2) $x^*$ cannot be semi-asymptotically stable under $h$.*

*Proof.* Suppose: $f'(x^*) = -1$, by Taylor theorem, we have some small $\delta$ with

$$f(x^* + \delta) = f(x^*) + \delta f'(x^*) + \frac{\delta^2 f''(x^*)}{2!} + \ldots + x^* - \delta + 0(\delta^2)$$

So, for $x_0 = x^* + \delta > x^*$, $f(x_o) < x^*$, and
for $x_0 = x^* - \delta < x^*$, $f(x_o) > x^*$, in other words, for $x_0 \in (x^* - \delta, x^* + \delta)$ either
$f^{(2m)}(x^*) \in (x^*, x^* + \delta)$ and $f^{(2m+1)}(x^*) \in (x^* - \delta, x^*)$ for all $m \in \mathbb{Z}^+$ or
$f^{(2m)}(x^*) \in (x^* - \delta, x^*)$ and $f^{(2m+1)}(x^*) \in (x^*, x^* + \delta)$ for all $m \in \mathbb{Z}^+$.
Now, if $f^{(2m)}(x_o) \to x^*$ as $k \to \infty$ then $f^{(2m+1)}(x_0) \to x^*$ as $m \to \infty$
Hence, either $x^*$ is unstable or $x^*$ is asymptotically stable. More importantly, it cannot be semi-asymptotically stable. □

**Example 3.17.** *Consider the map $f(x) = -2 + 2x^2 - 4x^3$*
$f'(0) = 0$, $f''(0) = 4$,$f'''(0) = -24$
$Sf(0) = -f'''(0) - \frac{3}{2}(f''(0))^2 = 0$, *so this test falls*
*We will use*
$h(x) = f(f(x)) = 38 - 112x^2 + 224x^3 + 104x^4 - 416x^5 + 384x^6 + 128x^7 - 128^8 + 256x^9$
$h'(0) = 0$, $h''(0) = -224$, $h^{(''')}(0) = 1344 > 0$, *by theorem (3.15) then* $0$ *is unstable.*

But this algorithm requires a difficult computation
Hence, we will use theorem (3.15) and formula which published by $(FAA'DIBRUNO)$
This formula depend on the derivatives of $f(x)$ only.

**Theorem 3.18.** **[12]** $(FAA'DIBRUNO)$. *Let $f \in C^{(n)}$, then*

$\frac{d^n}{dx^n}h(x) = \frac{d^n}{dx^n}f(f(x)) = \sum \frac{n!}{b_1!b_2!...b_n!} f^{(b)}(f(x)) \left(\frac{f'(x)}{1!}\right)^{b_1} \ldots \left(\frac{f^{(n)}(x)}{n!}\right)^{b_n}$.
*Where $b = b_1 + b_2 + \ldots + b_n$ and the sum extends over all possible integer $b_i$ such that $0 \leq b_i \leq n$ and $n = b_1 + 2b_2 + \ldots + nb_n$.*

If we are evaluating it at equilibrium point $x^*$ with $h(x) = f(f(x))$ and $f'x^*) = -1$

$$h^{(n)}(x^*) = \sum \frac{(-1)^{h_1} n! f^{(b)}(x^*)}{b_1!b_2!...b_n!} \left(\frac{f^{(2)}(x^*)}{2!}\right)^{b_2} ... \left(\frac{f^n(x^*)}{n!}\right)^{b_n} \tag{3.4}$$

For example, if $n = 3$, there are three terms
$\{b_1 = 3, b_2 = b_3 = 0\}$,$\{b_1 = b_2 = 1, b_3 = 0\}$ and $\{b_1 = b_2 = 0, b_3 = 1\}$

$$h^3(x^*) = \left((-1)^3\frac{3!}{3!}\right) f^3(x^*) + \left((-1)^1\frac{3!}{1!1!}\right) f''(x^*)\left(\frac{f''(x^*)}{2!}\right) + \left((-1)^0\frac{3!}{1!}\right) f'(\bar{x})\frac{f^3(\bar{x})}{3!}$$

$$= f^3(x^*) - 3(f''(x^*))^2 - f^3(x^*) = 2Sf(x)$$

There is generalized analogs of the schwarzian derivative to use in our classification by using formula (3.4)
Take $S_m f(x) = \frac{1}{2}h^{(2m+1)}(x)$, and simplify using assumption that $S_i f(x) = 0$ for all $i < m$:
$S_1 f(x) = -f^{(3)}(x) - \frac{3}{2}(f''(x))^2$
$S_2 f(x) = -f^{(5)}(x) - \frac{15}{2}f''(x)f^{(4)}(x) + 15(f''(x))^{(4)}$
$S_3 f(x) = -f^{(7)}(x) - 14f''(x)f^{(6)}(x) + \frac{945}{2}(f''(x))^{(3)}f^{(4)}(x) - \frac{35}{4}(f^{(4)}(x))^2 - \frac{9045}{4}(f''(x))^6$.

