Arch. Math. 75 (2000) 380–388
0003-889X/00/050380-09 $ 3.30/0

Archiv der Mathematik

Norm attaining operators from $L_1(\mu)$ into $L_\infty(\nu)$

By

Rafael Payá *) and Yousef Saleh

Abstract. Given an arbitrary measure μ and a localizable measure ν, we show that the set of norm attaining operators is dense in the space of all bounded linear operators from $L_1(\mu)$ into $L_\infty(\nu)$.

1. Introduction. Bishop and Phelps [1] have asked the general question, for which Banach spaces X and Y is the collection of norm attaining operators dense in the space $L(X, Y)$ of all bounded linear operators from X into Y. An operator $T \in L(X, Y)$ attain its norm if there is $x \in B_X$ (the closed unit ball of X) such that

$$\|Tx\| = \|T\|.$$

We denote the set of all norm attaining operators by $NA(X, Y)$. After the pioneering work of J. Lindenstrauss [8], the question of the denseness of $NA(X, Y)$ in $L(X, Y)$ has received a lot of attention. Let us only mention the results dealing with the case when X and Y are L_1-spaces or $C(K)$-spaces. It was shown by J. Uhl [10] that given a strictly convex Banach space Y, $NA(L_1[0,1], Y)$ is dense in $L(L_1[0,1], Y)$ if and only if Y has the Radon-Nikodým property, and A. Iwanik [5] proved that $NA(L_1(\mu), L_1(\nu))$ is dense in $L(L_1(\mu), L_1(\nu))$ for arbitrary measures μ and ν. Moreover, J. Johnson and J. Wolfe [6] showed that $NA(C(K), C(L))$ is dense in $L(C(K)), C(L))$ for arbitrary compact spaces K and L. In [9] W. Schachermayer proved that $NA(L_1[0,1], C[0,1])$ is not dense in $L(L_1[0,1], C[0,1])$, and asked for a characterization of compact spaces K such that $NA(L_1[0,1], C(K))$ is dense in $L(L_1[0,1], C(K))$. Recently C. Finet and R. Payá [4] have shown that $NA(L_1(\mu), L_\infty[0,1])$ is dense in $L(L_1(\mu), L_\infty[0,1])$, for every σ-finite measure μ, so giving a new example of a compact Hausdorff space K such that $NA(L_1[0,1], C(K))$ is dense $L(L_1[0,1], C(K))$.

In this note we extend the result in [4], to prove the following

Theorem 1. *$NA(L_1(\mu), L_\infty(\nu))$ is dense in $L(L_1(\mu), L_\infty(\nu))$ for every measure μ and every localizable measure ν.*

By using the isometric classification of L_1-spaces and a technical lemma which deals with the denseness of norm attaining operators from an arbitrary ℓ_1-sum into an arbitrary ℓ_∞–sum of Banach spaces we reduce the proof of the above theorem to the case when μ is

Mathematics Subject Classification (1991): 46B25, 47B38.

*) Research partially supported by Spanish D.G.E.S. Project No. PB96–1406.

finite and ν is the product measure on a (possibly infinite) product of copies of $[0,1]$. For this case we use a natural representation of norm attaining operators and the martingale convergence theorem.

If a compact Hausdorff space K is hyperstonian, then there is a localizable measure ν such that $C(K)$ is isometric to $L_1(\nu)^* = L_\infty(\nu)$ (see [2; pp. 493].) Conversely, if the measure ν is localizable, then $L_\infty(\nu)$ is isometric to $C(K)$ for suitable hyperstonian K. Therefore, our result can be equivalently stated by saying that $NA(L_1(\mu), C(K))$ is dense in $L(L_1(\mu), C(K))$ for every measure μ and every hyperstonian compact Hausdorff space K.

