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NEW SUFFICIENT CONDITIONS FOR THE DENSENESS OF NORM ATTAINING MULTILINEAR FORMS

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ABSTRACT

This paper gives new sufficient conditions for the density of the set of norm attaining multilinear forms in the space of all continuous multilinear forms on a Banach space. The symmetric case is also discussed.

1. *Introduction*

Motivated by the Bishop–Phelps theorem [**6**] that the set of norm attaining linear functionals on an arbitrary Banach space is dense in the dual space, some authors have considered the question of the denseness of norm attaining multilinear forms. Given real or complex Banach spaces $X_1, \ldots, X_N$, we denote by $\mathscr{L}^N(X_1, \ldots, X_N)$ the Banach space of all continuous N-linear mappings from $X_1 \times \ldots \times X_N$ into the scalar field. We say that $\varphi \in \mathscr{L}^N(X_1, \ldots, X_N)$ *attains its norm* if there are $x_i \in B_{X_i}$ (the unit ball of X_i) for $i = 1, 2, \ldots, N$, such that

$$|\varphi(x_1, \ldots, x_N)| = \|\varphi\| := \sup\{|\varphi(y_1, \ldots, y_N)| : (y_1, \ldots, y_N) \in B_{X_1} \times \ldots \times B_{X_N}\},$$

and we denote by $\mathscr{AL}^N(X_1, \ldots, X_N)$ the set of all norm attaining N-linear forms. In the case where $X_1 = \ldots = X_N = X$, we write simply $\mathscr{L}^N(X)$ and $\mathscr{AL}^N(X)$.

R. M. Aron, C. Finet and E. Werner [**5**] posed the question of when $\mathscr{AL}^N(X)$ is dense in $\mathscr{L}^N(X)$, and gave sufficient conditions for this denseness to hold. The first example of a Banach space X such that $\mathscr{AL}^2(X)$ is not dense in $\mathscr{L}^2(X)$ was given in [**3**]. Shortly after, Y. S. Choi [**7**] showed that $\mathscr{AL}^2(L_1)$ is not dense in $\mathscr{L}^2(L_1)$. For further results on this problem, we refer the reader to [**2**, **4**, **8**, **12**].

In this paper we give some improvements on the results in [**5**]. More concretely, it was shown there that $\mathscr{AL}^N(X)$ is dense in $\mathscr{L}^N(X)$, provided that X satisfies either the Radon–Nikodým property (RNP) or the so-called 'property (α)'. The latter property (see below for the definition) was introduced by W. Schachermayer [**14**] in the study of norm attaining linear operators. We prove that a property strictly weaker than (α), which had been previously used by J. Lindenstrauss [**13**], namely *that the unit ball of X is the closed absolutely convex hull of a uniformly exposed set*, is already sufficient. We shall show that our result actually provides new examples of Banach spaces X with $\mathscr{AL}^N(X)$ dense in $\mathscr{L}^N(X)$. Concerning property (α), we obtain an alternative proof of the result given in [**5**]. Our proof is demonstrated to be more effective when dealing with different spaces. For example, we find that $\mathscr{AL}^2(X, Y)$ is dense in $\mathscr{L}^2(X, Y)$ provided that X satisfies (α), with no assumption on Y, while the proof in [**5**] requires that both X and Y satisfy (α).

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We show, with examples, that the analogous improvement fails to be true when (α) is replaced either by the above-mentioned property of Lindenstrauss, or by the Radon–Nikodým property. Finally, we obtain a new sufficient condition for the denseness of the set $\mathscr{A}\mathscr{L}_s^N(X)$ of norm attaining symmetric N-linear forms in the space $\mathscr{L}_s^N(X)$ of all continuous symmetric N-linear forms. Very little is known about this denseness (see **[2]** and **[8]** for some positive results).

2. *Results*

Let us start by recalling the definition of property (α).

Definition 1. A Banach space X has property (α) if there exist a family $\{(x_\lambda, x_\lambda^*) : \lambda \in \Lambda\} \subset X \times X^*$ and a real number ρ with $0 \leqslant \rho < 1$ satisfying the following requirements.

(1) $x_\lambda^*(x_\lambda) = \|x_\lambda^*\| = \|x_\lambda\| = 1$, for all $\lambda \in \Lambda$.