**Theorem 3.19.** **[13]** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $f'(x^*) = -1$ and $f(x^*) = x^*$ let $m \geq 1$ be minimal such that $S_m f(x^*) = B \neq 0$, then $x^*$ is classified as follows:*

*(1) The $x^*$ is unstable , if $B > 0$.*

*(2) The $x^*$ is stable, if $B < 0$.*

**Example 3.20.** *Consider $f(x) = -x + x^2 - x^3 + \frac{2}{3}x^4$*
*$f'(0) = -1$, $f''(0) = 2$,$f^3(0) = -6$, and $f^4(0) = 16$*
*$S_1 f(0) = S_2 f(0) = 0$ and $S_3 f(0) = -86480 < 1$*
*Hence, 0 is stable fixed point*

## 3.2 Explicit criteria for stability of two-dimensional systems

In serval applications explicit criteria are needed on the entries of the matrix for the eigenvalues to lie inside the unit disk. To illustrate this, the following matrix should be taken into account

$$A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

The equation of the matrix is given by

$$\lambda^2 - (b_{11} + b_{22})\lambda + (b_{11}b_{22} - b_{12}b_{21}) = 0,$$

or

$$\lambda^2 - (trA)\lambda + \det A = 0. \tag{3.5}$$

where $p_1 = -trA$, $p_2 = \det A$, it is concluded that the eigenvalues of $A$ lie inside the unit disk if and only if

$$1 + trA + \det A > 0, \quad 1 - trA + \det A > 0, \quad 1 - \det A > 0$$

or, equivalently,

$$|trA| < 1 + \det A < 2. \tag{3.6}$$

Under condition (3.6), the zero solution of the equation

$$x(n+1) = Ax(n)$$

is asymptotically stable. The situation can be described when some eigenvalues of $A$ in (3.5) are inside the unit disk and some eigenvalues are not inside the unit disk. The result shown below is called the (Stable Subspace (Manifold)) Theorem. The result does not need that $A$ is invertible.

Let $\lambda$ be an eigenvalue of $A$ of multiplicity $k$ and let $\xi_1, \xi_2, \ldots, \xi_k$ be the generalized eigenvectors corresponding to $\lambda$. Then for each $i$, $1 \leq i \leq k$, either

$$A\xi_i = \lambda\xi_i$$

($\xi_i$ is an eigenvector of $A$), or

$$A\xi_i = \lambda\xi_i + \xi_{i-1}.$$

The general eigenvectors conforming to $\lambda$ are the answers of the equation

$$(A - \lambda J)^k\xi = 0.$$

All linear combinations, or the period of the generalized eigenvectors matching to $\lambda$ is invariant under $A$ and is called the generalized eigenspace $E_\lambda$ of the eigenvalue of $A$. Clearly, if $\lambda_1 \neq \lambda_2$, then $E_{(\lambda_1)} \bigcap E_{(\lambda_2)} = \{0\}$. Each eigenspace $E_\lambda$ includes the zero vector.

Suppose that $A$ is hyperbolic, that is, none of the eigenvalues of $A$ lie on the unit circle. Arrange the eigenvalues of $A$ such that $\Delta_z = \{\lambda_1, \lambda_2, ..., \lambda_r\}$ are all the eigenvalues of $A$ with $\mid \lambda_i \mid < 1, \;\; 1 \leq i \leq r$ and $\Delta_v = \{\lambda_{r+1}, \lambda_{r+z}, ..., \lambda_m\}$ are all the eigenvalues of $A$ with $\mid \lambda_i \mid > 1, \;\; r+1 \leq i \leq m$. The eigenspace spanned by the eigenvalues in $\Delta_z$ is denoted by $W^z$, where $W^z = \bigcup_{i=1}^{r} \lambda_i$ and the eigenspace spanned by the eigenvalues in $\Delta_v$ is denoted by $W^v$, where $W^v = \bigcup_{i=1}^{r} \lambda_i$.