2. Proof of the main result. We start by proving a lemma which will allow a reduction to the case when μ and ν are finite measures. Given an arbitrary family $\{X_i : i \in I\}$ of Banach spaces, we denote by $\left(\bigoplus_{i\in I} X_i\right)_{\ell_1}$ its ℓ_1-sum, i.e, the Banach space of all families $(x_i)_{i\in I}$ such that $x_i \in X_i$ for all i, and

$$\|(x_i)_{i\in I}\| := \sum_{i\in I} \|x_i\| < \infty .$$

Similarly $\left(\bigoplus_{j\in J} Y_j\right)_{\ell_\infty}$ denotes the ℓ_∞-sum of a family $\{Y_j : j \in J\}$ of Banach spaces, i.e, the Banach space of all families $(y_j)_{j\in J}$ with $y_j \in Y_j$ for all j and

$$\|(y_j)_{j\in J}\| := \sup\{\|y_j\| : j \in J\} < \infty .$$

We recall for later use the well known fact that the dual of an ℓ_1-sum is the ℓ_∞-sum of the dual spaces:

$$\left(\bigoplus_{i\in I} X_i\right)_{\ell_1}^* = \left(\bigoplus_{i\in I} X_i^*\right)_{\ell_\infty} .$$

Lemma 2. *Let $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ be arbitrary families of Banach spaces, $X = \left(\bigoplus_{i\in I} X_i\right)_{\ell_1}$ and $Y = \left(\bigoplus_{j\in J} Y_j\right)_{\ell_\infty}$. Then the following statements are equivalent*

(1) *$NA(X, Y)$ is dense in $L(X, Y)$*

(2) *$NA(X_i, Y_j)$ is dense in $L(X_i, Y_j)$ for every $(i,j) \in I \times J$.*

Proof. The key fact here is the natural identification of $L(X, Y)$ with the space $\left(\bigoplus_{(i,j)\in I\times J} L(X_i, Y_j)\right)_{\ell_\infty}$ which we now describe. For $i \in I$ and $j \in J$ let E_i and F_j denote the natural isometric embeddings of X_i and Y_j into X and Y, respectively. Similarly, let P_i and Q_j denote the natural norm-one projections from X and Y onto X_i and Y_j, respectively. For $T \in L(X, Y)$, it follows clearly from the definition of the ℓ_∞-sum that $\|T\| = \sup\{\|Q_j T\| : j \in J\}$. On the other hand, every $x \in X$ can be written as $x = \sum_{i\in I} E_i P_i x$, the family $\{E_i P_i x\}$ being absolutely summable with $\|x\| = \sum_{i\in I} \|E_i P_i x\| = \sum_{i\in I} \|P_i x\|$. Thus, for any Banach space Z and any operator $S \in L(X, Z)$ we have $\|Sx\| \leqq \sum_{i\in I} \|S E_i P_i x\| \leqq \sup\{\|S E_i\| : i \in I\} \sum_{i\in I} \|P_i x\|$, which shows that $\|S\| = \sup\{\|S E_i\| : i \in I\}$. In particular we have $\|Q_j T\| = \sup\{\|Q_j T E_i\| : i \in I\}$ for every $j \in J$, so

$$\|T\| = \sup\{\|Q_j T E_i\| : i \in I, j \in J\}$$

and $T \mapsto (Q_j T E_i)_{(i,j)\in I\times J}$ is the above mentioned identification. More concretely, if we are given operators $T_{ij} \in L(X_i, Y_j)$ with $\sup\{\|T_{ij}\| : i \in I, j \in J\} < \infty$, then there is a unique $T \in L(X, Y)$ such that $Q_j T E_i = T_{ij}$ for every $(i,j) \in I \times J$.

(1) $\Rightarrow$ (2) Let $h \in I$ and $k \in J$ be fixed. To show that $NA(X_h, Y_k)$ is dense in $L(X_h, Y_k)$, fix $u \in L(X_h, Y_k)$, assume without loss of generality that $\|u\| = 1$ and let $0 < \varepsilon < 1$ be given. Consider the operator $U = F_k u P_h \in L(X, Y)$. Note that $Q_j U = 0$ unless $j = k$ and $Q_k U E_i = 0$ unless $i = h$, while $Q_k U E_h = u$, so $\|U\| = \|u\| = 1$. By (1) we get an operator $T \in NA(X, Y)$ such that $\|T\| = 1$ and $\|T - U\| < \varepsilon$. Then we take $t = Q_k T E_h \in L(X_h, Y_k)$, which clearly satisfies $\|t\| \leqq 1$, $\|t - u\| \leqq \|T - U\| < \varepsilon$ and we are left with showing that t attains its norm. Let $x \in S_X$ be such that $\|Tx\| = \|T\| = 1$. For $j \in J, j \neq k$ we have

$$\|Q_j Tx\| = \|Q_j Tx - Q_j Ux\| \leqq \|T - U\| < \varepsilon < 1$$

so $\|Tx\| = \|Q_k Tx\| = 1$, which shows that $\|Q_k T\| = 1$ and $Q_k T$ also attains its norm at x.