(2) $|x_\lambda^*(x_\mu)| \leqslant \rho$ for $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$.

(3) The unit ball of X is the closed absolutely convex hull of the set $\{x_\lambda : \lambda \in \Lambda\}$.

If it is necessary to be more precise, we say that X *satisfies* (α) *with constant* ρ. The typical example of a Banach space with property (α) is $\ell_1(\Gamma)$. Actually, it is easy to check that a Banach space satisfying property (α) with constant 0 must be isometric to $\ell_1(\Gamma)$ for some set Γ. Property (α) was first used by W. Schachermayer **[14]**, who proved that if X has property (α), then the set of norm attaining operators from X into any other Banach space Y is dense in the space of all bounded linear operators. Concerning multilinear forms, it was shown in **[5]** that $\mathscr{A}\mathscr{L}^N(X)$ is dense in $\mathscr{L}^N(X)$ whenever X satisfies property (α). We give next an alternative proof of this result, and extend it in the following way.

Theorem 2. *Let $X_1, \dots, X_N$ be Banach spaces, and assume that $X_1, \dots, X_{N-1}$ satisfy property* (α). *Then $\mathscr{A}\mathscr{L}^N(X_1, \dots, X_N)$ is dense in $\mathscr{L}^N(X_1, \dots, X_N)$.*

Proof. To simplify the notation, we consider the case $N = 2$. The proof for the general case is exactly the same. So let X and Y be Banach spaces, and assume that X satisfies property (α), in order to prove that $\mathscr{A}\mathscr{L}^2(X, Y)$ is dense in $\mathscr{L}^2(X, Y)$. Consider the set $\{(x_\lambda, x_\lambda^*) : \lambda \in \Lambda\}$ and the constant $0 \leqslant \rho < 1$ fulfilling the requirements in the definition of property (α).

Fix $\varphi \in \mathscr{L}^2(X, Y)$, with $\|\varphi\| = 1$ and $\varepsilon > 0$. Then φ can be identified with a bounded linear operator T from X into Y^*, namely

$$[T(x)](y) = \varphi(x, y), \qquad x \in X,\ y \in Y.$$

We clearly have $1 = \|T\| = \sup\{\|Tx_\lambda\| : \lambda \in \Lambda\}$, so we may find $\mu \in \Lambda$ such that $\|Tx_\mu\| > 1 - \delta$, where δ is chosen to satisfy

$$0 < \delta < \frac{\varepsilon}{2} \quad \text{and} \quad 1 + \rho\left(\frac{\varepsilon}{2} + \delta\right) < \left(1 + \frac{\varepsilon}{2}\right)(1 - \delta).$$

By the Bishop–Phelps theorem, there exists $y^* \in Y^*$ which attains its norm and satisfies

$$\|y^*\| = \|Tx_\mu\| \quad \text{and} \quad \|y^* - Tx_\mu\| < \delta.$$

Now define a new operator $S : X \to Y^*$ by

$$S(x) = T(x) + x_\mu^*(x)\left[\left(1 + \frac{\varepsilon}{2}\right)y^* - Tx_\mu\right].$$

We have, clearly,

$$\|S - T\| \leqslant \frac{\varepsilon}{2}\|y^*\| + \|y^* - Tx_\mu\| \leqslant \frac{\varepsilon}{2} + \delta < \varepsilon.$$

Moreover, $Sx_\mu = (1 + \varepsilon/2)y^*$, so

$$\|Sx_\mu\| = \left(1 + \frac{\varepsilon}{2}\right)\|y^*\| = \left(1 + \frac{\varepsilon}{2}\right)\|Tx_\mu\| > \left(1 + \frac{\varepsilon}{2}\right)(1 - \delta),$$

while, for $\lambda \in \Lambda$, $\lambda \neq \mu$,

$$\|Sx_\lambda\| \leqslant \|Tx_\lambda\| + |x_\mu^*(x_\lambda)| \left[\frac{\varepsilon}{2}\|y^*\| + \|y^* - Tx_\mu\|\right] \leqslant 1 + \rho\left(\frac{\varepsilon}{2} + \delta\right).$$