**Theorem 3.21.** *(The Stable Subspace (Manifold) Theorem)*[**20**]
*If $A$ is hyperbolic, then the following statements should be true:*

*i) If $x(n)$ is a solution of (2.5) with $x(0) \in W^z$, then for each $n$, $x(n) \in W^z$. Furthermore,*

$$\lim_{n\to\infty} x(n) = 0.$$

*ii) If $x(n)$ is a solution of (2.5) with $x(0) \in W^v$, then $x(n) \in W^v$. for each $n$. Moreover,*

$$\lim_{n\to\infty} x(n) = 0.$$

*Proof.* i) Let $x(n)$ be a solution of (2.5) with $x(0) \in W^z$. Since $AE_\lambda = E_\lambda$, it follows that $AW^z = W^z$. Hence $x(n) \in W^z$ for all $n \in Z^+$. To prove the second statement, observe that $x(0) = \sum_{i=1}^{r} c_i \xi_i$, where $1 \leq \xi_i \leq r$ are the generalized eigenvectors corresponding to elements in $\Delta_z$. Let $J = P^{-1}AP$ be the Jordan form of $A$. Then $J$ may be written in the form

$$J = \begin{pmatrix} J_z & 0 \\ 0 & J_v \end{pmatrix}$$

where $J_z$ has the eigenvalues in $\Delta_z$ and $J_v$ has the eigenvalues in $\Delta_v$, the corresponding generalized eigenvectors $\tilde{\xi}_i, 1 \leq i \leq r$, of $J_z$ are of the form $\tilde{\xi}_i = P^{-1}\xi_i = (b_(i_1), b_(i_2), ..., b_(i_r), 0, 0, ..., 0)^T$. Now

$$\begin{aligned} x(n) &= A^n x(0) \\ &= PJ^nP^{-1}\sum_{i=1}^{r} c_i\xi_i \\ &= PJ^n\sum_{i=1}^{r} c_i\xi_i \\ &= P\sum_{i=1}^{r} c_i \end{aligned}$$

$$\begin{pmatrix} J_z^n & \tilde{\xi}_i \\ 0 & 0 \end{pmatrix}.$$

Thus $\lim_{n\to\infty} x(n) = 0$ since $J_z^n \to 0$ as $n \to \infty$. □

We now use the stability of the periodic system

$$x(n+1) = A(n)x(n), \quad A(n+N) = A(n). \tag{3.7}$$

Recall from (Chapter2) that if $\Phi(n, n_0)$ is a fundamental matrix of (3.7), then there exist a constant matrix $D$ whose eigenvalues are called the Floquet exponents and a periodic matrix $P(n, n_o)$ such that $\Phi(n, n_o) = P(n, n_o)D^{n-n_o}$, where $P(n+N, n_0) = P(n, n_0)$. Thus if $D^n$ is bounded, then so is $\Phi(n, n_0)$, and if $D^n \to 0$ as $n \to \infty$, then it follows that $\Phi(n, n_0) \to 0$ as $n \to \infty$.

**Theorem 3.22. [20]**
*The zero solution of (3.7) is:*

*i) The Floquet exponents have modulus less than or equal to* 1 *if and only if it is stable ; those of modulus of* 1 *are semisimple.*

*ii) All the Floquet exponents lie inside the unit disk if and only if it is asymptotically stable .*

For practical purposes, the following corollary is of paramount importance.

**Corollary 3.23. [10, 20]** *The zero solution of (3.7) is:*

*i) Each eigenvalue of the matrix* $C = A(N-1)A(N-2)...A(0)$ *has modulus less than or equal to* 1 *if and only if it is stable ; those solutions with modulus of value* 1 *are semisimple;*

*ii) each eigenvalue of* $C = A(N-1)A(N-2)...A(0)$ *has modulus less than* 1, *if and only if it is asymptotically stable.*

To summarize what we have discussed so far, first, for the autonomous (time-invariant) linear system $x(n+1) = Ax(n)$, the eigenvalues of $A$. But for a periodic system $x(n+1) = A(n)x(n)$, the eigenvalues of $A(n)$ do not play any role in the determination of the stability properties of the system.
However, the Floquet multipliers of $A(n)$ determine those properties. The following

example should dismiss any wrong ideas regarding the role of eigenvalues in a non-autonomous system.

**Example 3.24.** *Let consider the periodic system where*

$$A(n) = \frac{1}{8}\begin{pmatrix} 0 & 9+(-1)^n.7 \\ 9-(-1)^n.7 & 0 \end{pmatrix}.$$

*Here the eigenvalues of $A$ are $\pm 2^{-\frac{1}{2}}$, and thus $\rho[A(n)] < 1$. By applying Corollary (3.23), one may quickly check the stability of this system. We have*

$$C = A(1)A(0) = \frac{1}{8}\begin{pmatrix} 0 & 2 \\ 16 & 0 \end{pmatrix}\begin{pmatrix} 0 & 16 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 32 \end{pmatrix}$$

