HEBRON UNIVERSITY

FACULTY OF GRADUATE STUDIES

DEPARTMENT OF MATHEMATICS

$\mathcal{L}$-stable Rings and their Properties

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A dissertation submitted in partial fulfillment of the requirements for the degree of Master of Science

Hebron, Palestine

June, 2022

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Abstract

A unital ring R is called SR1 if for any element $a \in R$ and any left ideal L of R, $Ra + L = R$ implies $a - u \in L$ for some unit u in R. From this perspective, for some specific set $\mathcal{L}(R)$ of left ideals of R, the condition still hold. These rings will be called then $\mathcal{L}$-stable rings. For elements, an element $a \in R$ is $\mathcal{L}$-stable if $Ra+L = R$, $L \in \mathcal{L}(R)$, implies that $a - u \in L$ for some unit u of R. Then R is an $\mathcal{L}$-stable ring if each element of R is $\mathcal{L}$-stable. A class $\mathfrak{C}$ of rings is afforded by $\mathcal{L}$ if $\mathfrak{C} = \{\mathcal{L}\text{-stable}\}$-the class of all $\mathcal{L}$-stable rings, and $\mathfrak{C}$ is affordable if this happens for some certain set of left ideals of R. The class of all SR1 rings is a prototypical example of an affordable class of rings. Some other well-known examples of $\mathcal{L}$-stable rings are mentioned. It turns out that affordable classes of rings share many interesting properties.

الملخص

الحلقة ذات الوحدة R تسمى حلقة ذات مدى ثبات ١ $(SR1)$ إذا كان كل عنصر $a \in R$ و كل مثالي يساري L من R يحققان أنه إذا كان $Ra+L = R$ فإن $a - u \in L$ لبعض $u \in U(R)$. من هذا المنطلق، بعض مجموعات من المثاليات اليسارية $\mathcal{L}(R)$ من حلقة R لا يزال يتحقق هذا الشرط. سوف نطلق على هذه الحلقات بالحلقات الثابتة الثابتة بـ $\mathcal{L}$ $(\mathcal{L}-stable\ rings)$. أما بالنسبة للعناصر، نقول أن العنصر من الحلقة $a \in R$ ثابت بـ $\mathcal{L}$ $(\mathcal{L}-stable)$ إذا كان $Ra + L = R$، $L \in \mathcal{L}(R)$ يؤدي إلى أن $a - u \in L$ لبعض $u \in U(R)$. و بالتالي الحلقة تكون ثابتة بـ $\mathcal{L}$ إذا كان كل عنصر من عناصرها ثابت بـ $\mathcal{L}$. نقول عن الصف $\mathfrak{C}$ متأثر بـ $\mathcal{L}$ إذا كان $\{\mathcal{L} - stable\} = \mathfrak{C}-$ صف كل الحلقات الثابتة بـ $\mathcal{L}$، و $\mathfrak{C}$ قابل للتأثير إن وجدت مجموعة محددة من المثاليات اليسارية من R. صف الحلقات ذات مدى الثبات ١ هو مثال أساسي على صف حلقات قابل للتأثير. بعض الأمثلة المعروفة على صفوف حلقات قابلة للتأثير تم ذكرها. تبين أنه صفوف الحلقات القابلة للتأثير تشترك في العديد من الصفات المثيرة للاهتمام.

Acknowledgement

I would like to thank my supervisor **Dr. Ayman Horoub** for his massive support, patience and his trust in me. His guidence lead me to finish this work so perfectly. And I feel honored being his student.

I would also like to thank the committee members **Dr. Mahmoud Shalalfeh** (Internal Examiner) and **Dr. Iyad Hribat** (External Examiner) for their valuable comments and instructions.

Dedication

I dedicate this work to my wings, father (May Allah bless you in Hereafter.), mother, brothers, sisters, cousins and niblings; without them I would never fly.

To all my friends, electronic friends, colleagues and mates.

To all my students.

To all my professors, doctors, lecturers, school teachers for giving me what I'm supposed to know and more.

Contents

List of Symbols

$x :=$	x is defined to be equal to
$a \mid b$	a divides b
$\gcd(a, b)$	the greatest common divisor of a and b
$\Longrightarrow$, $\Longleftarrow$	implies, implied by
$\Longleftrightarrow$	if and only if
$\in$	belong(s) to
$\ni$	contains as element
$\subseteq$	is subset of or equal
$\subset$	is subset of
$\supseteq$	is superset of or equal
$\supset$	is superset of
$\cap$	intersection
$\cup$	union
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{H}$	the ring of quaternions over the real numbers
$\mathbb{H}(R)$	the ring of quaternions over the ring R
$\mathbb{Z}_n$	the ring of integers modulo n
$\mathbb{Z}_{(p)}$	the localization of integers at the prime ideal $\langle p \rangle$
$\overline{\mathbb{Z}}$	the ring of all algebraic integers
$\overline{\mathbb{Q}}$	the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$
$J(R)$	the Jacobson radical of a ring R
$C(R)$	the center of a ring R
$I(R)$	the idempotents of a ring R
$U(R)$	the group of units of a ring R
$\mathbb{M}_n(R)$	the $n \times n$ full matrix ring over a ring R

$\mathbb{T}_n(R)$	the $n \times n$ (upper or lower) triangular matrix ring over R
a^{-1}	the multiplicative inverse of an element a in a ring R
R/I	R modulo I
$\mathtt{l}(a)$	the left annihilator of the ring element a
$\mathtt{r}(a)$	the right annihilator of the ring element a
$\operatorname{ann}(a)$	the two-sided annihilator of the ring element a
$\subseteq^{ess}$	essential submodule
$\operatorname{End}(M)$, $\operatorname{End}_R(M)$	the endomorphism ring of a module M
R^{op}	the opposite ring of the ring R
$\oplus$	direct sum
$\cong$	isomorphism
$R[X]$	the polynomial ring over R in the set of inderminates X
$R[[X]]$	the power series ring over R in the set of inderminates X
Ra	the principal left ideal of R generated by a.
$\langle a\rangle$,(a)	the principal ideal generated by a.
$\operatorname{reg}(R)$	the set of regular elements of a ring R.
$\operatorname{ureg}(R)$	the set of unit-regular elements of a ring R.
$[a, b]$	the closed interval from a to b.
$\mathrm{C}(X)$	the ring of all continuous functions from X into $\mathbb{R}$
$\mathrm{C}^*(X)$	the ring of all bounded continuous functions from X into $\mathbb{R}$
$\operatorname{supp}(f)$	the support of the real valued continuous function f
$\operatorname{zer}(f)$	the zero set of the real valued continuous function f
$\operatorname{coz}(f)$	the cozero set of the real valued continuous function f
$A \ltimes E$	the trivial ring extension of A by E
$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$	the set $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a \in A, b \in B, c \in C, d \in D \right\}$
$\mathbb{M}_n[R_i, V_{ij}]$	the generalized upper triangular matrix over the rings R_i
$\mathbb{T}_n[R_i, V_{ij}]$	generalized upper triangular matrix over the rings R_i
$CN_n[R_i, V_{ij}]$	context-null extension of the rings R_i
$\Pi_{i=1}R_i$	product of rings, $i \in I$, (I is indexing set)
$\Pi_{i=1}a_i$	product of elements, $i \in I$, (I is indexing set)
$\overline{\mathcal{L}}$	closure of the left idealtor $\mathcal{L}$
K_p	Kaplansky's subring
$\mathcal{L}$	a left-ideal-map
$\operatorname{Hom}_R(A, B)$	the set of all homomorphisms from the left R-module A to the left R-module B
$\operatorname{diag}(a_1, a_2, \ldots, a_n)$	the diagonal matrix with entries $a_{ij} = a_i$ whenever $1 \leq i = j \leq n$

Introduction

Throughout, all rings are assumed to be unital and associative unless otherwise stated.

In 1964, in his seminal work, Hyman Bass invented the concept of stable range in his investigation of the stability properties of the general linear group in algebraic K-theory [14]. A ring R is defined to have **stable range 1** if for any $a \in R$, $Ra + L = R$, where L is an arbitrary ideal of R implies $a - u \in L$ for some unit u of R. Vaserstein has proved that this notion is left-right symmetric for rings.

In 1949, in his work on elementary divisors [70], Irving Kaplansky invented the concept of **left uniquely generated** rings, that is, if every $a \in R$ satisfying $Ra = Rb$, $b \in R$, implies $b = ua$ for some $u \in U(R)$. Lately in 2017, Nicholson defined an element a in a ring R to be left annihilator-stable (left AS element) if the following condition holds if $Ra + \mathtt{l}(b) = R$, $a, b \in R$, then $a - u \in \mathtt{l}(b)$ for some unit $u \in R$. The well-known result of Canfell [27, Corollary 4.4] applies to the rings R, and yet we conclude that a ring is left UG if and only if it is left AS, while it is not the case for elements, because it is shown that neither of the conditions AS and UG implies the other in general. Moreover, it is shown that every SR1 ring is Left UG (equivalently, Left AS).

In 2003, Song Guang-tian, Chu Cheng-hao, Zhu Min-xian defined "regular version" of the SR1 condition in [96]. A ring R has **regular stable range 1** (written $\mathrm{rsr}(R) = 1$) if every $a \in \mathrm{reg}(R)$ has stable range 1. Since this condition applies only on regular elements of the ring R, and not every element, this implies that for a ring R, we have $sr(R) = 1 \implies \mathrm{rsr}(R) = 1$. In 2002, Huanyin Chen [30, Lemma 1] proved that a ring R is partially unit-regular (that is, when regularity implies unit-regularity) if and only if R has regular stable range 1. A module M is said to have internal cancellation if, whenever $M = K \oplus N = K_0 \oplus N_0$ as modules where $K \cong K_0$, then necessarily $N \cong N_0$. in 2005, Khurana and Lam [73] called these rings **IC rings**. In 1976, G. Ehrlich [40] proved that partially unit regular rings are precisely the IC rings. For completeness, Khurana and Lam [73, Theorem 4.2] stated a short proof of the statement "R is IC $\iff$ $\mathrm{rsr}(R) = 1$". Moreover, it is shown that any left UG ring is IC ring.

More trivial condition, but larger class of rings, the class of **directly finite** rings, that is, the class in which each left unit of its rings is right unit, i.e., R is directly finite if and only if $Ra = R$, $a \in R$, implies $aR = R$. This notion is obviuosly left-right symmetric. An obvious observation is that any IC ring is DF. So we have these implications for a ring R.

$$\text{SR1} \implies \text{left UG} \implies \text{IC} \implies \text{DF}$$

The aforementioned classes of rings relations are studied under some certain conditions, like regularity, exchange, self injectivity and more. Some of the classes has a module-theoretic characterization, for example, a ring R is SR1 if and only if ${}_RR$ has the substitution property, and is IC if and only if ${}_RR$ has the internal canellation property, and DF if and only if ${}_RR$ is direcly finite module (An R-module M is called **Dedekind-finite** if $M \cong M \oplus N$ for some module N, then $N = 0$.). It is shown that any module satisfying **substitution property** (an R-module A has **substitution** if $M \cong A_1 \oplus H \cong A_2 \oplus K$ with $A \cong A_1 \cong A_2$ implies that, for a suitable submodule C of M, $M = C \oplus H = C \oplus K$ holds, here again H, K are R-modules.) is cancellable (A is said to be **cancellable** (or has the **cancellation property**) if, for any R-modules B, C , $A \oplus B \cong A \oplus C$ implies $B \cong C$)., and since cancellaation is clearly a stronger condition than internal cancellation, so for a module we obtain the hierarchy of conditions

$$\text{Substitution} \implies \text{Cancellation} \implies \text{Internal Cancellation} \implies \text{Dedekind-Finite}$$

The module theoretic characteriztion of left UG rings is still not discovered untill this day.

Lately, the concept of $\mathcal{L}$-stability, it was first declared in 2018 by Ayman Horoub in his seminal work [62] influenced by H. Bass the one who invented the concept of stable range in [14], Irving Kaplansky, who invented the concept of left UG rings in [70] and William Keith Nicholson who defined and characterized left AS rings [86]. Also with Dinesh Khurana and Tsit-Yuen Lam by their generous survey about IC rings in [73].

This thesis is set in order to discuss the details of the classes: SR1, left UG, IC and DF rings that turned to be $\mathcal{L}$-stable rings. In fact, they are the only four known classes of rings that are affordable. They, of course, may differ in some aspects. Also, they share many properties.