Our choice of δ shows that $\|S\| = \|Sx_\mu\|$, but the functional Sx_μ is a multiple of y^*, so it also attains its norm at some $y \in B_Y$; that is, $\|S\| = [Sx_\mu](y)$. Finally, if $\psi \in \mathscr{L}^2(X, Y)$ is the bilinear form associated to S, we clearly have $\|\psi\| = \psi(x_\mu, y)$, so $\psi \in \mathscr{A}\mathscr{L}^2(X, Y)$, and $\|\psi - \varphi\| = \|S - T\| < \varepsilon$. □

REMARKS 3. (1) It follows from the above theorem that $\mathscr{A}\mathscr{L}^2(\ell_1(\Gamma), Y)$ is dense in $\mathscr{L}^2(\ell_1(\Gamma), Y)$ for all Banach spaces Y, where Γ is an arbitrary set.

(2) Property (α) is very weak from the isomorphic point of view. Schachermayer [**14**] proved that every weakly compactly generated Banach space admits an equivalent norm with property (α). This result was later improved by M. V. Godun and S. L. Troyanski [**10**], who showed that every Banach space containing a long biorthogonal system can be renormed to satisfy property (α). Therefore, most Banach spaces X can be equivalently renormed to satisfy the condition that $\mathscr{A}\mathscr{L}^2(X, Y)$ is dense in $\mathscr{L}^2(X, Y)$ for all Banach spaces Y.

(3) In the proof of Theorem 2, we used the natural identification of $\mathscr{L}^2(X, Y)$ with the space $L(X, Y^*)$ of all bounded linear operators from X into Y^*. Under this identification, $\mathscr{A}\mathscr{L}^2(X, Y)$ becomes a subset (in general, a proper subset) of the set $NA(X, Y^*)$ of norm attaining operators. If X satisfies property (α), then $\overline{NA(X, Z)} = L(X, Z)$ for every Banach space Z [**14**], and therefore $\overline{NA(X, Y^*)} = L(X, Y^*)$ for every Y. However, the denseness of $NA(X, Y^*)$ is not enough to ensure that $\mathscr{A}\mathscr{L}^2(X, Y)$ is dense. For $X = L_1[0, 1]$, it was shown in [**9**] that $NA(X, X^*)$ is dense in $L(X, X^*)$, while $\mathscr{A}\mathscr{L}^2(X)$ is not dense in $\mathscr{L}^2(X)$ (see [**7**]).

Next, we shall prove the denseness of norm attaining multilinear forms under an assumption strictly weaker than property (α), which had been previously used by Lindenstrauss. More concretely, he proved the denseness of norm attaining operators under the assumption that the unit ball in the domain space is the absolutely convex closure of a uniformly exposed set [**13**, Proposition 1]. As mentioned above, the denseness of norm attaining multilinear forms does not follow directly from the result given by Lindenstrauss. However, we shall slightly modify his arguments to get the desired conclusion. Let us explain what is meant by a uniformly exposed set.

DEFINITION 4. Let X be a Banach space, and let $\{x_\lambda : \lambda \in \Lambda\} \subset B_X$. We say that this set is *uniformly exposed* if there exists another set $\{x_\lambda^* : \lambda \in \Lambda\} \subset B_{X^*}$, satisfying the following requirements.

(1) $x_\lambda^*(x_\lambda) = 1$, for all $\lambda \in \Lambda$.

(2) For every $\varepsilon > 0$, we can find $\delta > 0$, such that:

$$\text{if } \lambda \in \Lambda,\ x \in B_X, \text{ and } \operatorname{Re} x_\lambda^*(x) > 1 - \delta, \text{ then } \|x - x_\lambda\| < \varepsilon.$$

Note that each x_λ is a strongly exposed point of B_X. The fact that δ depends only on ε and not on λ accounts for the term 'uniform'. If X satisfies property (α) and $\{(x_\lambda, x_\lambda^*) : \lambda \in \Lambda\}$ fulfils the requirements in Definition 1, then it is easy to check that $\{x_\lambda : \lambda \in \Lambda\}$ is a uniformly exposed set, so X satisfies the assumption in the next statement.