*With $n_0 = 0$, the fundamental matrix is*

$$\phi(n,0) = \begin{pmatrix} 2^{-2n} & 0 \\ 0 & 2^n \end{pmatrix}$$

*if $n$ is even and*

$$\phi(n,0) = \begin{pmatrix} 0 & 2^n \\ 2^{-2n} & 0 \end{pmatrix}$$

*if $n$ is odd. In any case, this is a solution that will grow exponentially away from the origin.*

## 3.3 Stability of nonlinear systems of difference equation

Consider the nonlinear systems of difference equations

$$y(n+1) = A(n)y(n) + g(n, y(n)) \tag{3.8}$$

The linear component is

$$z(n+1) = A(n)Z(n) \tag{3.9}$$

Where $A(n)$ is $m \times m$ matrix for all $n \in \mathbb{Z}^+$
$g : \mathbb{Z}^+ \times G \to \mathbb{R}^m$, $G \subset \mathbb{R}^m$, is a continuous function.

System (3.8) arise from the linearization method which applied to the following system.

$$x(n+1) = h(n, x(n)) \tag{3.10}$$

Where $g : \mathbb{Z}^{+} \times G \to \mathbb{R}^{k}$, $G \subset \mathbb{R}^{m}$, is continuously differentiable at an equilibrium point $y^{*}$
(i.e $\frac{\partial h}{\partial y_i} \mid y^{*}$ exists and is continuous on an open neighbored of $y^{*}$ for $1 \leq i \leq m$.)
Let us write $h = (h_1, h_2, \ldots, h_m)^{T}$

$$\frac{\partial h(n,y)}{\partial y}\Big|_{y=0} = \frac{\partial h(n,0)}{\partial y} \begin{bmatrix} \frac{\partial h_1(n,0)}{\partial y_1} & \frac{\partial h_1(n,0)}{\partial y_2} & \cdots & \frac{\partial h_1(n,0)}{\partial y_m} \\ \frac{\partial h_2(n,0)}{\partial y_1} & \frac{\partial h_2(n,0)}{\partial y_2} & \cdots & \frac{\partial h_2(n,y)}{\partial y_m} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial_m(n,0)}{\partial y_1} & \frac{\partial h_m(n,0)}{\partial y_2} & \cdots & \frac{\partial h_m(n,0)}{\partial y_m} \end{bmatrix}$$

Let $\frac{\partial h(n,x^{*})}{\partial x}$ is denoted by $Dh(n, x^{*})$, and

$$y(n) = x(n) - x^{*}$$

By substituting in (3.10), we have

$$\begin{aligned} y(n+1) &= h(n, y(n) + x^{*}) - x^{*} \\ &= \frac{\partial h}{\partial x}(n, x^{*})y(n) + g(n, y(n)) \end{aligned}$$

Where $g(n, y(n)) = h(n, y(n) + x^{*}) - x^{*} - \frac{\partial h}{\partial x}(n, x^{*})y(n)$
Let $A(n) = \frac{\partial h}{\partial x}(n, x^{*})$, we obtain (3.8)
From the assumptions on $h$ we conclude that $g(n, y) = 0(\parallel y \parallel)$ as $\parallel y \parallel \to 0$ this means. given $\varepsilon > 0$, there exist $\delta > 0$, such that $\parallel g(n, y) \parallel \leq \varepsilon \parallel y \parallel$ whenever $\parallel y \parallel < \delta$, for all $n \in \mathbb{Z}^{+}$. When $x^{*} = 0$, we have:

$$\begin{aligned} g(n, y(n)) &= h(n, y(n)) - Dh(n, 0)y(n) \\ &= h(n, y(n)) - A(n)y(n) \end{aligned}$$

Special case of system (3.10) is the autonomous system

$$y(n+1) = h(y(n)) \tag{3.11}$$

Which can be written as

$$y(n+1) = Ay(n) + g(y(n)) \tag{3.12}$$

Where $A = h'(0)$. is the jacobian matrix $h$ at 0, and $g(y) = h(y) - Ay$
Because $h$ is differentiable at 0, it follows that $g(y) = o(y)$ as $\| y \| \to 0$,
Equivalently

$$\lim_{\|y\|\to 0} \frac{\| g(y) \|}{\| y \|} = 0$$

**Lemma 3.25.** *( Discrete Gronwall Inquality)*[**18, 1**] *Let $z(n)$ and $h(n)$ be two sequences of real numbers,*
*$n \geq n_0 \geq 0$ and $h(n) \geq 0$, if $z(n) \leq M \left[z(n_0) + \sum_{j=n_0}^{n-1} h(j)z(j)\right]$ for some $M > 0$, then*

$$z(n) \leq z(n_0) \prod_{j=n_0}^{n-1} [1 + Mh(j)], \quad n \geq n_0 \tag{3.13}$$

$$z(n) \leq z(n_0) \exp\left[\sum_{j=n_0}^{n-1} Mh(j)\right], \quad n \geq n_0 \tag{3.14}$$