The skeleton of this thesis is as follows:

- **Chapter 1**: This chapter consists of preliminary results with no proofs at all. And of some specific subclasses of exchange rings like regular, π-regular rings which will be useful in our invistigations in later chapters.
- **Chapter 2**: We focus only on the four major key classes of rings, namely, SR1 rings, left UG rings, IC rings and DF rings and give sufficient information about them.
- **Chapter 3**: We introduce the notion of $\mathcal{L}$-stability and give examples of $\mathcal{L}$-stable rings and discuss their main properties.
- **Chapter 4**: We focus on the ring-theoretic constructions concerning $\mathcal{L}$-stable rings: Corners, direct products, factor rings, unital subrings, ideal extentions, polynimal rings, rings of formal power series and matrix rings.

Chapter 1

Preliminary Results and Basic Concepts

As the name of this chapter suggests. In this chapter, we recall some definitions and propositions that are worthy to mention in this context.

1.1 Basic Elementary Ring Theory and Module Theory Facts

By a **homomorphism** or a **morphism**, we mean, a structure-preserving map between two algebraic structures of the same type, once it is a one-to-one correspondence, it is called an **isomorphism**. An **epimorphism** is a morphism $f : X \mapsto Y$ that is right-cancellative in the sense that, for all structured set Z and all morphisms $g_1, g_2 : Y \mapsto Z, g_1 \circ f = g_2 \circ f \implies g_1 = g_2$. Dually, a **monomorphism** is a morphism $f : X \mapsto Y$ that is left-cancellative in the sense that, for all structured set Z and all morphisms $g_1, g_2 : Z \mapsto X, f \circ g_1 = f \circ g_2 \implies g_1 = g_2$. An **endomorphism** is a mapping from X into itself. The **endomorphism ring**, denoted by $\text{End}(R)$, the set of all homomorphisms of a ring R into itself. Addition of endomorphisms arises naturally in a pointwise manner and multiplication via endomorphism composition. Moreover, endomorphism rings always have additive and multiplicative identities, namely, the zero map and identity map respectively. An isomorphism endomorphism is called **automorphism**.

A module ${}_RM$ is called **simple** if the only submodules of M are 0 and M itself.

Lemma 1.1.1. (Schur's Lemma) Assume that ${}_RM$ and ${}_RN$ are both simple modules. If $\alpha :_R M \mapsto_R N$ is R-linear, then $\alpha = 0$ or α is an isomorphism. Also, $\text{End}_R({}_RM)$ is a division ring.

However, even if $\text{End}_R({}_RM)$ is a division ring, it not neccessary that ${}_RM$ is simple; because if $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ and $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, where D is a division ring, and e is clearly an idempotent of R, then $Re = \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}$, and $eRe = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$ while Re is not simple since it contains the ideal $I := \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$.

We say that a (left, right, two-sided) ideal of R is **left quasi-regular** if all of its elements are left quasi-regular[1]. Similarly, a (left, right, two-sided) ideal of R is **right-quasi-regular** if all of its elements are right quasi-regular. If I is a left quasi-regular left ideal of R, then $I \subseteq J(R)$. Every element of the Jacobson radical of a ring is quasiregular. If an element, $r \neq 0$, of a ring is idempotent, it cannot be a member of the ring's Jacobson radical. This is because non-zero idempotent elements cannot be quasiregular. A (right, left, two-sided) ideal I is **nil** if all elements of I are nilpotents. An ideal I is **nilpotent** if there exists $n \in \mathbb{N}$ such that $I^n = 0$. It is equivalent to say that there exists $n \in \mathbb{N}$ such that $a_1a_2a_3 \cdots a_n = 0$, $a_i \in I$. All nilpotent ideals are nil ideals. If a left or right ideal I of R is nil, then $I \subseteq J(R)$. Every nilpotent element of a ring R is left quasi-regular. The Jacobson radical of a ring does not contain nonzero idempotents. A subset of a ring is called **left $\boldsymbol{T}$-nilpotent**[2] if for every sequence of elements $\{a_1, a_2, a_3, \dots\}$ in the subset there is some positive integer n such that $a_1a_2 \cdots a_n = 0$. To be right T-nilpotent requires instead that $a_n \cdots a_2a_1 = 0$. We will call a subset **$\boldsymbol{T}$-nilpotent** if it is both left and right **$\boldsymbol{T}$-nilpotent**[3]. We say that a set $S \subseteq R$ is **locally nilpotent** if for any subset $\{s_1, s_2, \dots, s_n\} \subseteq S$, there exists an integer t, such that any product of t elements from $s_1, s_2, \dots, s_n$ is zero. Denote the sum of the nilpotent ideals of R, called the **Wedderburn radical**, by $W(R)$. we let $\mathrm{Nil}^*(R), \mathrm{Levi}(R)$, and $\mathrm{Nil}_*(R)$ denote, respectively, the **upper nilradical** (the sum of all nil ideals), the **Levitsky radical** (the sum of all locally nilpotent ideals), and the **lower nilradical**[4] (the intersection of all prime ideals). One has the containments $W(R) \subseteq \mathrm{Nil}_*(R) \subseteq \mathrm{Levi}(R) \subseteq \mathrm{Nil}^*(R) \subseteq J(R)$, where each containment may be proper. More generally, let I be a right ideal in R. The following implications hold for some common nilpotence conditions on I.[5]. Moreover, we have the following:

$$\begin{tikzcd}
\text{nilpotent} \arrow[r, Rightarrow] \arrow[drr, Rightarrow] & T\text{-nilpotent} \arrow[r, Rightarrow] & \text{left } T\text{-nilpotent} \arrow[r, Rightarrow] & \text{locally nilpotent} \arrow[r, Rightarrow] & \text{nil} \\
 & & \text{nil of bounded index} \arrow[ur, Rightarrow] & &
\end{tikzcd}$$

Let R be a ring and let $e, f \in I(R)$ The idempotents e and f are **isomorphic** (in the ring R) if $eR \cong fR$ as right R-modules. In this case we write $e \cong_R f$.[6]. The idempotents e and f are **conjugate** (in the ring R) if $f = u^{-1}eu$ for some unit $u \in U(R)$. In this case we write $e \sim_R f$. The idempotents e and f are **left associate** if $Re = Rf$. In this case we write $e \sim_l f$. **Right associate** idempotents are defined by the condition $eR = fR$, and the relation is written $e \sim_r f$. Finally, the idempotents e and f are **equivalent** if there exist invertible elements $u, v \in R$ such that $f = uev$.

[1] A ring element $a \in R$ is said to be **quasi-regular**, if $1 - a$ is a unit in R, that is, invertible under multiplication. The notions of right or left quasiregularity correspond to the situations where $1 - a$ has a right or left inverse, respectively.

[2] The "T" in "T-nilpotency" stands for "transfinite".

[3] this notion is not left-right symmetric

[4] The lower nilradical is also known as the **prime radical**.

[5] For more on nilpotency conditions, one good reference is [26].

[6] This relation is also commonly called the **Murray-von Neumann equivalence**, and the idempotents are then said to be **algebraically equivalent**.

In any ring R we have that 0 and 1 are always idempotents, and called **trivial** idempotents. Not surprisingly, $e^2 = e \in R$ with $e \neq 0, 1$ is called **nontrivial**[7] (or, **proper**) idempotent. If I is an ideal of R, and if $r + I$ is an idempotent in R/I, we say that $r + I$ can be **lifted** to R if there exists an idempotent $e^2 = e \in R$ such that $e + I = r + I$, that is if $e - r \in I$. We say that **idempotents can be lifted modulo** I, or that A is **lifting**, if every idempotent in R/I can be lifted. **units lift** modulo an ideal $I \lhd R$ if $x \in U(R/I)$ implies that $x = u + A$ for some $u \in U(R)$. This holds whenever $A \subseteq J(R)$. Letting I be a one-sided ideal of a ring R, we say that $x \in R$ is **regular** modulo I if there exists $y \in R$ such that $x - xyx \in I$. If x is a regular element modulo I, then we say that x**lifts regularly** modulo I if there exists a regular element $a \in R$ such that $x - a \in I$.[8]. when we say **isomorphic idempotents lift** modulo an ideal $I \leq R$, this means that given any $x, y \in R$ such that their images in the factor ring R/I are isomorphic idempotents, then there exist isomorphic idempotents $e, f \in R$ such that $x - e, y - f \in I$. Similarly, **conjugate idempotents lift** modulo I when given $x, y \in R$ such that their images in R/I are conjugate idempotents, then there exist conjugate idempotents $e, f \in R$ such that $x - e, y - f \in I$. A one-sided ideal I of a ring R is said to be **strongly lifting** if whenever $x^2 - x \in I$ for some $x \in R$, there is an idempotent $e \in xR$ such that $e - x \in I$. Strong lifting is left-right symmetric, in the sense that we can always replace the conclusion $e \in xR$ with $e \in Rx$, or even $e \in xRx$. Let e be a nonzero idempotent of a ring R. The set $\{e_1, \ldots, e_n\}$ of idempotents in a ring R is said to be **orthogonal** if $e_i e_j = 0$ for any $i \neq j$.

We say that a matrix with exactly one entry equal to 1 and all other entries equal to 0 is a **matrix unit** and is denoted by E_{ij} when the entry in the ith row and jth column is 1. Notice that the set of $n \times n$ matrix units $\{E_{ii}\}_{i=1}^n$ is a finite set of mutually orthogonal idempotents in the full matrix ring $\mathbb{M}_n(R)$ for any ring R, and notice that the sum $E_{11} + E_{22} + \cdots + E_{nn}$ is equal to the $n \times n$ identity matrix. In general, a finite set of mutually orthogonal idempotents whose sum is equal to the identity 1 is said to be a **complete set of orthogonal idempotents**.

The following lemma is a collection of well-known results concern aforementioned relations

Lemma 1.1.2. ([74]) Let R be a ring and let $e, f \in I(R)$.[9].

1. The following are equivalent:

    (a) $eR \cong fR$ as right R-modules (that is, $e \cong_R f$).

    (b) $Re \cong Rf$ as left R-modules.

    (c) There exist elements $a \in eRf$, $b \in fRe$ satisfying $e = ab$ and $f = ba$.

    (d) There exist elements $a, b \in R$ satisfying $e = ab$ and $f = ba$.

[7] A ring in which all idempotents are trivial is called **connected**. Any domain or local ring would be an example.

[8] One highly recommended reference in which these concepts are discussed is [74]

[9] Note that each of relations is an equivalence relation on the set $I(R)$

2. The following are equivalent:

   (a) $e \sim_R f$.

   (b) $e \cong_R f$ and $1 - e \cong_R 1 - f$.

3. The following are equivalent:

   (a) $e \sim_l f$.

   (b) $f = ue$ for some unit $u \in U(R)$.

   (c) $f = e + (1 - e)xe$ for some $x \in R$.

   (d) $ef = e$ and $fe = f$.

A ring R is called **simple** if it contains no nontrivial[10] ideals. A ring R is is called **local** if R contains only one left maximal ideal (equivalently, $R/J(R)$ is a division ring). A ring R is **semilocal**[11][12] if $R/J(R)$ is a semisimple ring. A ring R is called **semiperfect** if R is semilocal, and idempotents of $R/J(R)$ can be lifted to R. A **left perfect** ring is a semilocal ring R whose Jacobson radical, $J(R)$, is left T-nilpotent. semilocal rings with a nilpotent Jacobson radical are called **semiprimary** rings. A ring R is called **left artinian** if, whenever we have $L_1 \supseteq L_2 \supseteq \cdots \supseteq L_i \supseteq \cdots$ where each L_i is a left ideal of R, then $L_n = L_{n+1}$ for some $n \in \mathbb{N}$. A **left Noetherian** ring is a ring that satisfies the ascending chain condition on left ideals, that is, given any increasing sequence of left ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ there exists $n \in \mathbb{N}$ such that $I_n = I_{n+1}$. A ring is called **semiprimitive** (or, **Jacobson semisimple**) if its Jacobson radical is the zero ideal. A ring R is called **prime** if, for ideals A and B of R, $AB = 0$ implies $A = 0$ or $B = 0$. A ring is called **semiprime** if $A^n = 0$, $n \geq 1$ implies $A = 0$ where A is an ideal. A ring R is said to be **reduced** if R has no nonzero nilpotent elements. If M is a left R-module and $\mathtt{l}({}_RM) = 0$, then M is called a **faithful**. A ring R is called **left primitive** if it has a simple faithful left R-module.