THEOREM 5. *Let X be a Banach space such that B_X is the closed absolutely convex hull of a uniformly exposed set. Then $\mathscr{A}\mathscr{L}^N(X)$ is dense in $\mathscr{L}^N(X)$, for every N.*

Proof. We just consider the bilinear case. Let E be a uniformly exposed set whose closed absolutely convex hull is B_X. Given $0 < \varepsilon < 1$, we choose a decreasing sequence (δ_n) of positive numbers satisfying the following conditions:

$$4\sum_{n=1}^{\infty}\delta_n < \varepsilon < 1, \quad \text{and} \quad \lim_{n\to\infty}\frac{\eta_n}{\delta_n} = 0 \quad \text{where} \quad \eta_n = \sum_{k=n+1}^{\infty}\delta_k\,. \tag{1}$$

Now, given $\varphi \in \mathscr{L}^2(X)$ with $\|\varphi\| = 1$, we construct by induction a sequence (φ_n) in $\mathscr{L}^2(X)$. We start with $\varphi_1 = \varphi$. Assume that φ_n has been defined, and note that $\|\varphi_n\| = \sup\{|\varphi(x, y)| : x, y \in E\}$, so we may find points $x_n, y_n \in E$ such that

$$|\varphi_n(x_n, y_n)| > \|\varphi_n\| - \delta_n^2\,. \tag{2}$$

Now consider the functionals $x_n^*, y_n^* \in X^*$ associated to x_n, y_n by the definition of a uniformly exposed set. Then define φ_{n+1} by

$$\varphi_{n+1}(x, y) = \varphi_n(x, y) + \delta_n x_n^*(x)\varphi_n(x_n, y) + \delta_n y_n^*(y)\varphi_n(x, y_n). \tag{3}$$

It is easy to check by induction that

$$\|\varphi_n\| \leqslant 1 + 4\sum_{k=1}^{n-1}\delta_k \leqslant 2$$

for all n, so

$$\|\varphi_{n+1} - \varphi_n\| \leqslant 2\delta_n\|\varphi_n\| \leqslant 4\delta_n$$

and

$$\|\varphi_{n+k} - \varphi_n\| \leqslant 4\sum_{i=n}^{n+k-1}\delta_i\,, \tag{4}$$

for all n, k. It follows that the sequence (φ_n) converges in norm to a bilinear form ψ which satisfies

$$\|\psi - \varphi_n\| \leqslant 4\sum_{k=n}^{\infty}\delta_k < \varepsilon$$

for all n. In particular, $\|\psi - \varphi\| < \varepsilon$, and we now have only to show that ψ attains its norm. To this end, we shall show that the sequences (x_n) and (y_n) have norm convergent subsequences. For arbitrary n and k, we have

$$\begin{aligned}
\|\varphi_{n+1}\| &\leqslant \|\varphi_{n+k}\| + \|\varphi_{n+1} - \varphi_{n+k}\| \\
&\leqslant |\varphi_{n+k}(x_{n+k}, y_{n+k})| + \delta_{n+k}^2 + 4\eta_n \\
&\leqslant |\varphi_{n+1}(x_{n+k}, y_{n+k})| + \delta_n^2 + 8\eta_n \\
&\leqslant \|\varphi_n\| + \delta_n|x_n^*(x_{n+k})|\,\|\varphi_n\| + \delta_n|y_n^*(y_{n+k})|\,\|\varphi_n\| + \delta_n^2 + 8\eta_n,
\end{aligned} \tag{5}$$

where we have used (2), (3) and (4).

On the other hand, we have also

$$\|\varphi_{n+1}\| \geqslant |\varphi_{n+1}(x_n, y_n)| = (1 + 2\delta_n)|\varphi_n(x_n, y_n)| \geqslant (1 + 2\delta_n)(\|\varphi_n\| - \delta_n^2). \tag{6}$$

By combining inequalities (5) and (6), we get, after some computation,

$$|x_n^*(x_{n+k})| + |y_n^*(y_{n+k})| \geqslant 2 - \frac{2\delta_n^2 + 2\delta_n^3 + 8\eta_n}{\delta_n\|\varphi_n\|}.$$