**Theorem 3.26.** [**1**] *Suppose that $g(n, y) = o(\| y \|)$ uniformly as $\| y \| \to 0$. If the zero solution of the linear system (3.9) is uniformly asymptotically stable, then the zero solution of the nonlinear system(3.8) is exponentially stable.*

*Proof.* we have $\| \phi(n, m) \| \leq M\eta^{n-m}$, $n \geq m \geq n_0$ , for some $M \geq 1$ and $\eta \in (0, 1)$.
By variation of constant formula is given by

$$y(n, n_0, y_0) = \Phi(n, n_0)y_0 + \sum_{j=n_0}^{n-1} \Phi(n, j+1)g(j, y(j)).$$

Thus

$$\| y(n) \| \leq M\eta^{n-n_0} \| y_0 \| + M\eta^{-1} \sum_{j=n_0}^{n-1} \eta^{(n-j)} \| g(j, y(j)) \| \tag{3.15}$$

For a given $\varepsilon > 0$, there is $\delta > 0$ such that $\| g(j, y) \| < \varepsilon \| y \|$ when ever $\| y \| < \delta$, so as long as $\| y(j) \| < \delta$, (3.15) becomes

$$\eta^{-n} \| y(n) \| \leq M \left[\eta^{-n_0} \| y_0 \| + j + \sum_{j=n_o}^{n-1} \varepsilon\eta^{-j-1} \| y(j) \|\right] \tag{3.16}$$

Letting $z(n) = \eta^{-n} \parallel y(n) \parallel$ and then applying the Gronwall inequality (3.13) one obtains

$$\eta^{-n} \parallel y(n) \parallel \leq \eta^{-n_0} \parallel y_0 \parallel \prod_{j=n_0}^{n-1} [1 + \varepsilon \eta^{-1} M]$$

Thus

$$\parallel y(n) \parallel \leq \parallel y_0 \parallel (\eta + \varepsilon M)^{(n-n_0)} \tag{3.17}$$

Choose $\varepsilon < \frac{(1-\eta)}{M}$. Then $\eta + \varepsilon M < 1$, thus $\parallel y(n) \parallel \leq \parallel y_0 \parallel < \delta$ for all $n \geq n_0 \geq 0$. Therefore, formula (3.16) holds and consequently by formula (3.17), we obtain exponential stability. □

**Example 3.27.** *Consider the following system*

$$y_1(n+1) = \frac{\alpha y_2(n)}{[1 + y_1^2(n)]} \tag{3.18}$$

$$y_2(n+1) = \frac{\beta y_1(n)}{[1 + y_2^2(n)]}. \tag{3.19}$$

Let $h = (h_1, h_2)^T$, where $h_1 = \frac{\alpha y_2(n)}{[1+y_1^2(n)]}$, $h_2 = \frac{\beta y_1(n)}{[1+y_2^2(n)]}$ computing jacobian matrix.

$$\frac{\partial h}{\partial y} \mid (0,0) = \begin{bmatrix} \frac{\partial h_1}{\partial y_1}(0,0) & \frac{\partial h_1}{\partial y_2}(0,0) \\ \frac{\partial h_2}{\partial y_1}(0,0) & \frac{\partial h_2}{\partial y_2}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}$$

this yields, the following system

$$\begin{bmatrix} y_1(n+1) \\ y_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix} \begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} + \begin{bmatrix} \frac{\partial h_1}{\partial y_1}(0,0) & \frac{\partial h_1}{\partial y_2}(0,0) \\ \frac{\partial h_2}{\partial y_1}(0,0) & \frac{\partial h_2}{\partial y_2}(0,0) \end{bmatrix}$$

Or

$$y(n+1) = Ay(n) + g(y(n))$$

The eigenvalues of $A$ are $\lambda_1 = \sqrt{\alpha\beta}$, $\lambda_2 = -\sqrt{\alpha\beta}$, hence. if $\mid \alpha\beta \mid < 1$, this shows that the system is asymptotically stable $|\alpha\beta| < 1$. The following theorem for the cases $\rho(A) = 1$, and $\rho(A) > 1$.

**Theorem 3.28.** [**1**] *The following statement*

*(1) the zero solution of (3.8) may be stable or unstable, if* $\rho(A) = 1$.

*(2) the zero solution of (3.8) is unstable, if* $g(x)$ *is* $o(x)$ *as* $\| x \| \to 0$ *and* $\rho(A) > 1$.