Remark 1.1.3. ([78]) As a summing up, we have:

$$\begin{tikzcd}
 & \text{left artinian rings} \arrow[d, Rightarrow] & \\
 & \text{semiprimary rings} \arrow[d, Rightarrow] & \\
 & \text{left perfect rings} \arrow[d, Rightarrow] & \\
\text{local rings} \arrow[r, Rightarrow] & \text{semiperfect rings} \arrow[r, Rightarrow] & \text{semilocal rings}
\end{tikzcd}$$

And

$$\begin{tikzcd}
\text{division rings} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{domains} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{reduced rings} \arrow[d, Rightarrow] \\
\text{left primitive rings} \arrow[r, Rightarrow] & \text{prime rings} \arrow[r, Rightarrow] & \text{semiprime rings}
\end{tikzcd}$$

[10] Here "nontrivial" means neither 0 nor R.

[11] A commutative ring is semilocal if it has only finitely many maximal ideals.

[12] Some authers [14] call a ring R semilocal if $R/J(R)$ is artinian ring. Another equivalent definition [62] a ring R semilocal if $R/J(R)$ is a left artinian ring

The **opposite of a ring** (or simply, the **opposite ring**) is another ring with the same elements and addition operation, but with the multiplication performed in the reverse order. More explicitly, the opposite of a ring $(R, +, \cdot)$ is the ring $(R, +, \star)(= R^{op})$ whose multiplication $\star$ is defined by $a \star b = b \cdot a$ for all a, b in R. Howerver, if R is a ring, then it is not always the case that $R \cong R^{op}$ (of course it is the case whenever R is commutative).

Example 1.1.4. ([79, Ex. 1.22A, Ex. 1.22B.]) Consider the the upper triangular rings $R_1 = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ and $R_2 = \begin{bmatrix} \mathbb{Z}_{2^k} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ where $k \geq 2$. Then $R_1 \ncong R_1^{op}$ and $R_2 \ncong R_2^{op}$.

Every unital ring may be regarded as the endomorphism ring[13] of a module, thus, making all the module-theoretical statements available More precisely, we lose no generality in our assumption that R is an endomorphism ring.[14]

Homomorphic images and quotients of a ring are the same up to isomorphism. Every ideal is the kernel of a ring homomorphism and vice versa.

Proposition 1.1.5. ([52]) Every quotient ring of a ring R is a homomorphic image of R. And every homomorphic image of R is isomorphic to a quotient ring of R. Moreover, every ideal of a ring R is the kernel of a ring homomorphism of R. In particular, an ideal I is the kernel of the mapping $r \mapsto r + I$ from R to R/I.

Lemma 1.1.6. ([81], [46],[64],[73],[69], [78]) Let D be a division ring, R, S any rings, I a left ideal of R, $e^2 = e \in R$, V a countably infinite dimensional vector space, M, N left R-modules , $m, n \in \mathbb{N}$, and ω a cardinal. Then the following statements are true:

1. $\begin{bmatrix} R & R \\ 0 & R \end{bmatrix} \cong \begin{bmatrix} R & 0 \\ R & R \end{bmatrix}$.
2. If $M \cong N$ as R-modules, then $\text{End}_R(M) \cong \text{End}_R(N)$ as rings.
3. $\mathbb{T}_n(\mathbb{M}_m(R)) \cong \mathbb{M}_m(\mathbb{T}_n(R))$.
4. $\mathbb{M}_m(\mathbb{M}_n(R)) \cong \mathbb{M}_{mn}(R)$.
5. $\text{End}_R({}_RR) \cong R^{op}$. Analogously, $\text{End}_R(R_R) \cong R$.
6. $\mathbb{M}_n(R) \cong \text{End}_R(R^n)$.
7. $\mathbb{M}_\omega(D) \cong \text{End}({}_DV)$.
8. $\text{End}_R(eR) \cong eRe$.
9. $R \cong \begin{bmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix}$.[15]
10. $\mathbb{T}_n(D)/J(\mathbb{T}_n(D)) \cong \Pi_{i=1}^n D$.

[13]Endomorphism rings play an important role in both module theory and ring theory. There are numerous ring-theoretical properties of the endomorphism ring which can be reflected by properties of the module and vice versa.

[14]Non-unital rings cannot be endomorphism rings.

[15]No wonder the ring in the upper left corner eRe is called the **Peirce corner ring** of R (or simply, corner ring), and its unity is $1_{eRe} = e$.

11. $\mathbb{M}_n(R)[x] \cong \mathbb{M}_n(R[x])$.

12. $\mathbb{M}_n(D[y])[x] \cong \mathbb{M}_n(D[y,x]) \cong \mathbb{M}_n(D[x,y]) \cong \mathbb{M}_n(D[x])[y]$.

13. $\mathbb{M}_n(R)/J(\mathbb{M}_n(R)) = \mathbb{M}_n(R/J(R))$.

14. $J(R/I) = J(R)/I$ if $I \subseteq J(R)$.

15. $J(R/J(R)) = 0$.

16. $J(I) = I \cap J(R)$.

17. $R = Re \oplus R(1-e)$.

18. $\mathbb{Z}_{(p,q)}/J(\mathbb{Z}_{(p,q)}) \cong \mathbb{Z}_p \times \mathbb{Z}_q$.

19. $\mathbb{H}(\mathbb{Z}_{(3)})/J(\mathbb{H}(\mathbb{Z}_{(3)})) \cong \mathbb{M}(\mathbb{Z}_3)$.

1.2 Regular Rings

Regularity captures important ring-theoretic and module-theoretic information. To give one example, if R is the endomorphism ring of some module (for instance, by identifying R in the natural way with $\mathrm{End}(R_R)$), then regular elements correspond to those endomorphisms whose kernels and images are direct summands. As each direct sum decomposition of a module is determined by an idempotent in the endomorphism ring, we see that regular elements are intricately connected to idempotents.

We follow von Neumann [85], by starting with the formal definition of von Neumann regular rings which will be called simply, regular rings.

Definition 1.2.1. An element a in a ring R is called **regular** if $a = aba$ for some $b \in R$, and a ring R is called a **regular** ring if every element in R is regular.[16]

It is quite remarkable that that if $aba = a$, then $(ab)^2 = ab$ and $(ba)^2 = ba$, that is, both ba and ab are idempotents. Also, for each regular element $a \in R$ there exists an element $z \in R$ such that $aza = a$ and $zaz = z$ because if we let $a = axa$ and define $z = xax$, then $aza = axaxa = axa = a$ and $zaz = xaxaxax = xaxax = xax = z$.

Now we are ready for some examples.

Example 1.2.2. Any field or division ring is regular.

Proof. For any $a \neq 0$, we can choose $b = a^{-1}$ in that field or division ring. □

Lemma 1.2.3. ([16]) Every right (left) ideal of R generated by an idempotent is a direct summand of R.

Proof. Let $e^2 = e \in R$ be an idempotent. Then notice that e and $1-e$ are both idempotents, $e(1-e) = 0$ and $e + (1-e) = 1$. So every element $a \in R$ can be written as $ea + (1-e)a = a$. So we have that $eR + (1-e)R = R$. In order to show that this sum is a direct sum consider $x \in eR \cap (1-e)R$. So then $x = er_1 = (1-e)r_2$. Multiplying on the left by e we get that $x = er_1 = 0$. So then $R = eR \oplus (1-e)R$ and in particular any ideal generated by an idempotent is a direct summand. □

[16] Oftenly, b is sometimes called the **inner inverse**. Moreover, b need not be unique.

Theorem 1.2.4. ([16]) Every finitely generated ideal of a regular ring R is a direct summand.

Proof. Let $a \in R$. First we shall show that $aR = eR$ for some idempotent e. Let $a = axa$, our observation above gives us ax is an idempotent, call it e, then $aR = axaR = eaR \subseteq eR$. The other containment is clear since $eR = axR \subseteq aR$. So every one sided ideal generated by a regular element is generated by an idempotent and thus a direct summand. So consider the ideal generated by $\{a_1, a_2, \dots, a_n\}$. Consider the case where $n = 2$ and then we can use induction from there. Since every principal ideal in a regular ring is generated by an idempotent, we can rewrite $a_1R + a_2R$ as $eR + fR$ for some idempotents e and f . We will start by showing $eR + fR = eR + (1-e)fR$. Note that by multiplying out and grouping, $eR + (1-e)fR \subseteq eR + fR$. For the other containment let $er_1 + fr_2 \in eR + fR$, and notice

$$er_1 + fr_2 = er_1 + efr_2 - efr_2 + fr_2 = e(r_1 + fr_2) + (1-e)fr_2 \in eR + (1-e)fR$$

Now since $(1-e)fR$ is principally generated $(1-e)fR = e'R$ for some idempotent e'. Note that $ee' \in e(1-e)fR = 0$ and thus $(e + e')e' = e'$. Now, since $eR + fR = eR + (1-e)fR$. We can see that both eR and fR can be written as elements of $(e+e')R$ our ideal is principally generated, thus a direct summand. □

Example 1.2.5. Any product of regular rings is again regular.

Proof. Clear by elementary component-wise calculations. □

Lemma 1.2.6. ([83]) If a is an element of R such that $a \in aRa$, then there exists an idempotent e such that $aR = eR$ and $e - a \in (a - a^2)R$.

Proof. Say $a = axa$. Then $a(x + 1 - ax)a = a(xa + a - a) = a$ so the element $e = a(x + 1 - ax)$ is an idempotent such that $aR = eR$ and $e - a = (a - a^2)x$. □

Example 1.2.7. Regular domain is neccessary a division ring.

Proof. If $0 \neq a = aba \in R$, then $a - aba = a(1 - ba) = (1 - ab)a = 0$, and so $ab = ba = 1$. Hence, a and b are both units. □

The following gives two more additional ways to describe a regular ring.

Theorem 1.2.8. ([55]) For a ring R, the following conditions are equivalent:

1. R is regular.
2. Every principal right (left) ideal of R is generated by an idempotent.
3. Every finitely generated right (left) ideal of R is generated by an idempotent.

Proof. (1) $\implies$ (2): Given $x \in R$, there exists $y \in R$ such that $xyx = x$. Then xy is an idempotent in R such that $xyR = xR$.

(2) $\implies$ (3): It suffices to show that $xR + yR$ is principal for any $x, y \in R$. Now, $xR = eR$ for some idempotent $e \in R$, and since $y - ey \in xR + yR$, we see that $xR + yR = eR + (y - ey)R$. There is an idempotent $f \in R$ such that $fR = (y - ey)R$, and we note that $ef = 0$. Consequently, $g = f - fe$ is an idempotent orthogonal to e. Observing that

$fg = g$ and $gf = f$, we see that $gR = fR = (y - ey)R$, whence $xR + yR = eR + gR$. Inasmuch as e and g are orthogonal, we conclude that $xR + yR = (e + g)R$.

(3) $\implies$ (1) Given $x \in R$, there exists an idempotent $e \in R$ such that $eR = xR$. Then $e = xy$ for some $y \in R$ and $x = ex = xyx$. $\square$

Not all domains are regular because

Example 1.2.9. The ring of integers $\mathbb{Z}$ is not regular.

Proof. Since $2\mathbb{Z}$ is a principal ideal but not generated by an idempotent of $\mathbb{Z}$. $\square$

Also a regular ring need not be domain because the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ exists.

Example 1.2.10. ([3]) The rings $R[x]$ and $R[[x]]$ are never regular.

Proof. The indeterminate x is not a regular element in either ring. $\square$

A ring R is called **Abelian ring**[17][18] if all idempotents in R are central.

Proposition 1.2.11. ([2]) Any left non-zero-divisor regular element in an Abelian ring is a unit.

Proof. Let R be an Abelian ring and x a left non zero-divisor regular element of R. Let $y \in R$ be such that $xyx = x$. Then $x(1 - yx) = 0$ implies that $yx = 1$. On the other hand, since $e = xy$ is an idempotent, $x = x^2y$, and $x(1 - xy) = 0$. Hence $xy = 1$, and so x is a unit. $\square$

Theorem 1.2.12. ([79]) If R is regular, then so is the corner eRe.

Proof. Assume now R is regular. Let $a \in eRe$ and write $a = axa$ where $x \in R$. Since $ae = a = ea$, we have $a = (ae)x(ea) = aya$ where $y = exe \in eRe$. This verifies that eRe is also regular. $\square$

Regularity condition passes to matrix rings, and so regularity is Morita invariant[19] property of rings.

Theorem 1.2.13. ([104]) For any $n \in \mathbb{N}$, a ring R is regular if and only if so does $\mathbb{M}_n(R)$.

Proof. The proof is omitted—see [104, Theorem 2.14]. $\square$

Theorem 1.2.14. ([55]) The center of a regular ring is regular.