Now, for $i \in I, i \neq h$ we have

$$\|Q_k T E_i\| = \|Q_k T E_i - Q_k U E_i\| \leqq \|T - U\| < \varepsilon < 1, \qquad \text{so}$$

$$1 = \|Q_k T\| = \|Q_k T E_h\| = \|t\|.$$

By writting $x = \sum_{i\in I} E_i P_i x$ with $\sum_{i\in I} \|P_i x\| = \|x\| = 1$, we get

$$\begin{aligned} 1 = \|Q_k Tx\| &\leqq \sum_{i\in I} \|Q_k T E_i P_i x\| \\ &= \|t P_h x\| + \sum_{i\in I\setminus\{h\}} \|Q_k T E_i P_i x\| \\ &\leqq \|P_h x\| + \varepsilon \sum_{i\in I\setminus\{h\}} \|P_i x\| \\ &\leqq \|P_h x\| + \sum_{i\in I\setminus\{h\}} \|P_i x\| = 1. \end{aligned}$$

It follows that $P_i x = 0$ for $i \in I \setminus \{h\}$, so $\|P_h x\| = 1$ and the equality $\|t P_h x\| = \|P_h x\| = 1$ shows that t attains its norm, as required.

(2)$\Longrightarrow$(1). Let $T \in L(X, Y)$ with $\|T\| = 1$ and $0 < \varepsilon < 1$ be given. We first find $h \in I$ and $k \in J$ such that $1 - \frac{\varepsilon}{3} < \|Q_k T E_h\| \leqq 1$. Then we use (2) to find an operator $s_0 \in NA(X_h, Y_k)$ such that $\|s_0 - Q_k T E_h\| \leqq \frac{\varepsilon}{3}$ and it follows that

$$1 - \frac{2\varepsilon}{3} \leqq \|s_0\| \leqq 1 + \frac{\varepsilon}{3}.$$

By taking $s = \frac{s_0}{\|s_0\|}$ we have $s \in NA(X_h, Y_k), \|s\| = 1$ and $\|s - Q_k T E_h\| \leqq \varepsilon$. Finally let $S \in L(X, Y)$ be such that $Q_k S E_h = s$ and $Q_j S E_i = Q_j T E_i$ for $(i,j) \neq (h,k)$. We have clearly $\|S\| = 1$, $\|S - T\| \leqq \varepsilon$, and we are left with showing that $S \in NA(X, Y)$. Actually if $x_h \in X_h$ is such that $1 = \|x_h\| = \|s x_h\|$, for $x = E_h x_h$ we have

$$\begin{aligned} \|Sx\| &\geqq \|Q_k Sx\| = \|Q_k S E_h x_h\| \\ &= \|s x_h\| = 1, \end{aligned}$$

and S attains its norm, as required. $\square$

Recall that if μ is an arbitrary measure, $L_1(\mu)$ can be decomposed in the form

$$L_1(\mu) = \Big(\bigoplus_{i\in I} L_1(\mu_i)\Big)_{\ell_1}$$

where μ_i is a finite measure for all $i \in I$ (see [2, Appendix B] for example). On the other hand if ν is a localizable measure we have that $L_\infty(\nu) = L_1(\nu)^*$ and we get also finite measures $\{\nu_j : j \in J\}$ such that

$$L_\infty(\nu) = \Big(\bigoplus_{j\in J} L_\infty(\nu_j)\Big)_{\ell_\infty}.$$

In view of the above lemma, to prove Theorem 1 we may assume without loss of generality that μ and ν are finite measures. In this case we have a characterization of the operators in $NA(L_1(\mu), L_\infty(\nu))$, which was essentially obtained in [4]. For the sake of completeness we state and prove it for our formally more general case. To this end we recall a well known representation of the operator space $L(L_1(\mu), L_\infty(\nu))$.