## 3.4 Application

### 3.4.1 One Species with Two Age Classes

you can consider a single-species, two-age-class system, with $X(n)$ being the number of young and $Y(n)$ that of adults, in the $n^{th}$ time interval:

$$X(n+1) = dY(n), \tag{3.20}$$

$$Y(n+1) = gX(n) + sY(n) - DY^2(n). \tag{3.21}$$

A percentage $g$ of the young become adult, and the rest will die before they could reach maturity. The grownups or the adults have a fertility rate b and a density-dependent existence rate $sY(n) - DY^2(n)$. thus, the following Equation (3.20) may be written in a more convenient form by supposing $X(n) = \frac{DX(n)}{b}$ and $\tilde{Y}(n) = DX(n)$. Then we have

$$\tilde{X}(n+1) = \tilde{Y}(n),$$

$$\tilde{Y}(n+1) = a\tilde{X}(n) + sY(n) - Y^2(n). \tag{3.22}$$

with $a = cb > 0$.
The nontrivial fixed point is $(\tilde{X}^*, \tilde{Y}^*)$, with $\tilde{X}^* = \tilde{Y}^*$ and $\tilde{Y}^* = d + s - 1$.
Note that the equilibria $\tilde{X}^*$ and $\tilde{Y}^*$ must be positive in order for the model to make sense biologically. This result suggests that $d + s - 1 > 0$. Since it is easier to do stability analysis on the zero equilibrium point, we let $x(n) = \tilde{X}(n) - \tilde{X}^*$ and $y(n) = \tilde{Y}(n) - \tilde{Y}^*$. This yields the system

$$x(n+1) = y(n),$$

$$y(n+1) = dx(n) + ry(n) - y^2(n), \quad r = 2 - 2d - s. \tag{3.23}$$

The fixed point $(0, 0)$ matches with the fixed point $(\tilde{X}^*, \tilde{Y}^*)$. Local stability can now be obtained by investigating the linearized system

$$x(n+1) = y(n),$$

$$y(n+1) = dx(n) + ry(n),$$

whose eigenvalues are the roots of the characteristic equation

$$\lambda^2 - r\lambda - d = 0.$$

By criteria (3.6), the trivial answer is asymptotically stable if and only if:

(i) $1 - r - d > 0$ or $d + s > 1$ , and

(ii) $1 + r - d > 0$ or $3d + s < 3$.

In the light of the above, the range of values of $d$ and $s$ for which the trivial solution is asymptotically stable is limited by the region $d = 1, s = 1, d + s = 1$, and $3d + s = 3$. The shaded region represents the range of parameters $d$, $s$ for which the trivial solution is asymptotically stable. To find the region of stability (or the basin of attraction) of the trivial solution the methods of Liapunov functions has to be taken into account. Let

$$V(x, y) = d^2x^2 + \frac{2drxy}{1-d} + y^2.$$

Recall from calculus that $Ax^2 + 2Bxy + Cy^2 = D$ is an ellipse if $AC - B^2 > 0$, or $d^2 - \frac{d^2r^2}{(1-d)^2} > 0$, or $d-1 < r < 1-d$. This reduces to $s+d > 1$ and $s < 3-3d$, the mixed term xy can be removed to obtain $A'x^2 + C'y^2 = D$, with $A' + C' = d^2 + 1 > 0$. Moreover, $A'C' > 0$. Hence both $A'$ and $C'$ are positive and, consequently, $D$ is positive. $V(x, y)$ is positive definite. After some computation we obtain

$$\Delta V(x, y) = y^2 W(x, y),$$

where

$$W(x, y) = (y - r)^2 - 2dx - \frac{2ar(r - y)}{1 - d} + d^2 - 1.$$

Hence, $\Delta V(x, y) \leq 0$ if $W(x, y) < 0$, that is, if $(x, y)$ is in the region

$$J = \{(x, y) : (y - r)^2 - 2dx - \frac{2dr(r - y)}{1 - d} + d^2 - 1 < 0\}.$$

The area $J$ is limited by the parabola $W(x, y) = 0$. Now, in the area $J$, $\Delta V(x, y) = 0$ on the $x$-axis $y = 0$. So $E$ is the $x$-axis. But since $(J, 0)$ is mapped to $(0, dg)$, the largest invariant set $M$ in $E$ is the origin. Consequently, by Theorem (LaSalle′s Invariance Principle) every bounded solution that remains in $J$ will converge to the origin.