Proof. Let R be a regular ring with center S, and let $x \in S$. There exists $y \in R$ such that $xyx = x$, and we set $z = yxy$. Note that $xzx = x$. Given any $r \in R$, we have $zr = yxyr = y^2rx = y^2rxyx = yxyxry = yxry$. By symmetry, $rz = yrxy = yxry = zr$, whence $z \in S$. Therefore, S is regular. $\square$

[17] Abelian rings are also known as **normal rings**

[18] Unlike groups, where Abelian group means that the group is commutative. Abelian ring does not mean that the ring is commutative. The ring of quaternions $\mathbb{H}$ is an example of an Abelian ring that is not commutative. However, since all elements in a commutative ring are central (especially idempotent ones), we have that any commutative ring is Abelian.

[19] a ring-theoretic property $\mathcal{P}$ is said to be Morita invariant if and only if, whenever a ring R enjoys $\mathcal{P}$, so do eRe for any full idempotent $e \in R$, i.e., $ReR = R$ and $\mathbb{M}_n(R)$ (for any $n \geq 2$).

A ring R is **indecomposable** if R cannot be written as $R \cong R_1 \times R_2$ with non-zero R_1 or R_2.

Corollary 1.2.15. ([55]) A nonzero regular ring R is indecomposable (as a ring) if and only if its center is a field.

Proof. Assume that R is indecomposable. Let S denote the center of R, and let x be any nonzero element of S. By Theorem 1.2.14, $xyx = x$ for some $y \in S$, whence xy is a nonzero central idempotent in R. Since R is indecomposable, $xy = 1$. Therefore S is a field. □

In 1968, Ehrlich [39] introduced the notion of unit-regular rings as follows.

Definition 1.2.16. A ring R is called **unit-regular** if every element a in R is **unit-regular**, that is, $a = aua$ for some unit u in R.[20]

Clear that every unit-regular ring is regular. The following example proves the existence of a regular ring that is not unit regular.

Example 1.2.17. ([39]) Let M_D be an infinite-dimensional vector space over a division ring D. Then the endomorphism ring $R = \mathrm{End}(M_D)$ is regular but not unit-regular.

Proof. Let $A \in R$ be a linear transformation which is surjective but not injective. Let A^{-1} be a right inverse of A. If E is an idempotent in R such that $XE = A$ for X is a unit of R, then $E \neq I$ since A is not a unit. But $E = X^{-1}A$ and $EA^{-1} = X^{-1}$. This is impossible since E is not surjective. Thus R is not unit-regular ring. □

Example 1.2.18. Any division ring is unit-regular.

Note that Example 1.2.17 says also that not all units in a regular ring are two sided.

The following result determines exactly when the ring of integers modulo n becomes unit-regular.

Theorem 1.2.19. ([39]) For $n > 1$, the ring $\mathbb{Z}_n$ of integers modulo n is regular (hence unit-regular[21]) if and only if n is squarefree.

Proof. Clearly, $\mathbb{Z}_n$ is regular if and only if, for every integer a, there is an integer x such that $a^2x \equiv a \mod n$. This congruence has a solution for each $a \in \mathbb{Z}$ if and only if $\gcd(a^2, n)$ divides a for each $a \in \mathbb{Z}$. But this is the case if and only if n is squarefree. □

The ring $\mathbb{Z}_4$ has a unique maximal ideal $2\mathbb{Z}_4$, thus, local, but not unit-regular by Theorem 1.2.19 because 4 is not squarefree. While $\mathbb{Z}_6$ is unit-regular because 6 is squarefree and not local because 6 is not prime power. As a result, since squarefree prime powers are just the primes, we obtain:

Corollary 1.2.20. $\mathbb{Z}_n$ is a field if and only if $\mathbb{Z}_n$ is both local and unit-regular ring.

Theorem 1.2.21. ([60],[61]) A ring R is unit-regular if and only if so is $\mathbb{M}_n(R)$.

Proof. The proof is omitted—see the if part in [61, Corollary 7] and the only if part in [60, Theorem 7] □

[20] Sometimes u is said to be the **unit inner inverse** of a

[21] Note that a commutative regular ring is always unir-regular.

And so, unit-regularity is a Morita invariant property of rings.

Lemma 1.2.22. ([55]) Let I be an ideal in a regular ring R, then R is unit regular if and only if

(1) R/I is unit regular.

(2) If e and f are idempotents in I such that $(1-e)R \cong (1-f)R$, then $eR \cong fR$.

Proof. The proof is omitted —see [55, Lemma 4.15] $\square$

Lemma 1.2.23. ([55]) Let I be an ideal of a unit-regular ring S, and let R be a subring of S that contains I. If R/I is unit-regular, then so is R.

Proof. Assume that I and R/I are unit-regular, then clearly both are regular. It follows that R enjoys regularity condition, and so if e and f are idempotents in I with $(1-e)R \cong (1-f)R$, we have $(1-e)S \cong (1-f)S$, hence, $eS \cong fS$. Now, as $e, f \in I$, we get $eR = eS$ and $fR = fS$ whenever $eR \cong fR$. Henceforth, by Lemma 1.2.22, we have that R is unit-regular. $\square$

Theorem 1.2.24. ([55]) The product of two unit-regular rings is again unit-regular (and so any finite product of such family).

Proof. Let R_1 and R_2 be two unit-regular rings. We are proceeding to show that this is the case for $R = R_1 \times R_2$. To finish this, consider the ideal $R_1 \times 0$ of R, we get that $R/(R_1 \times 0) \cong R_2$. It follows by Lemma 1.2.23 that $R = R_1 \times R_2$ is unit-regular as required. $\square$

Definition 1.2.25. ([63],[25]) A ring R is said to be **casilocal** if $R/J(R)$ is unit-regular and called **semi-unit-regular** (SUR) if, in addition, $J(R)$ is lifting.

So it is clear that every SUR ring R is casilocal since it enjoys one more additional property, that is $J(R)$ is lifting. So, the class of SUR rings is contained in the class of casilocal rings. Moreover, the following example shows that this containment is proper.

Example 1.2.26. There exists a casilocal ring that is not SUR.

Proof. Consider the ring $\mathbb{Z}_{(2,3)} = \left\{\frac{a}{b} \in \mathbb{Q} \mid 2 \nmid b, 3 \nmid b\right\}$, then $\mathbb{Z}_{(2,3)}/J(\mathbb{Z}_{(2,3)}) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. Clear that $\mathbb{Z}_{(2,3)}/J(\mathbb{Z}_{(2,3)})$ is unit-regular being isomorphic copy of direct product of two unit-regular rings, and so, $\mathbb{Z}_{(2,3)}$ is casilocal. But since idempotents do not lift modulo $J(\mathbb{Z}_{(2,3)})$, we have that $\mathbb{Z}_{(2,3)}$ is not SUR. $\square$

Recall that **Wedderburn-Artin Theorem** states that a ring R is semisimple if and only if $R \cong \mathbb{M}_{n_1}(D_1) \times \mathbb{M}_{n_2}(D_2) \times \cdots \times \mathbb{M}_{n_k}(D_k)$ where each D_i is a division ring. In particular, if R is commutative, then R is a finite direct sum of fields.

As an application of Wedderburn-Artin Theorem, we have that

Example 1.2.27. Semisimple rings are unit-regular.

Proof. By Wedderburn-Artin Theorem, if R is semisimple[22] ring we have that $R \cong \mathbb{M}_{n_1}(D_1) \times \mathbb{M}_{n_2}(D_2) \times \cdots \times \mathbb{M}_{n_k}(D_k)$ for some $n_1, n_2, \ldots, n_k \in \mathbb{N}$ where each D_i is a division ring. But since any full matrix ring over a unit-regular ring is again unit-regular by Theorem 1.2.21 any division ring is unit-regular by Example 1.2.18 and unit-regularity is closed under finite product by Theorem 1.2.24, we have that R is unit-regular. $\square$

The class containment of semisimple rings in the class of unit-regular rings is proper. See [55, Example 5.15] which states that there exist unit-regular rings which contain uncountable direct sums of nonzero pairwise isomorphic left ideals.

Lemma 1.2.28. ([79]) If $a \in R$ is unit-regular, then a can be written as a product of a unit and idempotent.

Proof. Let $a \in R$ be unit-regular where $u \in U(R)$, then $a = aua$. Now, consider $(ua)^2 = uaua = u(aua) = ua$, so that ua is an idempotent, name it e such that $e = ua$, so $a = u^{-1}e$. Henceforth, a can be written as a product of unit and an idempotent. $\square$

Theorem 1.2.29. ([61]) In a unit-regular ring R, left units are right units.

Proof. If u is a unit of R such that $aua = a$, then $au = (aua)b = ab = 1$ so $ua = 1$, whence $b = u$ is the (two-sided) inverse of a. $\square$

Theorem 1.2.30. ([60]) Let R be a unit-regular ring and e be an idempotent element in R. Then eRe is unit-regular.

Proof. Let $ere \in eRe$ and $u = (ere + 1 - e)^{-1}$ be a unit. Since $(1-e)u(1-e) = 1-e$, $ereu(1-e) = 0$, $(1-e)uere = 0$, $eu(1-e) = u(1-e)-(1-e)$ and $(1-e)ue = (1-e)(u-1)$, we have $ere(e(u - u(1-e)u)e)ere = ere$ and

$$(u - u(1-e)u)e \cdot eu^{-1}e = e = eu^{-1}e \cdot e(u - u(1-e)u)e.$$

$\square$

Definition 1.2.31. A ring R is said to be **strongly regular** if for every element $r \in R$ there is some element $x \in R$ such that $r = r^2x$.

Its worthwhile noting the following remark.

Remark 1.2.32. Any commutative regular ring is strongly regular.

Theorem 1.2.33. ([67]) If R is a strongly regular, then R is unit-regular.

Proof. Let r be any element in a strongly regular ring R. Then there is some element $z \in R$ such that $rzr = r$ with $rz = zr$ according to Azumaya [10, Lemma 1]. Notice that $e = rz$ is idempotent, and that $u = r - (1-e)$ is a unit of R with inverse $v = ze - (1-e)$. It now follows that r can be written as the product of a unit and an idempotent of R by writing $r = ue$. Therefore, by 1.2.28 we have that R is unit-regular. $\square$

[22]Note that the class of semisimple rings does not contain every simple ring as one may expect. One of most famous examples is the the $\mathbb{Q}$-algebra, $A = \mathbb{Q}[x, y]/\langle xy - yx - 1\rangle$. Moreover, A is a noncommutative domain.

So we have the following irreversible implications for rings (and elements).

$$\text{commutative regular} \implies \text{strongly regular} \implies \text{unit-regular} \implies \text{regular}$$

Strong regularity property for rings is inherited by corners.

Theorem 1.2.34. ([79]) If R is a strongly regular ring, then so is the corner eRe.

Proof. Assume R is strongly regular. Let $a \in eRe$ and write $a = a^2x$ where $x \in R$. Since $ae = a = ea$, we have $a = (a^2x)e = a^2(exe) \in a^2eRe$, so eRe is strongly regular. □

Under some certain idempotency conditions, regularity implies unit-regularity. But before recognizing that condition, we need the following lemma first.

Lemma 1.2.35. ([20]) Let e and e' be isomorphic idempotents in a ring R and $e = ab$, $e' = ba$ for elements $a, b \in R$. If $bab = (bab)u(bab)$ holds for a unit $u \in U(R)$ and $c = (1 - ue'b)u(1 - e')$, $d = (1 - e')u^{-1}(1 - e)$ then $cd = 1 - e$ and $dc = 1 - e'$.

Proof. By left and right multiplication with a, from $bab = (bab)u(bab)$, we obtain $ab = abubab$ and $aba = abuba$. From the first we derive $e(1 - ue'b) = 0$ or $(1 - e)(1 - ue'b) = 1 - ue'b$. From the second we deduce

$$(1 - ue'b)ue' = uba - ubabuba = uba - ubaba = uba - uba = 0$$

Thus

$$\begin{aligned} cd &= (1 - ue'b)u(1 - e')u^{-1}(1 - e) \\ &= (1 - ue'b)[1 - e - ue'u^{-1}(1 - e)] \\ &= 1 - e - ue'b(1 - e) - (1 - ue'b)ue'u^{-1}(1 - e) \\ &= 1 - e - 0 - 0 \\ &= 1 - e \end{aligned}$$

because $e'b = be$. Finally,

$$\begin{aligned} dc &= (1 - e')u^{-1}(1 - e)(1 - ue'b)u(1 - e') \\ &= (1 - e')u^{-1}(1 - ue'b)u(1 - e') \\ &= (1 - e')u^{-1}[(1 - ue'b)u - (1 - ue'b)ue'] \\ &= (1 - e')u^{-1}(1 - ue'b)u \\ &= 1 - e' - e'bu + e'bu \\ &= 1 - e' \end{aligned}$$

□

Theorem 1.2.36. ([20])**(Ehrlich-Handelman)** A regular ring R is unit-regular if and only if for every two idempotents, $e \cong e'$ implies $1-e \cong 1-e'$.[23]

Proof. If $e \cong e'$, there are elements $a, b \in R$ with $e = ab$, $e' = ba$. Choose $u \in U(R)$, c and d as in Lemma 1.2.35. Then $cd = 1-e$ and $dc = 1-e'$ and so $1-e \cong 1-e'$. Conversely, let $a \in R$ be an arbitrary element. Since the ring is supposed to be regular, there is an element $x \in R$ such that $a = axa$. Without restriction of generality, we can assume that also $xax = x$. Clearly, ax and xa are isomorphic idempotents in R. Hence there exist elements c , $d \in R$ such that$1-ax = cd$ and $1-xa = dc$. By left and right multiplication with a and x, respectively, we obtain $cda = 0 = adc$ and $xcd = 0 = dcx$. Now consider $u = x + dcd$ and $v = a + cdc$. It is readily checked (notice that both cd and dc are idempotents) that $a = aua$ and $uv = 1 = vu$, that is, $u \in U(R)$, as desired. □

Theorem 1.2.37. ([7]) A ring R is strongly regular if and only if it is Abelian regular.