If $(\Omega, \mathscr{A}, \mu)$ and $(K, \mathscr{B}, \nu)$ are finite measure spaces we have an isometric isomorphism

$$L(L_1(\mu), L_\infty(\nu)) \equiv L_\infty(\mu \otimes \nu)$$

where $\mu \otimes \nu$ denotes the product measure on $\Omega \times K$, the operator $\hat{h}$ corresponding to an essentially bounded function h being given by

$$[\hat{h}(f)](t) = \int_\Omega h(w,t) f(w) d\mu(w)$$

for $[\nu]$-a.e. $t \in K$ and every $f \in L_1(\mu)$ (see [4].) We now characterize those functions h such that the operator $\hat{h}$ attain its norm.

Proposition 3. *Let $(\Omega, \mathscr{A}, \mu)$ and $(K, \mathscr{B}, \nu)$ be finite measure spaces and let $h \in L_\infty(\mu \otimes \nu)$. Consider the following statements.*

(1) *There is a set $A \in \mathscr{A}, \mu(A) > 0$, and a sequence (B_n) in $\mathscr{B}$, $\nu(B_n) > 0$ for every n, such that*

$$\lim_{n\to\infty} \frac{1}{\mu(A)\nu(B_n)} \Big| \int_{A\times B_n} h d(\mu\otimes\nu) \Big| = \|h\|_\infty.$$

(2) *There are sets A, B_n like in* (1) *and a measurable function φ on Ω such that $|\varphi(w)| = 1$ for all $w \in \Omega$ and*

$$\lim_{n\to\infty} \frac{1}{\mu(A)\nu(B_n)} \Big| \int_{A\times B_n} h(w,t)\varphi(w) d\nu(t) d\mu(w) \Big| = \|h\|_\infty.$$

(3) *The operator $\hat{h} \in L(L_1(\mu), L_\infty(\nu))$ corresponding to h attains its norm.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). *Moreover, in the real case all three statements are equivalent.*

Proof. (1) $\Rightarrow$ (2) is clear. For (2) $\Rightarrow$ (3), just consider the function $f = \dfrac{\varphi\chi_A}{\mu(A)}$ where χ_A denotes the characteristic function of A. Then $\|f\|_1 = 1$, and it follows clearly from (2) that $\|\hat{h}(f)\|_\infty = \|h\|_\infty = \|\hat{h}\|$.

(3) $\Rightarrow$ (2). Let us start with a general remark on norm attaining functionals on $L_1(\mu)$. Suppose that a function $g \in L_\infty(\mu)$ attains its norm as a functional on $L_1(\mu)$ and, without

loss of generality, that $\|g\|_\infty = 1$. Then there is a function f in the unit sphere of $L_1(\mu)$ such that

$$1 = \int_\Omega fg d\mu \leqq \int_\Omega |fg| d\mu \leqq \int_\Omega |f| d\mu = 1$$

hence $fg = |f|$, $[\mu]$-almost everywhere. Let us write $f = \varphi|f|$ where φ is a measurable function on Ω and $|\varphi| = 1$. Then $\varphi g = 1$, $[\mu]$-a.e in the set $C := \{w \in \Omega : f(w) \neq 0\}$. Therefore, if A is any measurable subset of C with $\mu(A) > 0$, we have

$$\int_A \varphi g d\mu = \mu(A),$$

which shows that g also attains its norm at the function $\varphi_A := \mu(A)^{-1}\varphi\chi_A$, which is in the unit sphere of $L_1(\mu)$. In the real case, since φ only takes the values ± 1, we may choose the set A so that φ is constant on it.