We now can give a basic estimate of the basin of attraction, that is, the set of all points in $J$ that converges to the origin. Define

$$\begin{aligned} V_{min} &= min\{V(x_0, y_0) : (x_0, y_0) \in \partial J\}, \\ L_n &= \{\tilde{X}, \tilde{Y}\} : \tilde{X} = x_0 + \tilde{X}^*, \tilde{Y} = y_0 + \tilde{Y}^*, \end{aligned}$$

$$V(x(n), y(n)) < V_{min}, \ \ n = 0, 1, 2, \ldots$$

If $(x_0, y_0) \in L_0$, then $V(x(1), y(1)) \leq V(x_0, y_0) < V_{min}$, and hence $(x(1), y(1)) \in L_0$. Likewise $(x(n), y(n)) \in L_0$ for $n = 1, 2, 3, \ldots$ and, therefore, as $n \to \infty$, $(x(n), y(n)) \to (0, 0)$. Now, if $(x_0, y_0) \in L_n$, then

$$V(x(n+1), y(n+1)) \leq V(x(n), y(n)) < V_{min},$$

the argument continues as before to demonstrate that $x(n), y(n)) \to (0, 0)$ as $n \to \infty$. So the sets $l_n$ are estimates of the basin of attraction of $(\tilde{X}, \tilde{Y})$.

### 3.4.2 A Business Cycle Model

One of the best formal mathematical models for business cycles refers to Paul Samuelson (1939). Sir John Hicks (1950) modified the model later on. Let I(n) denote the investment at time period n and Y (n) is the income at time period n. according to the Samuelson-Hicks model, it is presumed that investment is relational to income change, i.e.,

$$W(n) = u(Y(n-1) - Y(n-2)).$$

similarly, one may consumption $G(n)$ is proportional to income $Y(n-1)$ in the previous period, i.e.,

$$G(n) = (1-s)Y(n) \tag{3.24}$$

where $0 \leq s \leq 1$ is the ″complementary″ proportion used. Introducing the accounting identity for a closed economy:

$$Y(n) = G(n) + I(n) \tag{3.25}$$

we derive a simple second-order difference equation

$$Y(n) = (1 + u - s)Y(n-1)uY(n-2). \tag{3.26}$$

The linear model (3.26) does not adequately represent a business since it does not produce oscillatory solutions (or periodic cycles) except for special cases (such as $u = 1$). A nonlinear cubic model

$$W(n) = u(Y(n-1) - Y(n-2)) - u(Y(n-1) - Y(n-2))^3, \tag{3.27}$$

$$G(n) = (1-s)Y(n-1) + \varepsilon s Y(n-2), \tag{3.28}$$

was proposed in Puu [**9**] and [**14**]. A fraction $0 \leq \varepsilon \leq 1$ of savings was suppose to be spent after being saved for one period. So for $\varepsilon = 0$, the original Hicks model (3.24) is recovered.
Let us introduce a new variable

$$\tilde{Z}(n-1) = \frac{W(n)}{u} = Y(n-1) - Y(n-2). \tag{3.29}$$

Adding (3.11) and (3.28) and using (3.25) yields

$$\begin{aligned} Y(n+1) &= W(n+1) + G(n+1) \\ &= u(Y(n) - Y(n-1)) + (1-s)Y(n) + \varepsilon s Y(n-1) - u(Y(n) - Y(n-1))^3. \end{aligned}$$

Subtracting $Y(n)$ from both sides yields

$$\tilde{Z}(n) = (u - \varepsilon s)\tilde{Z}(n-1) - u\tilde{Z}^3(n) + (\varepsilon - 1)sY(n-1).$$

Let

$$\tilde{Z}(n) = \sqrt{\frac{1 + u - \varepsilon s}{u}} Z(n).$$

Now

$$Z(n) = (u - \varepsilon s)Z(n-1) - (1 + u - \varepsilon s)Z^3(n-1) + (\varepsilon - 1)sY(n-1).$$

consider $d = (v - \varepsilon s), (\varepsilon - 1)s = b$. We get

$$Z(n+1) = dZ(n) - (1+d)Z^3(n) + bY(n) \tag{3.30}$$

where $b = (1 - \varepsilon)s$ represents a sort of eternal rate of saving. Using (3.29) and (3.30) we now have the two-dimensional system

$$Y(n+1) = Y(n) + Z(n),$$

$$Z(n+1) = dZ(n) - (d+1)Z^3(n) - bY(n). \tag{3.31}$$

System (3.31) has a single equilibrium point $X^* = (Y^*, Z^*) = (0,0)$. Local stability can now be obtained by examining the linearized system

$$\begin{pmatrix} Y(n+1) \\ Z(n+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -b & -d \end{pmatrix} \begin{pmatrix} Y(n) \\ Z(n) \end{pmatrix}. \tag{3.32}$$

The eigenvalues are given by

$$\lambda_{1,2} = \frac{d+1 \pm \sqrt{(d-1)^2 - 4b}}{2}.$$

By criteria (3.26), the trivial solution is asymptotically stable if and only if:

(i) $2 + 2d + b > 0$,

(ii) $b > 0$,

(iii) $1 - d - b > 0$.