Proof. Let R be Abelian regular. Given any $x \in R$, there is $y \in R$ such that $xyx = x$. Since xy is an idempotent and, thus, is central in R, it follows that $x = (xy)x = x^2x$. Conversely, let R be strongly regular. Obviously, an element $x \in R$ can satisfy x^2 only if $x = 0$, from which we infer that R has no nonzero nilpotent elements. □

Theorem 1.2.38. ([79]) The following conditions on a ring R are equivalent:

1. R is strongly regular.
2. R is regular and reduced.
3. R is regular and Abelian.
4. Every principal right ideal of R is generated by a central idempotent.

Proof. (1) $\implies$ (2). Assume R is strongly regular. We have already observed in the last Exercise that R is reduced. For any $a \in R$, write $a = a^2x$ where $x \in R$. Then

$$(a-axa)^2 = a^2 + axa^2xa - a^2xa - axa^2 = a^2 + axa^2 - a^2 - axa^2 = 0, \text{ so } a = axa$$

(2) $\implies$ (3). Automatic since any reduced ring is Abelian.
(3) $\implies$ (4). Trivial.
(4) $\implies$ (1). Let $a \in R$. By (4), $aR = eR$ for a central idempotent $e \in R$. Write $e = ax$, $a = ey$, where $x, y \in R$. Then (1) follows since $a^2x = aax = eye = e^2y = ey = a$. □

Following [99], An element q of a ring R is called **quasi-idempotent** if $q^2 = uq$ for some central unit u of R. A ring R is called a **quasi-Boolean** ring if every element of R is quasi-idempotent. Boolean rings and any direct product of fields are quasi-Boolean.

Theorem 1.2.39. ([99]) A quasi-idempotent is a strongly regular element.

Proof. If q is a quasi-idempotent, then $(u^{-1}q)^2 = u^{-1}q$ is an idempotent. So we have that an element q is a quasi-idempotent if and only if $q = ue$, where e is an idempotent and u is a unit in R. □

Corollary 1.2.40. Quasi-Boolean rings are unit-regular.

[23] In fact, this theorem characterizes the IC rings, which is a topic to discuss in later chapters.

A ring R is said to be π-regular if for every element $a \in R$, there exist an element $b \in R$ and a positive integer n with $a^n = a^n ba^n$. Clearly, the notion of π-regularity is left-right symmetric and it generalizes the notion of regularity. An element $a \in R$ is called left π-regular if the chain $Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \cdots$ terminates, and right π-regular if the chain $aR \supseteq a^2R \supseteq a^3R \supseteq \cdots$ terminates, and is called strongly π-regular if it is both right and left π-regular. Dischinger [38] proved that if every element of R is right π-regular, then every element of R is left π-regular, that is, the notion of strong π-regularity is also left-right symmetric. It also generalizes the notion of strong regularity.

Now we start with the basic definition

Definition 1.2.41. ([10]) A ring R is said to be **π-regular** if for every element $a \in R$, there exist an element $b \in R$ and a positive integer n with $a^n = a^n ba^n$.

Fixing n at 1 implies that

Remark 1.2.42. Every regular ring is π-regular.

Definition 1.2.43. ([10]) A ring R is called **strongly π-regular** if for every element $x \in R$ there is some $y \in R$ such that $x^n y x^n = x^n$ with $yx = xy$ for some positive integer n (or equivalently, if for any $x \in R$ there exist $n \in \mathbb{N}$, $y \in R$ such that $x^n = x^{n+1}y$). [24]

By definition it is clear that

Remark 1.2.44. Every strongly π-regular ring is π-regular.

The class of strongly π-regular rings is contained properly in the class of π-regular rings.

Example 1.2.45. ([67]) The upper triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_2)$ is stronglyπ-regular but not regular and and hence a π-regular ring may not be regular in general.

Also, a unit-regular ring need bot be strongly π-regular ring as the following example exhibits.

Example 1.2.46. ([34]) There exists a unit-regular ring R which is not a strongly π-regular ring.

Proof. Let F be a field and $R = \prod_{n=1}^{\infty} \mathbb{M}_n(F)$. Then R is unit regular since every $\mathbb{M}_n(F)$ is unit-regular. We prove that R is not strongly π-regular. Assume to the contrary, then $a = (a_1, a_2, ..., a_n, ...)$ is strongly π-regular, where, $a_n = (a_{ij})_{n \times n} \in \mathbb{M}_n(F)$ for all $n \in \mathbb{N}$ with $a_{ij} = 0$ when $i \geq j$, and $a_{ij} = 1$ when $i \leq j$. Hence there exist $b \in R$ and a positive integer m such that $a^m = a^{2m}b$. It follows that $a_{m+1}^m \neq 0$ and $a_{m+1}^{2m} \neq 0$, which is impossible. $\square$

In the following we see a semi-unit-regular ring but not unit-regular.

[24]If the natural number $n \in \mathbb{N}$ and a unique $y \in R$ both depending on x such that the equality $x^n = x^{n+1}y$ is valid, then R is said to be **uniquely strongly π-regular**. **Unique π-regularity** and **Unique regularity** are defined in similar manner. Moreover, we shall not discuss these classes because in [37] it is shown that for a ring R, we have that R is uniquely π-regular if and only if R is uniquely strongly π-regular if and only if R is uniquely regular if and only if R is a division ring, thus, a "trivial" example.

Example 1.2.47. ([8]) Let K be a field and $R = K[[x]]$ be the (formal) power series ring with indeterminate x over K. Note that R is not π-regular and $J(R) = xK[[x]]$. So $R/J \cong K$ is unit-regular. Let $f(x)^2 - f(x) \in J(R)$ and $f(x) = a_0 + a_1x + \cdots \in R$. Then $a_0^2 = a_0$ and this yields that $a_0 = 0$ or $a_0 = 1$. When $a_0 = 0$, $0 - f(x) \in J(R)$. When $a_0 = 1$, $f(x) = 1 + a_1x + \cdots$ and so $1 - f(x) = a_1x + \cdots \in J(R)$. These imply that idempotents lift modulo $J(R)$.

Let A is an algebra over a field F. An element a of an algebra A over a field F is said to be **algebraic** over F if a is the root of some non-constant polynomial in $F[x]$. A is said to be an **algebraic algebra** over F if every element of A is algebraic over F. An algebra over a field F that is finite dimensional as a vector space over F is called a **finite dimensional algebra** over F. For instance, the 2×2 matrix ring over $F[x]/\langle x^2\rangle$, where F is any field, is a finite dimensional algebra. If A is a finite dimensional algebra over a field, then A is an algebraic algebra. but A being algebraic over F does not necessarily imply that A is finite dimensional over F. For example, if $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, then it is easily seen that $\dim_{\mathbb{Q}} \overline{\mathbb{Q}} = \infty$. Thus the matrix ring $A = \mathbb{M}_n(\overline{\mathbb{Q}})$ is an algebraic $\mathbb{Q}$-algebra which is not finite dimensional over $\mathbb{Q}$.

Example 1.2.48. ([31]) Any algebraic algebra over a field is strongly π-regular.

Proof. Let A be an algebraic algebra over a field F, and let $a \in A$. Then a is the root of some non-constant polynomial in $F[x]$. Thus, there exist some $a_m, \ldots, a_n \in F$ such that $a_na^n + a_{n-1}a^{n-1} + \cdots + a_ma^m = 0$, where $a_m \neq 0$. Thus,

$$x^m = -a_m^{-1}(a_na^n + \cdots + a_{m+1}a^{m+1}) = -a_m^{-1}(a_na^{n-m-1} + \cdots + a_{m+1})a^{m+1}$$

Set $b = -a_m^{-1}(a_na^{n-m-1} + \cdots + a_{m+1})a^{m+1}$. Then $a^m = ba^{m+1}$, and so A is strongly π-regular. □

Following Badawi [12]. An element $e \in R$ is said to be a **near idempotent** if e^n is an idempotent for some positive integer n. Clearly, every idempotent is a near idempotent. We say that R is **Euler** if every element of R is a near idempotent. If there exists a fixed positive integer n such that x^n is an idempotent for every $x \in R$, then R is said to be **exact-Euler**. In particular, an exact-Euler ring is Euler.

Theorem 1.2.49. ([12]) If a ring R is Euler, then R is strongly π-regular.

Proof. Let $x \in R$ and let n be a positive integer such that x^n is an idempotent. Let $y = x^n$. Then $x^{2n}y = x^n$ and $xy = yx$. Hence R is strongly π-regular. □

Let R be a commutative ring. The **Krull dimension** of R, denoted $\dim(R)$ is the supremum over all n for which there exist strictly descending chains of prime ideals $P_0 \supset P_1 \supset \ldots \supset P_n$. **Zero-dimensional**[25] ring is a ring whose Krull dimension is zero. Any integral domain which is not a field must have dimension at least one. In particular, an integral domain is a field if and only if its Krull dimension is zero.

Theorem 1.2.50. ([79]) For a commutative ring R, the following are equivalent:

1. R has Krull dimension 0.

2. $J(R)$ is nil and $R/J(R)$ is regular.

[25]Note that if R is commutative and $\dim(R)$= 0, then all prime ideals in R are maximal ideals.

3. For any $a \in R$, the descending chain $Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \cdots$ stabilizes.

4. For any $a \in R$, there exists $n \geq 1$ such that a^n is regular (i.e. such that $a^n \in a^n R a^n$).

Proof. The proof is omitted —see [79, Ex. 4.15] $\square$

Example 1.2.51. ([41]) Let $p \in \mathbb{N}$ be a prime number. We consider the localization at the prime ideal, $\langle p \rangle$, $\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \text{ is not divisible by } p \right\}$. Then $\mathbb{Z}_{(p)}$ is not strongly π-regular.

Proof. It follows from the fact that $\mathbb{Z}_{(p)}$ is an integral domain which is not a field. $\square$

Following Cang Wu and Liang Zhao in[108], a ring R is called to be an **RS** (resp., π-**RS**) ring if all regular elements (resp., π-regular elements) in R are strongly regular (resp., strongly π-regular). Let R be a ring. Then, R is strongly regular if and only if R is RS and regular. R is strongly π-regular if and only if R is π-RS and π-regular. RS rings are π-RS rings. However, we have a strongly π-regular ring (which is also a regular ring) need not be an RS ring.

Example 1.2.52. ([108]) Let $R = \mathbb{M}_n(\mathbb{C})$ for $n \geq 2$. Then it is well-known that R is a strongly π-regular ring as well as a regular ring. However, regular elements such as $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ are not strongly regular, since $A^2 X = 0 \neq A$ for any $X \in R$. Thus R is not an RS ring.

1.3 Exchange Rings

Exchange rings were first defined by Warfield [105]. Later, Nicholson [87] showed these rings are exactly those for which idempotents lift modulo all ideals. Moreover, his notion of suitability, captures at an element-wise level much of the information needed to lift idempotents. Following Nicholson [87], given a ring R, an element $a \in R$ is said to be exchange (or sometimes, suitable) if there is an idempotent $e \in a + R(a - a^2)$. This is a left-right symmetric notion.

Definition 1.3.1. ([62],[87],[35]) A ring R is called an **exchange ring** if any element a in R is **left exchange**, that is, if $Ra + L = R$, L is a left ideal of R, implies $e^2 = e \in Ra$ exists with $1 - e \in L$. Equivalently, a ring R is exchange if it satisfies any of the following conditions:

1. For any $x \in R$. There exists $e^2 = e \in R$ with $e - x \in R(x - x^2)$.
2. For any $r \in R$, there is an idempotent e in R with $e \in rR$ and $1 - e \in (1 - r)R$.
3. Idempotents lift modulo L for every left ideal L of R.[26]
4. $R/J(R)$ is an exchange ring and idempotents lift modulo $J(R)$.