By using the Hahn-Banach Theorem we can now extend the above remark to operators. Let T be a norm attaining operator from $L_1(\mu)$ into an arbitrary Banach space (say Y) with $\|T\| = 1$. Then there are $f \in L_1(\mu)$ and $y^* \in Y^*$ such that $\|f\|_1 = \|y^*\| = 1$ and $y^*(T(f)) = 1$, but this means that the functional $T^*(y^*)$ attains its norm, where T^* is the adjoint operator. Thus, there are a measurable set $A \subset \Omega$ with $0 < \mu(A)$ and a measurable function φ on Ω with $|\varphi| = 1$ such that

$$1 = |y^*(T(\varphi_A))| \leqq \|T(\varphi_A)\|.$$

Back to the proof of the desired implication, assume that (3) holds and find a set A and a function φ as above, such that

$$\|h\|_\infty = \|\hat{h}\| = \|\hat{h}(\varphi_A)\|_\infty .$$

Now recall that the unit ball of $L_1(\nu)$ is the closed absolutely convex hull of the set

$$\{\nu(B)^{-1}\chi_B : B \in \mathscr{B}, \nu(B) > 0\},$$

equivalently,

$$\|g\|_\infty = \sup\left\{\frac{1}{\nu(B)}\left|\int_B g d\nu\right| : B \in \mathscr{B}, \nu(B) > 0\right\}$$

for every $g \in L_\infty(\nu)$. Thus we may find a sequence (B_n) in $\mathscr{B}$ with $\nu(B_n) > 0$ for every n, such that

$$\begin{aligned}\|h\|_\infty = \|\hat{h}(\varphi_A)\|_\infty &= \lim_{n\to\infty} \frac{1}{\nu(B_n)}\left|\int_{B_n} \hat{h}(\varphi_A) d\nu\right| \\ &= \lim_{n\to\infty} \frac{1}{\mu(A)\nu(B_n)}\left|\int_{A\times B_n} h(w,t)\varphi(w) d\mu(w) d\nu(t)\right|\end{aligned}$$

which shows that (2) holds. Recall that in the real case we can arrange that φ is constant on A, hence we actually get that (1) holds. In [4] the reader may find an example showing that (2) is strictly weaker than (1) in the complex case. □

In the special case $h = \chi_E$, the characteristic function of a measurable set, we get

Corollary 4. *Given a measurable set $E \subset \Omega \times K$, with positive measure, the operator $\hat{\chi_E}$ attains its norm if and only if*

$$\lim_{n\to\infty} \frac{[\mu\otimes\nu]((A\times B_n)\cap E)}{\mu(A)\nu(B_n)} = 1$$

for some measurable set $A \subset \Omega$, with $\mu(A) > 0$ and some sequence (B_n) of measurable subsets of K with $\nu(B_n) > 0$ for every n.

In a special case, namely when K is a product of unit intervals, we will show that every measurable set $E \subseteqq \Omega \times K$ satisfies the condition in the above corollary, and this will provide a fairly large set of norm attaining operators. Let us first show that we can reduce the proof of our theorem to this special case. This is a consequence of the representation theory for L_1-spaces. Indeed, if ν is a finite measure, we may write

$$L_1(\nu) = \Big(\underset{i\in I}{\oplus} X_i\Big)_{\ell_1}.$$

Where each space X_i is either 1-dimensional or of the form $L_1([0,1]^\Lambda)$ where Λ is a finite or infinite set, for each coordinate interval we consider Lebesgue measure on the Borel subsets of $[0,1]$, and $[0,1]^\Lambda$ is provided with the product measure on its Borel σ-algebra (see [7].) It follows that

$$L_\infty(\nu) = \Big(\underset{j\in J}{\oplus} Y_j\Big)_{\ell_\infty}$$

where each Y_j is either 1-dimensional or of the form $L_\infty([0,1]^\Lambda)$. By Lemma 2, to show that $NA(L_1(\mu), L_\infty(\nu))$ is dense in $L(L_1(\mu), L_\infty(\nu))$ it is enough to show that $NA(L_1(\mu), Y_j)$ is dense in $L(L_1(\mu), Y_j)$ for every $j \in J$, and the Bishop-Phelps Theorem takes care of the case when Y_j is 1-dimensional, so we may restrict our attention to the case $L_\infty(\nu) = L_\infty[0,1]^\Lambda$. Therefore, *in what follows we will take $K = [0,1]^\Lambda$ where Λ is an arbitrary set, $\mathcal{B}$ will be the Borel σ-algebra on K and ν will be the product measure on K each coordinate interval being provided with Lebesgue measure.*