Taking into account that $d > 0$ and $0 < b < 1$, the region of stability $S$ is given by

$$S = \{(b,d) | 0 < b < 1, 0 < d < 1 - b\}.$$

We conclude that if $(b,d) \in S$, then equilibrium $X^* = (0,0)$ is asymptotically stable. Notice that the eigenvalues $\lambda_1, \lambda_2$ are complex numbers if $1 - 2\sqrt{b} < d < 1 + 2\sqrt{b}$. But this is fulfilled if $(b,d) \in S$. Thus if $(b,d) \in S$, the equilibrium point $X^* = (0,0)$ is a stable focus.

At $d = 1 - b$, the equilibrium point $X^* = (0,0)$ loses it stability and possible appearance of cycles. For example, for $d = 0, b = 1$, an attracting cycle of period 6 and a saddle cycle of period 7 appear. At $d = \sqrt{2} - 1, b = 2 - \sqrt{2}$ and attracting and saddle cycles of period 8 appear and so on [**16**].

# Bibliography


[1] **Elaydi** S,(2005),An Introduction to Difference Equations, Third Edition.Springer,NewYork.

[2] **Scheinerman** E,(2000),Invitation To Dynamical System, (Lecture notes), Congress Information, Johns Hopkins University.

[3] **Berglund** N,(2002),Geometrical Theory of Dynamical Systems (Lecture notes)

[4] **Evan Dummit's**,Chaos, Dynamics, and Fractals, Fall 2015 :(part 1): Introduction to Dynamics.v.1.07

[5] **Geoffrey R. Goodson**LECTURE NOTES ON DYNAMICAL SYSTEMS, CHAOS AND FRACTAL GEOMETRY, 2016

[6] **D. C. Munson, Jr., and A.C. Singer** ECE 310:Digital Signal Processing (Course Notes, 2013).

[7] **Arne Jensen**Department of Mathematical Sciences,Aalborg University,Lecture Notes on Difference Equations Fr. Bajers Vej 7G,DK-9220 Aalborg Ã˜, Denmark,Version 1, July 18, 2011

[8] **Kulenovic, M.R.S., and G. Ladas**, Dynamics of Second-Order Rational Difference Equations with Open Problems and Conjectures, Chapman Hall/CRC Press, Boca Raton, FL, 2003.

[9] **Puu, T.**, Nonlinear Economic Dynamics, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin, 1989.

[10] **Agarwal R,**(2000), Difference Equations and Inequalities, Theory, methods and Application ,second edition, Marcel Dekker,Inc. New york.

[11] **Dannan F,Elaydi S, Ponomarenko V**,(2003),stability of Hyperbolic And Non Hyperbolic Fixed points of one-dimensional Maps,J.of Difference equations and Application ,vol,(95),pp,449-457.

[12] **johnson W.P**(2002), The curios history of Faadi Brounos formula J. of Am Math .Monthly , vol.109(3),pp.217-234.

[13] **Ponommarenko V**,(2004), Faa di Brunos Formula and non Hyperbolic Fixed points of one-dimensional Maps , Hindawi publishing corp. vol.(29), pp,1543-1552

[14] **Puu, T.**, Chaos in business cycles, Chaos Solitons Fractals 1 (1991), 457-473.

[15] **Carlson, D.C.**, The Stability of Finite Difference Equations, Masters thesis, University of Colorado, Colorado Springs, 1989.

[16] **Puu, T., and I. Sushko**, A business cycle model with cubic nonlinearity, CERUM Working Paper 47, 2002.

[17] **LaSalle, J.P.**, The Stability and Control of Discrete Processes, Applied Mathematical Sciences, Vol. 82, Springer-Verlag, New York, 1986.

[18] **V. Lakshmikantham and D. Trigiante**:(1988)THEORY OF DIFFERENCE EQUATIONS NUMERICAL METHODS AND APPLICATIONS Second Edition

[19] **WALTER G. KELLEY.ALLAN C. PETERSON**;Difference Equations An Introduction with Applications second Edition

[20] **Klaus Neusser**February 20, 2019 Difference Equations for Economists1

[21] **Elaydi, S.**, Asymptotics for linear difference equations I: Basic theory, J. Difference Equ. Appl. 5 (1999), 563-589.

[22] **Robinson, C.**, Dynamical Systems, 2nd ed., CRC Press, Boca Raton, FL,1999.

[23] **Cushing, J.**, An Introduction to Structural Population Dynamics, SIAM,Philadelphia, 1999.

[24] **Klaus Neusser**,Difference Equations for Economists1,April 15, 2019

[25] **Mario Tirelli**Linear Difference Equations,Still a preliminary version ,March 2, 2014