The notion of exchange is left-right symmetric for rings in the case that someone would call a ring **right exchange** if it satisfies condition (1) and **left exchange** if it satisfies condition (2) in the following next Theorem 1.3.2

Theorem 1.3.2. ([35]) For any ring R the following are equivalent:

1. For any $r \in R$, there is an idempotent e in R with $e \in rR$ and $1 - e \in (1 - r)R$.
2. For any $r \in R$, there is an idempotent f in R with $f \in Rr$ and $1 - f \in R(1 - r)$.

Proof. The proof is omitted —see [35, 11.16. Characterisations I] □

The exhange property for rings passes to corners.

Theorem 1.3.3. ([87]) If R is exchange and $e^2 = e \in R$ the ring eRe is exchange.

Proof. If $x \in eRe$ choose $f^2 = f \in Rx$ such that $1 - f \in R(1 - x)$. Then $fe = f$ so $(ef)^2 = ef \in (eRe)$ and $e - ef = e(1 - f)e \in eRe(e - x)$. □

As useful implications, we have some examples on exchange rings

Example 1.3.4. ([97]) Every π-regular ring is an exchange ring.

Proof. Let R be π-regular and let $a \in R$ be given. Choose $x \in R$, $n \in \mathbb{N}$ with $a^n = a^n x a^n$. Then $g = xa^n$ and $e = g + (1 - g)a^n g$ are idempotents, where $e \in Ra$ and $(1 - e) = (1 - g)(1 - a^n g) = (1 - g)(1 - a^n) \in R(1 - a)$. □

Recall that a ring R is **semiregular** if $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$.

[26] This is equivalent to saying that every left ideal of R is strongly lifting.

Example 1.3.5. ([87]) Every semiregular ring is exchange

Proof. We may assume R is regular. If $x \in R$ choose $y \in R$ such that $xyx = x$ and write $f = yx$. If $e = f + (1-f)xf$ then $e^2 = e \in Rx$ and $1 - e = (1-f)(1-x)$. $\square$

The class containment of the class of semiregular rings in the class of exchange rings is proper because of the following

Example 1.3.6. ([31]) Let $\mathbb{Q}$ be the field of rational numbers and L be the ring of all rational numbers with odd denominators. Define

$$R(\mathbb{Q}, L) = \{(r_1, \ldots, r_n, s, s, \ldots) \mid 1 \leq n \in \mathbb{N}, r_1, \ldots, r_n \in \mathbb{Q}, s \in L\}$$

With componentwise operations, then $R(\mathbb{Q}, L)$ is a commutative exchange ring, while it is not semiregular.

Following Nicholson in [87], a ring R is called **clean** if every element of R is the sum of a unit and an idempotent. Clean rings are exchange, thus, a gate for examples, the converse is not true in general. However, it is well known that abelian exchange rings are clean.

Definition 1.3.7. ([87]) Let R be a ring. An element $a \in R$ is **clean** if we can write $a = u + e$, where $u \in U(R)$ is a unit and $e \in R$ is an idempotent.If all the elements of a ring are clean, we say the ring is a **clean** ring. If in addition, we pick u and e so that they commute, we say that a is **strongly clean**. If all the elements of a ring are strongly clean, we say the ring is a **strongly clean** ring.

Observe that a is clean if and only if $1 - a$ is clean, because if $a = u + e$ where u is a unit and e is an idempotent, then $1 - a = 1 - (u + e) = (-u) + (1 - e)$ is a sum of a unit $-u$ and an idempotent $1 - e$.

Example 1.3.8. ([92]) As examples of clean elements, we have:

- Units: $u = u + 0$.
- Nilpotents: $x = (x - 1) + 1$.
- Idempotents: $e = (2e - 1) + (1 - e)$.
- Quasi-regular: $x = -(1 - x) + 1$.

Example 1.3.9. Boolean rings, division rings, local rings.

Proof. Each consists of types of elements mentioned in Example 1.3.8. $\square$

Theorem 1.3.10. ([88]) Every strongly π-regular ring is strongly clean.[27]

Proof. Since a is strongly π-regular, there exists a natural number $n \geq 0$ such that $a^n = fw = wf$ where $f^2 = f$, $w \in U(R)$ and f, w and a all commute. If we show that $u = n - (1 - f)$ is a unit, we are done with $e = 1 - f$. Define

$$v := a^{n-1}x^{-1}f - (1 + a + a^2 + \cdots + a^{n-1})(1 - f)$$

[27] The original proof of this theorem of Burgess and Menal can be seen in [18, Proposition 2.6(iii)]

Now, $uv = vu$ and $u = af - (1-a)(1-f)$ imply that

$$\begin{aligned} uv &= (af - (1-a)(1-f))(a^{n-1}x^{-1}f - (1 + a + a^2 + \cdots + a^{n-1})(1-f)) \\ &= a^n w^{-1} f + (1-a)(1 + a + a^2 + \cdots + a^{n-1})(1-f) \\ &= f + (1-a^n)(1-f) \\ &= 1 \end{aligned}$$

because $a^n f = a^n$. Clearly e, u and a all commute. $\square$

The covnerse fails as the followng example exhibits

Example 1.3.11. ([88]) Let $R = \{\frac{m}{n} \in \mathbb{Q} \mid n \text{ is odd}\}$. Then , R is local, thus, clearly, strongly clean. But it is not strongly π-regular because $J(R)$ is not nil.

Example 1.3.12. The ring of integers $\mathbb{Z}$ is not clean.

Proof. The units of $\mathbb{Z}$ are -1 and 1, the idempotents of $\mathbb{Z}$ are 0 and 1, thus, the set of clean elements of $\mathbb{Z}$ is $\{-1, 0, 1, 2\}$ which is, obviously, not the whole ring. $\square$

Theorem 1.3.13. ([58]) Let I be an ideal of a ring R such that $I \subseteq J(R)$. Then R is clean iff the quotient ring R/I is clean and idempotents lift modulo I.

Proof. If R is clean so is R/I being an image of R. If $r^2 - r \in I$, write $r = e + u$ where $e^2 = e$ and u is a unit in R. Then $r - u^{-1}(1-e)u = u^{-1}(r^2 - r) \in I$. so $r + I$ lifts to $u^{-1}(1-e)u$. Conversely, let $\overline{x}$ denote $x + I$ in the ring R/I. If $r \in R$, write $\overline{r} = \overline{e} + \overline{u}$ where $\overline{e}^2 = \overline{e}$ and $\overline{u}$ is a unit in R/I. By hypothesis we may assume that $e^2 = e$. Since $r - e$ is a unit in R/I it follows that $r - e$ is a unit in R because $I \subseteq J(R)$. $\square$

So we note that

Corollary 1.3.14. A ring R is clean if and only if $R/J(R)$ is clean and idempotents can be lifted modulo $J(R)$.

Moreover, since idempotents lift modulo any nil ideal and since every nil ideal of a ring R is contained in its Jacobson radical, we have

Corollary 1.3.15. If N is any nil ideal of a ring R. Then R is clean if and only if the quotient ring R/N is clean.

Another type of clean elements is defined as follows

Definition 1.3.16. ([22]) We call an element a in a ring R **special clean**, if there exists a decomposition $a = e + u$, such that $aR \cap eR = 0$, where $e \in I(R)$ and $u \in U(R)$. A ring R is called **special clean**, if every element of R is special clean.

Theorem 1.3.17. ([22])**(Camillo-Khurana)** A ring R is unit regular if and only R is a special clean ring.

Proof. The proof is omitted —see [22, Theorem 1] $\square$

However, for elements, the case is more sensitive. For instance, considering the matrix ring $\mathbb{M}_2(\mathbb{Z})$, we have that $\begin{bmatrix} 12 & 5 \\ 0 & 0 \end{bmatrix}$ is unit-regular since it can decompose as follows: $\begin{bmatrix} 12 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 5 & -12 \end{bmatrix} \begin{bmatrix} 12 & 5 \\ 0 & 0 \end{bmatrix}$. While it is proven to be not clean (See [72, Example 4.5]).

Fortunately, we have that the other direction is always true.

Theorem 1.3.18. ([29]) Every special clean element in a ring is unit-regular.

Proof. Let $a \in R$ be special clean. Then there exists an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $aR \cap eR = 0$. Hence, $au^{-1} = eu^{-1} + 1$. Thus, $au^{-1}e = eu^{-1}e + e \in aR \cap eR = 0$. This yields $au^{-1}(a - u) = 0$, and so $au^{-1}a = a$. Therefore, $a \in R$ is unit-regular. $\square$

Example 1.3.19. ([87]) Every clean ring is exchange.

Proof. If $x = e + u$ where $e^2 = e$ and u is a unit then $u\left(x - u^{-1}\left(1 - e\right)u\right) = ue + u^2 - u + eu = x^2 - x$ and the result follows. $\square$

Corollary 1.3.20. ([67]) Idempotents lift modulo every left (right) ideal of a clean ring.

The class containment of the class of clean rings in the class of exchange rings is proper because of the following

Example 1.3.21. ([59]) Let k be a field, and $A = k[[x]]$ the power series ring. Let K be the field of fractions of A. Define

$$R = \{r \in \mathrm{End}(A_k) : \exists\, q \in K \text{ and } \exists\, n > 0 \text{ with } r(a) = qa \;\forall\, a \in \langle x^n \rangle\}$$

Then R is an exchange ring but not a clean ring.

However, under some certain conditions, exchange rings become clean.

Theorem 1.3.22. ([87]) An Abelian exchange ring is clean.

Proof. If R is suitable and $x \in R$ choose $e^2 = e \in Rx$ with $1 - e \in R(1 - x)$. If $e - ax$ we may assume $ea = a$ so that $axa = a$. If the idempotents are central then $xa = x(ax)a - xa(ax) = (xa)ax = a(xa)x = ax$. Similarly write $1 - e = b(1 - x)$ where $(1 - e)b = b$ and $b(1 - x) = (1 - x)b$. Then an easy calculation shows that $a - b$ is the inverse of $x - (1 - e)$. $\square$

It turns out that there is even weaker conditions for an exchange ring to be clean than Abelian. But before we reach to this result, we need some definitions.

A ring R is called **left idempotent reflexive** if $aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in I(R)$. Clearly, Abelian rings are left idempotent reflexive. A ring R is called **quasi-normal** if $ae = 0$ implies $eaRe = 0$ for $a \in N(R)$ and $e \in I(R)$ and R is said to be **semiabelian** (cf. [34]) if every idempotent of R is either **left semicentral** or **right semicentral**, that is, if for every $x \in R$, $ex = exe$ (resp., $xe = exe$). And a ring R is called Abelian if every idempotent of R is central. Clearly, an Abelian ring is semiabelian, and a semiabelian ring is quasi-normal. Moreover, we also need the fact [107, Theorem 2.14] which asserts that if I is an ideal of a ring R and idempotents can be lifted modulo I. If R is quasi-normal, then so is R/I.

Theorem 1.3.23. ([107]) The following conditions are equivalent for a ring R:

1. R is Abelian
2. R is semiabelian and left idempotent reflexive
3. R is quasi-normal and left idempotent reflexive

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) automatically.

(3) $\Longrightarrow$ (1) Let $e \in I(R)$. Since R is quasi-normal, $eR(1-e)Re = 0$. Since R is left idempotent reflexive, $eReR(1-e) = 0$, that is, $eR(1-e) = 0$, and so $(1-e)Re = 0$. Hence e is central and this shows that R is abelian. □

Theorem 1.3.22 is generalized according to the following result.

Theorem 1.3.24. ([107]) Let R be a quasi-normal ring. Then R is clean if and only if R is exchange.

Proof. For the other direction, let R be an exchange ring, then $R/J(R)$ is exchange and idempotents can be lifted modulo $J(R)$. Since $R/J(R)$ is semiprime, $R/J(R)$ is left idempotent reflexive. By Theorem 1.3.23, $R/J(R)$ is abelian. Therefore, $R/J(R)$ is clean by Nicholson 1.3.22, so by Remark 1.3.14 , R is a clean ring. □

So now, we have

Corollary 1.3.25. Let R be a semiabelian ring. Then R is clean if and only if R is exchange.

Call a ring R **potent** if idempotents can be lifted modulo $J(R)$ and every left (equivalently right) ideal not contained in $J(R)$ contains a nonzero idempotent. It turns out that the class of potent rings is larger than the class of exchange rings as the following result shows.

Theorem 1.3.26. ([87]) Every exchange ring is potent.