Proposition 5. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $E \subseteqq \Omega \times K$ be a measurable set with positive measure: $E \in \mathcal{A} \otimes \mathcal{B}$, $(\mu\otimes\nu)(E) > 0$. Then there are measurable sets $A \in \mathcal{A}$, $B_n \in \mathcal{B}$, with $\mu(A) > 0$, $\nu(B_n) > 0$ for all n, such that*

$$\frac{[\mu\otimes\nu]((A\times B_n)\cap E)}{\mu(A)\nu(B_n)} \longrightarrow 1.$$

Proof. We start with the case when Λ is countable and use an analog of Lebesgue Density Theorem. Let us fix a sequence $(\Pi_n)_{n=1}^\infty$ of finite partitions of $K = [0,1]^\Lambda$ into sets of positive measure, such that Π_{n+1} is a refinement of Π_n for each n, and the σ-algebra generated by $\bigcup_{n=1}^{\infty} \Pi_n$ is the Borel σ-algebra $\mathcal{B}$. For each $y \in K$ and $n \in \mathbb{N}$, let $B(n,y)$ be the set in Π_n containing y.

Now, given a Borel set $F \subseteqq K$, define

$$\delta(F) = \Big\{ y \in K : \lim_{n\longrightarrow\infty} \frac{\nu(F\cap B(n,y))}{\nu(B(n,y))} = 1 \Big\}.$$

As a special case of the martingale almost everywhere convergence theorem (see [3; § 17], for example) we have that

$$(*) \qquad \nu(F \setminus \delta(F)) = \nu(\delta(F) \setminus F) = 0.$$

Let us now consider a measurable set $E \subseteqq \Omega \times K$ with $(\mu \otimes \nu)(E) > 0$. For each $w \in \Omega$ take the vertical section

$$E_w = \{x \in K : (w,x) \in E\}$$

and define a new set $\hat{E} \subseteqq \Omega \times K$ by

$$\hat{E} = \{(w,y) : w \in \Omega, y \in \delta(E_w)\}.$$

We claim that $\hat{E}$ is also measurable. Indeed, consider for each $n \in \mathbb{N}$ the measurable set $H^{(n)} \subseteqq \Omega \times K \times K$ given by

$$H^{(n)} = \left\{(w,y,z) : (w,z) \in E, (y,z) \in \bigcup_{B \in \Pi_n} (B \times B)\right\}.$$

Then, for each $(w,y) \in \Omega \times K$ the corresponding section of $H^{(n)}$ is given by

$$\begin{aligned} H^{(n)}_{(w,y)} &= \{z \in K : z \in E_w, B(n,y) = B(n,z)\} \\ &= E_w \cap B(n,y) \end{aligned}$$

and it follows from Fubini's theorem that the function φ_n defined on $\Omega \times K$ by

$$\varphi_n(w,y) = \nu(E_w \cap B(n,y))$$

is measurable. Since $\psi_n(w,y) = \nu(B(n,y)) > 0$ is measurable as well, we see that the set

$$\hat{E} = \left\{(w,y) : \liminf_{n \longrightarrow \infty} \frac{\varphi_n(w,y)}{\psi_n(w,y)} = 1\right\}$$

is measurable, as claimed. It follows from $(*)$ and Fubini's Theorem that

$$(\mu \otimes \nu)(\hat{E}) = \int_\Omega \nu(\delta(E_w))d\mu(w) = \int_\Omega \nu(E_w)d\mu(w) = (\mu \otimes \nu)(E) > 0$$

but we have also

$$(\mu \otimes \nu)(\hat{E}) = \int_K \mu((\hat{E})^y)d\nu(y)$$

where $\hat{E}^y$ is now the horizontal section:

$$\hat{E}^y = \{w \in \Omega : y \in \delta(E_w)\}.$$

It follows that $\mu((\hat{E})^{y_0}) > 0$ for some $y_0 \in K$. Take $A = (\hat{E})^{y_0}$ and note that $y_0 \in \delta(E_w)$ for every $w \in A$. Take also $B_n = B(n, y_0)$ for all n. Then we have

$$\lim_{n \longrightarrow \infty} \frac{\nu(E_w \cap B_n)}{\nu(B_n)} = 1$$

for every $w \in A$. By integrating over $w \in A$ and using the dominated convergence theorem we finally get

$$\frac{(\mu\otimes\nu)(E\cap(A\times B_n))}{\mu(A)\nu(B_n)}\longrightarrow 1$$

as required.