By (1), the last term on the right-hand side of the above inequality tends to zero, and we get

$$\lim_n \inf_{m>n} |x_n^*(x_m)| = \lim_n \inf_{m>n} |y_n^*(y_m)| = 1.$$

Since the set $\{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\} \subset E$ is uniformly exposed by the set of functionals $\{x_n^* : n \in \mathbb{N}\} \cup \{y_n^* : n \in \mathbb{N}\}$, the last assertion implies that the sequences (x_n) and (y_n) have norm convergent subsequences $(x_{\sigma(n)}) \to x$ and $(y_{\sigma(n)}) \to y$ (see [**1**, Lemma 6] for details). In view of (2), we have

$$|\varphi_{\sigma(n)}(x_{\sigma(n)}, y_{\sigma(n)})| \geqslant \|\varphi_{\sigma(n)}\| - \delta_{\sigma(n)}^2$$

for all n, and by letting $n \to \infty$ we get $|\psi(x, y)| \geqslant \|\psi\|$, so ψ attains its norm, as required. □

EXAMPLE 6. The above theorem actually gives new examples of Banach spaces satisfying the denseness of norm attaining multilinear forms. More precisely, there are Banach spaces satisfying the assumption of the above theorem, but failing property (α) and also failing the Radon–Nikodým property. To build a concrete example, recall (see [**14**]) that any separable Banach space can be equivalently renormed with property (α), so let Y be an isomorphic copy of c_0 with property (α), and take $X = \ell_2 \oplus_1 Y$. As mentioned above, B_Y is the closed absolutely convex hull of a uniformly exposed set E. On the other hand, it is clear that the unit sphere of a Hilbert space is uniformly exposed. Since we put the ℓ_1-norm on the direct sum, we clearly have $B_X = \mathrm{co}(B_{\ell_2} \cup B_Y)$, so B_X is the closed absolutely convex hull of $S_{\ell_2} \cup E$, and it is easy to check that this set is uniformly exposed. It is clear that X fails the Radon–Nikodým property, for it contains an isomorphic copy of c_0. To see that X also fails property (α), one looks at the extreme points in its unit ball, and realizes that $\mathrm{ex}(B_X) = S_{\ell_2} \cup \mathrm{ex}(B_Y)$. Had X property (α), we should easily deduce that ℓ_2 would also satisfy property (α), which is clearly false. With some additional effort, one can also prove that the space $X = \ell_2 \oplus_p Y$ with $1 < p < \infty$ has the same properties as in the case $p = 1$.

EXAMPLE 7. The proof of the above theorem can be easily modified to deal with different spaces and show that $\mathscr{A}\mathscr{L}^N(X_1, \ldots, X_N)$ is dense in $\mathscr{L}^N(X_1, \ldots, X_N)$, provided that all the spaces $X_1, \ldots, X_N$ satisfy the assumption of the theorem. It is worth mentioning that this time we cannot expect the same result that we obtained in Theorem 2. More concretely, there are Banach spaces X, Y such that X satisfies the assumption of the theorem but $\mathscr{A}\mathscr{L}^2(X, Y)$ is not dense in $\mathscr{L}^2(X, Y)$. Just take $X = \ell_2$ and $Y = d_*(w, 1)$, the predual of the Lorentz sequence space $d(w, 1)$, where $w = (1/n)$. It was shown by W. Gowers [**11**] that $NA(Y, X^*) \equiv NA(Y, X)$ is not dense in $L(Y, X^*)$. It follows that $\mathscr{A}\mathscr{L}^2(X, Y)$ is not dense in $\mathscr{L}^2(X, Y)$. Note also that X satisfies the Radon–Nikodým property.

We conclude this paper with a sufficient condition on a Banach space X for the denseness of the set $\mathcal{A}\mathcal{L}_s^2(X)$ of norm attaining symmetric bilinear forms in the space $\mathcal{L}_s^2(X)$ of all continuous symmetric bilinear forms. Very few results in this direction are known (see [**2**, **8**]). For instance, the question of whether this denseness holds when X satisfies the Radon–Nikodým property seems to be open. We assume property (α) with a sufficiently small constant.