Proof. It suffices to show that there is a nonzero idempotent in Rx for each $x \notin J(R)$. Suppose $x \in R$ is such that $e^2 = e \in Rx$ implies $e = 0$. Given $a \in R$ choose $e^2 = e \in Rax$ such that $1-e \in R(1-ax)$. Then $e = 0$ and so $1 \in R(1-ax)$. This means $x \in J(R)$. □

The class of exchange rings is contained properly in the class of potent rings.

Example 1.3.27. ([62]) Consider the ring $S = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \cdots$, and let R be the subring of S consisting of sequences of the form $(x_1, x_2, \cdots, x_n, m, m, \cdots)$ where $n \geq 1$, $m \in Z$ and $x_i \in \mathbb{Q}$. Then, R is a non-exchange potent ring.

$$\text{semiperfect} \Longrightarrow \text{clean} \Longrightarrow \text{exchange} \Longrightarrow \text{potent}$$

A ring R in which the Jacobson radical $J(R)$ is a nil ideal and every left ideal of R which is not contained in $J(R)$ contains a nonzero idempotent is called **Zorn** ring. Replacing "right ideal" with "left ideal" yields an equivalent definition. Left or right Artinian rings, left or right perfect rings, semiprimary rings and regular rings are all examples of associative[28] Zorn rings. The ring $\mathbb{Z}_{(2)}$ of all rational numbers with odd denominators (when written in lowest terms) is exchange but not Zorn. An arbitrary Zorn ring need not π-regular according to [106, Example 20].

We enclose this chapter by a summarization of the distinguished irreversible implications.

$$\begin{array}{ccccccccc}
\text{commutative regular} & \Longrightarrow & \text{strongly regular} & \Longrightarrow & \text{unit-regular} & \Longrightarrow & \pi\text{-regular} & \Longrightarrow & \text{Zorn} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{zero-dimensional comm} & \Longrightarrow & \text{strongly}\pi\text{-regular} & \Longrightarrow & \text{clean} & \Longrightarrow & \text{exchange} & \Longrightarrow & \text{potent}
\end{array}$$

[28]Kaplansky [71], named an alternative ring in which for every non-nilpotent x there exists an element y such that xy is a non-zero idempotent a **Zorn** ring after Max Zorn, this explains why associativity assumption is not superfluous. Every associative ring is alternative. The ring of octonions $\mathbb{O}$ is an example of an alternative ring that is not associative.

Chapter 2

Four Classes of Rings

In 1964, in his seminal work, Hyman Bass invented the concept of stable range in his investigation of the stability properties of the general linear group in algebraic K-theory [14]. A ring R is defined to have **stable range 1** if for any $a \in R$, $Ra + L = R$, where L is an arbitrary ideal of R implies $a - u \in L$ for some unit u of R. Vaserstein has proved that this notion is left-right symmetric for rings.

In 1949, in his work on elementary divisors [70], Irving Kaplansky invented the concept of **left uniquely generated** rings, that is, if every $a \in R$ satisfying $Ra = Rb$, $b \in R$, implies $b = ua$ for some $u \in U(R)$. Lately in 2017, Nicholson defined an element a in a ring R to be left annihilator-stable (left AS element) if the following condition holds if $Ra + \mathtt{l}(b) = R$, $a, b \in R$, then $a - u \in \mathtt{l}(b)$ for some unit $u \in R$. The well-known result of Canfell [27, Corollary 4.4] applies to the rings R, and yet we conclude that a ring is left UG if and only if it is left AS, while it is not the case for elements, because it is shown that neither of the conditions AS and UG implies the other in general.

In 2003, Song Guang-tian, Chu Cheng-hao, Zhu Min-xian defined "regular version" of the SR1 condition in [96]. A ring R has **regular stable range 1** (written $\mathrm{rsr}(R) = 1$) if every $a \in \mathrm{reg}(R)$ has stable range 1. Since this condition applies only on regular elements of the ring R, and not every element, this implies that for a ring R, we have $\mathrm{sr}(R) = 1 \implies \mathrm{rsr}(R) = 1$. In 2002, Huanyin Chen [30, Lemma 1] proved that a ring R is partially unit-regular (that is, when regularity implies unit-regularity) if and only if R has regular stable range 1. A module M is said to have internal cancellation if, whenever $M = K \oplus N = K_0 \oplus N_0$ as modules where $K \cong K_0$, then necessarily $N \cong N_0$. in 2005, Khurana and Lam [73] called these rings **IC rings**. In 1976, G. Ehrlich [40] proved that partially unit regular rings are precisely the IC rings. For completeness, Khurana and Lam [73, Theorem 4.2] stated a short proof of the statement "R is IC $\iff$ $\mathrm{rsr}(R) = 1$"

More trivial condition, but larger class of rings, the class of **directly finite** rings, that is, the class in which each left unit of its rings is right unit, i.e., R is directly finite if and only if $Ra = R$, $a \in R$, implies $aR = R$. This notion is obviuosly left-right symmetric. An obvious observation is that any IC ring is DF. So we have these implications for a ring R.

In this chapter, we shall start discussing the strongest condition among the aforementioned and then the weaker ones. So we start with

2.1 SR1 Rings

A sequence $\{a_1, \ldots, a_n\}$ in a ring R is said to be left unimodular if

$$Ra_1 + Ra_2 + \cdots + Ra_n = R.$$

In case $n \geq 2$, such a sequence is said to be reducible if there exist $r_1, \ldots,\ r_n \in R$ such that $R(a_1 +\ r_1 a_n) + R(a_2\ +\ r_2 a_n) + \cdots + R(a_{n-1}\ +\ r_{n-1} a_n) =\ R$. This reduction notion leads directly to the definition of stable range. A ring R is said to have left stable range $\leq n$ if every left unimodular sequence of length $> n$ is reducible. The smallest such n is said to be the left stable range of R, we write simply $\mathrm{sr}_l(R) = n$. (If no such n exists, we say $\mathrm{sr}_l(R) = \infty$). The right stable range is defined similarly, and is denoted by $\mathrm{sr}_r(R)$. Vaserstein has proved that $\mathrm{sr}_l(R) = \mathrm{sr}_r(R)$ for any ring R. So, we may write $\mathrm{sr}(R)$ for this common value, and call it simply the stable range of R. In fact we need Vaserstein's result only in the case of stable range 1 and so we call the ring R is an SR1 ring.

SR1 rings have been characterized by many mathematicians, and for ease of use, we avoid the the original definition by means of unimodularity and replace it with the following one.[1]

Definition 2.1.1. ([32]) A ring R has **stable range 1 (SR1)** if it satisfies the following equivalent conditions:

1. $Ra + Rb = R$ implies that $ua + tb = 1$ where $t \in R$ and $u \in R$ is a unit.
2. $ra + b = 1$ in R implies that $a + tb$ is a unit for some $t \in R$.
3. $Ra + L = R$ where $L \subseteq R$ is a left ideal implies that $a + c$ is a unit for some $c \in L$.

In 1984, L.N. Vaserstein showed that these conditions are equivalent to their left right analogues.

Theorem 2.1.2. ([103]) If $\mathrm{sr}_r(R) = 1$, then $\mathrm{sr}_l(R) = 1$ (and, of course, conversely).

Proof. Start with $Rb + Rd = R$. Then $ab + c = 1$ for some $c \in Rd$. From $aR + cR = R$, we have a right invertible element $u = a + cx$ (for some $x \in R$). Say $uv = 1$. For $w = a + x(1 - ba)$, we have

$$\begin{aligned} w(1 - bx) &= a + x(1 - ba) - abx - xb(1 - ab)x \\ &= a + x - xba - abx - xb(u - a) \\ &= a + x - abx - xbu \\ &= a + cx - xbu \\ &= (1 - xb)u \end{aligned}$$

Therefore, for $y = (1 - bx)v$, we have $wy = 1 - xb$.

It follows that $w(b + yc) = ab + x(1 - ba)b + (1 - xb)c = ab + xbc + (1 - xb)c = 1$. Thus, $R(b + yc) = R$, with $yc \in yRd \subseteq Rd$, as desired. $\square$

[1]Further results on SR1 rings can be found, for instance, in [102], [103],[31],[42],[77],[110],[111]

And so we always have that $R \cong R^{op}$ for any SR1 ring R.

Theorem 2.1.3. ([103]) In SR1 ring, one-sided inverses are two-sided.

Proof. Let $ax = 1$. For $b = 1 - xa$ we have $Ra + Rb = R$. Hence there exists t with $u = a + tb$ left invertible. Since $bx = x - xax = x - x$ =0, $1 = ux$ so that u is also right invertible. Thus, u is a unit, and so are x and a. $\square$

As an observation we have that

Example 2.1.4. Any division ring or field is SR1.

Proof. By exhaustion, let R be a division ring, L an ideal of R, then L is either 0 or R. If $L = 0$, then $Ra + L = Ra = R$, choosing $c = 0$ implies that $a + c$ is a unit. Else if $L = R$, then $Ra + L = Ra + R = R$ for $c \neq -a$ we have that $a + c$ is always a unit. Hence, R is SR1. (Fields are treated the same way). $\square$

Now we define a module-theoretic property which has something to do with SR1 rings.

Definition 2.1.5. ([47]) We say that an R-module A has **substitution** if $M \cong A_1 \oplus H \cong A_2 \oplus K$ with $A \cong A_1 \cong A_2$ implies that, for a suitable submodule C of M, $M = C \oplus H = C \oplus K$ holds, here again H, K are R-modules.

Substitution property passes to summands and back as the following result shows.

Theorem 2.1.6. ([77]) A direct sum of modules $A \oplus D$ has the substitution property iff A and D both do.

Proof. Suppose $A \oplus D$ has the substitution property. To see that A does, consider a module $M = A \oplus B = A' \oplus C$, where $A' \cong A$. In $D \oplus M$, the submodules B and C have complements isomorphic to $A \oplus D$, so they have a common complement X. Then $X \cap M$ is a common complement for B and for C in M. Conversely, suppose both A and D have the substitution property. To check that $A \oplus D$ also does, consider a module $N = (A \oplus D) \oplus B = (A' \oplus D') \oplus C$, where $A' \cong A$ and $D' \cong D$. Then $D \oplus B$ and $D' \oplus C$ have a common complement A_0 in N. But then $A_0 \oplus B$ and $A_0 \oplus C$ must have a common complement D_0 in N. Now $A_0 \oplus D_0$ gives a common complement for B and for C in N. $\square$

The linkage between SR1 property for rings and substitution property for modules is seen through the following.

Theorem 2.1.7. ([31]) Let A be a right R-module, and let $E = \mathrm{End}_R(A)$. Then the following are equivalent:

1. E is SR1.

2. Given any right R-module decompositions $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A \cong A_2$, there exists $C \subseteq M$ such that $M = C \oplus B_1 = C \oplus B_2$.

Proof. (1) $\implies$ (2) We are given $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A \cong A_2$. Using the decomposition $M = A_1 \oplus B_1 \cong A \oplus B_1$, we have projections $p_1 : M \mapsto A_1 \cong A$, $p_2 : M \mapsto B1$ and injections $q_1 : A \cong A_1 \mapsto M$, $q_2 : B_1 \mapsto M$ such that $p_1q_1 = 1A, q_1p_1 + q_2p_2 = 1_M$. Using the decomposition $M = A_2 \oplus B_2 \cong A \oplus B_2$, we have a projection $f : M \mapsto A_2 \cong A$ and an injection $g : A \cong A_2 \mapsto M$ such that $fg = 1_A$. As $(fq_1)(p_1g) + fq_2p_2g = 1A$, there exists some $y \in E$ such that $fq_1 + fq_2p_2gy \in U(E)$. This implies that $M = \ker(f) \oplus C$, where $C = \mathrm{Im}(q_1 + q_2p_2gy)$. As $p_1(q_1 + q_2p_2gy) = 1_A$, we also get $M = \ker(p_1) \oplus C$. Therefore $M = C \oplus B_1 = C \oplus B_2$.