It remains to consider the case when Λ is uncountable. This case can be easily reduced to the previous one by using well-known properties of the product measure. More concretely, if Λ is uncountable and $E\subseteqq\Omega\times[0,1]^\Lambda$ is measurable, then E differs by a null set of a measurable set F which only restricts a countable number of coordinates, that is

$$F=F_0\times[0,1]^{\Lambda\setminus J}$$

where $J\subseteqq\Lambda$ is a countable set and $F_0\subseteqq\Omega\times[0,1]^J$ is measurable. By the first part of our proof we have measurable sets $A\subseteqq\Omega$ and $B_n^{(0)}\subseteqq[0,1]^J$ such that

$$\frac{[\mu\otimes\nu](F_0\cap(A\times B_n^{(0)}))}{\mu(A)\nu(B_n^{(0)})}\longrightarrow 1.$$

By taking the same A and $B_n=B_n^{(0)}\times[0,1]^{\Lambda\setminus J}$ we have clearly

$$\frac{[\mu\otimes\nu](E\cap(A\times B_n))}{\mu(A)\nu(B_n)}=\frac{[\mu\otimes\nu](F\cap(A\times B_n))}{\mu(A)\nu(B_n)}=\frac{[\mu\otimes\nu](F_0\cap(A\times B_n^{(0)}))}{\mu(A)\nu(B_n^{(0)})}\longrightarrow 1,$$

as required. □

End of the proof of Theorem 1. Recall that the proof was reduced to the case when μ is finite and ν is the product measure on a product K of copies of $[0,1]$, so we just work in this case.

Let $h\in L_\infty(\mu\otimes\nu)$ be a simple function, that is, a linear combination of characteristic functions of measurable subsets of $\Omega\times K$. The set of these simple functions is a dense subspace of $L_\infty(\mu\otimes\nu)$, so we are left with showing that the operator $\hat{h}$ corresponding to h attains its norm. By normalizing h and up to rotation, we may assume without loss of generality that h has the form

$$h=\chi_E+k$$

where E is a measurable subset of $\Omega\times K$ with positive measure, k is another simple function such that $\|k\|_\infty\leqq 1$, and k vanishes on E. By Proposition 5, there are measurable sets $A\subseteqq\Omega$ and $B_n\subseteqq K$ with $\mu(A)>0$ and $\nu(B_n)>0$ for all n such that

$$\lim_{n\to\infty}\frac{[\mu\otimes\nu]((A\times B_n)\cap E)}{\mu(A)\nu(B_n)}=1$$

equivalently,

$$\lim_{n\to\infty}\frac{1}{\mu(A)\nu(B_n)}\int\limits_{A\times B_n}\chi_E d(\mu\otimes\nu)=1=\|h\|_\infty.$$

On the other hand, since the function k vanishes on E and $\|k\|_\infty\leqq 1$, we also have

$$\left|\frac{1}{\mu(E)\nu(B_n)}\int\limits_{A\times B_n}kd(\mu\otimes\nu)\right|\leqq\frac{[\mu\otimes\nu]((A\times B_n)\setminus E)}{\mu(A)\nu(B_n)}\longrightarrow 0.$$

Thus, we have finally shown that

$$\lim_{n\to\infty} \frac{1}{\mu(A)\nu(B_n)} \int\limits_{A\times B_n} hd(\mu \otimes \nu) = 1 = \|h\|_\infty .$$

and Proposition 3 tells us that the operator $\hat{h}$ attains its norm, as required. □

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Eingegangen am 12. 5. 1999*)

Anschrift der Autoren:

Rafael Payá and Yousef Saleh
Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071-Granada
Spain

*) Eine überarbeitete Fassung ging am 18. 10. 1999 ein.