THEOREM 8. *Let X be a Banach space satisfying property (α) with constant $\rho < \sqrt{2}-1$. Then $\mathcal{A}\mathcal{L}_s^2(X)$ is dense in $\mathcal{L}_s^2(X)$.*

Proof. Let $\{(x_\lambda, x_\lambda^*) : \lambda \in \Lambda\}$ be the set satisfying the requirements of property (α), and let $\varphi \in \mathcal{L}_s^2(X)$ with $\|\varphi\| = 1$ and $0 < \varepsilon < 1$ be given. Since $\rho < \sqrt{2}-1$, we have $\varepsilon\rho < (\varepsilon/2)(1-\rho^2)$, so we may find $\delta > 0$ such that

$$1 + \varepsilon\rho < 1 - \delta + \frac{\varepsilon}{2}\big(1-\rho^2\big).$$

Next we find $\lambda, \mu \in \Lambda$ such that $|\varphi(x_\lambda, x_\mu)| > 1 - \delta$, and define a new bilinear form ψ by

$$\psi(x, y) = \varphi(x, y) + \frac{\varepsilon}{2} t x_\lambda^*(x) x_\mu^*(y) + \frac{\varepsilon}{2} t x_\lambda^*(y) x_\mu^*(x)$$

where t is scalar such that $|t| = 1$ and $\varphi(x_\lambda, x_\mu) = t|\varphi(x_\lambda, x_\mu)|$. It is clear that ψ is symmetric, and that $\|\psi - \varphi\| \leqslant \varepsilon$, so we are left with showing that ψ attains its norm. If $\lambda \neq \mu$, we have

$$\begin{aligned} |\psi(x_\lambda, x_\mu)| &= \left|\varphi(x_\lambda, x_\mu) + \frac{\varepsilon}{2} t + \frac{\varepsilon}{2} t x_\lambda^*(x_\mu) x_\mu^*(x_\lambda)\right| \\ &\geqslant \left| t\left(|\varphi(x_\lambda, x_\mu)| + \frac{\varepsilon}{2}\right)\right| - \frac{\varepsilon}{2}\left|x_\lambda^*(x_\mu) x_\mu^*(x_\lambda)\right| \\ &\geqslant 1 - \delta + \frac{\varepsilon}{2}\left(1-\rho^2\right), \end{aligned}$$

while if $\lambda = \mu$ we have a still better estimate:

$$|\psi(x_\lambda, x_\mu)| = |\varphi(x_\lambda, x_\mu) + \varepsilon t| \geqslant 1 - \delta + \varepsilon.$$

On the other hand, for $\alpha, \beta \in \Lambda$ with $(\alpha, \beta) \neq (\lambda, \mu)$, and $(\alpha, \beta) \neq (\mu, \lambda)$, we have

$$|\psi(x_\alpha, x_\beta)| = \left|\varphi(x_\alpha, x_\beta) + \frac{\varepsilon}{2} t x_\lambda^*(x_\alpha) x_\mu^*(x_\beta) + \frac{\varepsilon}{2} t x_\lambda^*(x_\beta) x_\mu^*(x_\alpha)\right| \leqslant 1 + \rho\varepsilon.$$

The choice of δ gives

$$|\psi(x_\alpha, x_\beta)| \leqslant |\psi(x_\lambda, x_\mu)|$$

for every $\alpha, \beta \in \Lambda$, which shows that ψ attains its norm, as required. □

REMARK 9. A routine extension of the above proof shows that $\mathcal{A}\mathcal{L}_s^N(X)$ is dense in $\mathcal{L}_s^N(X)$, provided that X satisfies property (α) with constant $\rho < \rho_N$, where $\rho_N > 0$ depends only on N and can be easily calculated. Unfortunately, $\rho_N \to 0$ as $N \to \infty$, so we have to require property (α) with constant zero if we want $\mathcal{A}\mathcal{L}_s^N(X)$ to be dense in $\mathcal{L}_s^N(X)$ for every N. This is not completely satisfactory from the point of view of renorming. By the results in [**10**], if X is a Banach space containing a long biorthogonal system, then X can be renormed to satisfy property (α) with arbitrarily small constant $\rho > 0$. Therefore, for each N we find an equivalent norm on X

with which $\mathscr{A}\mathscr{L}_s^N(X)$ is dense in $\mathscr{L}_s^N(X)$, but this norm depends on N. We wonder, for example, whether any separable Banach space can be equivalently renormed to satisfy the denseness of $\mathscr{A}\mathscr{L}_s^N(X)$ for every N.

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