(2) $\Longrightarrow$ (1) Suppose that $ax + b = 1_A$ with $a, x, b \in E$. Set $M = 2A$, and let $p_i : M \mapsto A$, $q_i : A \mapsto M$ (for $i = 1, 2$) denote the projections and injections of this direct sum. Set $A_1 = q_1(A)$ and $B_1 = q_2(A)$, so that $M = A_1 \oplus B_1$ with $A_1 \cong A$. Define $f = ap_1 + bp_2$ from M to A and $g = q_1x + q_2$ from A to M. Observing that $fg = 1_A$, we get $M = \ker(f) \oplus g(A)$. Set $A_2 = g(A)$ and $B_2 = \ker(f)$, so that $M = A_2 \oplus B_2$ and $A_2 \cong A$. By assumption, $M = C \oplus B_1 = C \oplus B_2$ for some $C \subseteq M$. Let $h : A \cong A1 \cong C \mapsto M$ be the natural injection. Then $C = h(A)$. So $M = \ker(p1) \oplus h(A)$, we infer that p_1h is an isomorphism. On the other hand, $M = \ker(f) \oplus h(A)$. Hence, fh is an isomorphism. Clearly, $fh = (ap_1 + bp_2)h = (a + bp_2h(p_1h)^{-1})p_1h \in U(E)$. Therefore $a + bp_2h(p_1h)^{-1} \in U(E)$, as required. □

As another related result, we have

Theorem 2.1.8. ([31]) If $\mathrm{End}_R(M_1) \cdots , \mathrm{End}_R(M_n)$ are SR1, then so does $\mathrm{End}_R(M_1 \oplus \cdots \oplus M_n)$.

Proof. Given right R-module decompositions $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong M_1 \oplus \cdots \oplus M_n \cong A_2$, then we have $A_1 = A_{11} \oplus \cdots \oplus A_{1n}$ and $A_2 = A_{21} \oplus \cdots \oplus A_{2n}$ with $A_{1i} \cong M_i \cong A_{2i}$ $(1 \le i \le n)$. So $M = A_{11} (\oplus A_{12} \oplus \cdots \oplus A_{1n} \oplus B_1) = A_{21} \oplus (A_{22} \oplus \cdots \oplus A_{2n} \oplus B2)$ with $A_{11} \cong M_1 \cong A_{12}$. Since $\mathrm{End}_R(M_1)$ is SR1, by virtue of Theorem 2.1.7, we can find a submodule $C_1 \subseteq M$ such that $M = C_1 \oplus A_{12} \oplus \cdots \oplus A_{1n} \oplus B_1 = C_1 \oplus A_{22} \oplus \cdots \oplus A_{2n} \oplus B_2$. Likewise, we have submodules $C_2, \cdots , C_n \subseteq M$ such that $M = C_1 \oplus C_2 \oplus \cdots \oplus C_n \oplus B_1 = C_1 \oplus C_2 \oplus \cdots \oplus C_n \oplus B_2$. By Theorem 2.1.7 again, we conclude that $\mathrm{End}_R(M_1 \oplus \cdots \oplus M_n)$ is SR1. □

And so, we conclude that substitution is an ER-peoperty[2].

Corollary 2.1.9. Let M be a simple R-module, then M is substitutible.

Proof. By Schur's lemma[3], $\mathrm{End}_R({}_RM)$ is a division ring and so $\mathrm{End}_R({}_RM)$ is SR1 by Example 2.1.4, and hence M is substitutible by Theorem 2.1.7. □

Example 2.1.10. The ring of integers $\mathbb{Z}$ is not SR1.

Proof. For elements a, b and s in $\mathbb{Z}$, set $a = 2, b = 3$ and $s = -5$, then $(2)(3) + (-5) = 1$ implies that $2 - 5x \neq \pm 1$, thus, $2 - 5x \notin U(\mathbb{Z})$ for any $x \in \mathbb{Z}$. Therefore, $\mathbb{Z}$ is not SR1. □

Corollary 2.1.11. ([47])The ring $\mathbb{Z}$ of integers (as a module) fails to have the substitution property.

More generally, we have

Example 2.1.12. ([42]) The ring of algebraic integers of any finite field extension of $\mathbb{Q}$ is not SR1.

Proof. The proof is omitted —see [42, Corollary 7.7]. □

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[2]A module-theoretic property $\mathcal{P}$ is called an endomorphism ring property (**ER-property**, for short) if for any module M_R, M_R has $\mathcal{P}$ if and only if $\mathrm{End}_R(M)$ has $\mathcal{P}$ as a module over itself.

[3]Remember that Schur's lemma asserts that if ${}_RK$ and ${}_RN$ are simple modules and $\alpha :_R K \mapsto_R N$ is R-linear implies that either $\alpha = 0$ or α is an isomorphism (In particular, $\mathrm{End}(K)$ is a division ring).

However, on the other hand, we have:

Example 2.1.13. ([103]) The ring of all algebraic integers $\overline{\mathbb{Z}}$ is SR1 ring.[4]

Proof. The proof is omitted —see [103, Example 1.2] or alternatively [31, Corollary 10.1.11]. $\square$

We define another module theoretic property as follows.

Definition 2.1.14. ([77]) If A is an R-modules, A is said to be **cancellable** (or has the **cancellation property**) if, for any R-modules B, C , $A \oplus B \cong A \oplus C$ implies $B \cong C$.

Like substitution, cancellation property passes to summands as the following theorem verifies.

Theorem 2.1.15. ([77]) A module $A \oplus D$ is cancellable iff A and D themselves are.

Proof. First assume A and D are cancellable. If $(A \oplus D) \oplus B \cong (A \oplus D) \oplus C$, then we can cancel A first and then cancel D, to get $B \cong C$. Conversely, if $A \oplus D$ is cancellable, then from $D \oplus B \cong D \oplus C$, we can add A and cancel $A \oplus D$, to get $B \cong C$. This shows D is cancellable, and by symmetry the same holds for A. $\square$

Example 2.1.16. ([77]) If R is a Dedekind domain[5]. Then the module R_R is cancellable.

Proof. The proof is omitted —see [77, Proposition 3.6] or alternatively [77, Theorem 5.8]. $\square$

Theorem 2.1.17. ([35]) A substitutable module M is cancellable.

Proof. Let A, M_1, M_2, N_1, and N_2 be modules such that $A = M_1 \oplus N_1 \cong M_2 \oplus N_2$, where $M_1 \cong M \cong M_2$. Then $A = M_1 \oplus N_1 = M_3 \oplus N_3$, where $M_3 \cong M_2$ and $N_3 \cong N_2$. Since M is substitutable, this then gives $A = M_0 \oplus N_1 = M_0 \oplus N_3$ where $M_0 \cong M$ and so $N_1 \cong A/M_0 \cong N_3 \cong N_2$, as required. $\square$

In fact, Theorem 2.1.17 makes Evan's cancellation theorem crystal clear.

Corollary 2.1.18. ([43, Theorem 2]) If $\mathrm{End}({}_RM)$ is SR1, then ${}_RM$ is cancellable.

Unfortunately, the cancellation property on modules is not ER as the following exapmle illustrates.

Example 2.1.19. ([77]) The cancellation property on modules is not ER.

Proof. We work over the ring $k = \mathbb{Z}$, and use the $\mathbb{Z}$-module A constructed in [77, Example 3.2 (5)]. To be more specific, let A be the subgroup of $\mathbb{Q}$ generated by $\frac{1}{p}$, where p ranges over, say, the (infinite) set of primes $\equiv 3 \mod 4$. According to [77, Example 3.2 (5)], A is not cancellable. To compute $R = \mathrm{End}(A)$, note that any R is the restriction of an endomorphism of $\mathbb{Q}_{\mathbb{Z}}$ (since $\mathbb{Q}$ is injective over $\mathbb{Z}$), so is given by multiplication by a rational number r. But in order that $rA \subseteq A$, r must clearly be an integer. Thus, $R \cong \mathbb{Z}$, and according to Example 2.1.16, $R_R = \mathbb{Z}_{\mathbb{Z}}$ is cancellable. This shows that the cancellation property on modules is not ER. $\square$

[4] An algebraic integer is a complex number that serves to be a root for a monic polyniomal with coefficients from $\mathbb{Z}$.

[5] A **Dedekind domain** is a Noetherian integrally closed integral domain R in which every nonzero prime ideal of R is maximal.

The previous example shows also that the class of substitutable modules is contained properly in the class of cancellable modules, i.e., the following implication is irreversible:

$$\text{substitution} \implies \text{cancellation}$$

Now we back on track.

Theorem 2.1.20. ([103]) Let R be a ring and $I \subseteq J(R)$. Then $\mathrm{sr}\,(R) = \mathrm{sr}\,(R/I) = 1$.

Proof. ($\Longrightarrow$) Let $\overline{a}, \overline{b}, \overline{x} \in \overline{R} = R/I$ satisfying $\overline{ax} + \overline{b} = \overline{1}$. Since $I \subseteq J$, $ax + b$ is a unit in R. Let u be in R such that $(ax + b)u = 1$. By hypothesis, there exists $y \in R$ such that $a + buy$ is a unit. Hence, $\overline{a} + \overline{b}\overline{uy}$ is a unit.

($\Longleftarrow$) Let $a, b, x \in R$ such that $ax + b = 1$. Since $\overline{R}$ is SR1 1, there exists $\overline{y} \in \overline{R}$ such that $\overline{a} + \overline{b}\overline{y}$ is a unit. Assume that $\overline{a} + \overline{b}\overline{y}$ is a unit. Then there exists $u \in R$ such that $1 - (a + by)u \in I \subseteq J$. This implies that $a + by$ is a unit. $\square$

Which enables us to use the following useful tool.

Lemma 2.1.21. A ring R is SR1 if and only if $R/J(R)$ is SR1.

As an application of Lemma 2.1.21 we obtain

Example 2.1.22. Any local ring is SR1.

Proof. Assume that R is local and let I be the maximal ideal of R. Now since I is unique, it follows that $I = J(R)$. Maximality of I in R implies that $R/J(R)$ is a division ring. It follows by Example 2.1.4 that R is SR1. $\square$

An R-module M is called **strongly indecomposable** if the endomorphism ring $\mathrm{End}_R(M)$ is local. From which it follows that

Corollary 2.1.23. ([77]) Strongly indecomposable modules are substitutable and cancellable.

Theorem 2.1.24. ([62]) Every homomorphic image of any SR1 ring R is again SR1.

Proof. For simplicity, we prove the result for factor rings. Let R be SR1, and let $\bar{R} = R/X$ be its factor ring where X is an ideal of R. Assume that $\bar{R}\bar{a} + \bar{R}\bar{b} = R$ with $\bar{a}, \bar{b} \in \bar{R}$. Then, $\bar{r}\bar{a} + \bar{c} = \bar{1}$ where $\bar{r} \in \bar{R}$ and $\bar{c} \in \bar{R}\bar{b}$. Hence, $(ra + c) + X = 1 + X$, and then, $ra + c - 1 = x \in X$. So, $ra + (c - x) = 1$, which implies $Ra + R(c - x) = R$. And since R is already assumed to be SR1, then we have that $a - u \in R(c - x)$ for some $u \in U(R)$. That is, $a + t(c - x) = u$ for some $t \in R$, and so, $a + tc - u = tx \in X$. Thus, $\bar{a} + \bar{t}\bar{c} - \bar{u} = \bar{0}$, it follows that $\bar{a} - \bar{u} = -\bar{t}\bar{c} \in \bar{R}\bar{b}$ where $\bar{u} \in U(\bar{R})$. Therefore, $\bar{a}$ is an SR1 element in $\bar{R}$ and so $\bar{R}$ is SR1 as promised. $\square$

The following couple of observations are due to L. N. Vaserstein [103]

Theorem 2.1.25. ([103]) If R is the direct product of a family $\{R_\alpha\}$ of rings, then R is SR1 ring if and only if each R_α is SR1.

Proof. By component-wise calculations —see [103, Theorem 2.3]. $\square$

Theorem 2.1.26. ([103]) For any natural number n, a ring R is SR1 ring if and only if the full matrix ring over R, $\mathbb{M}_n(R)$ is SR1.[6]

Proof. Using Theorem 2.1.8, if $R \cong \mathrm{End}_R(R)$ is SR1, then so is $\mathrm{End}_R({}_RR^n) \cong \mathrm{End}_R(R^n_R) \cong \mathbb{M}_n(R)$. The converse is now clear by Theorem 2.1.6. □

Let **c** be a condition on an element in a ring R. We say that **c** is a **translation invariant** if, whenever $a \in R$ satisfies the condition **c**, then ua and au both satisfy **c** for every unit $u \in R$.

Lemma 2.1.27. ([86])The following statments hold:

1. SR1 condition is translation invariant.
2. Unit-regularity condition is translation invariant.

Proof.
1. Let a be SR1. If $Rua + Rb = R$, then $Ra + Rb = R$ so $a - v \in Rb$, v a unit. Hence $ua - uv \in Rb$. As to au, $Rau + Rb = R$ implies $Ra + Rbu^{-1} = R$, so $a - w \in Rbu^{-1}$, w a unit. Thus $au - wu \in Rb$.
2. Let a be unit-regular. Write $a = vf$ where $v \in U(R)$ and $f^2 = f$. Then $ua = (uv)f$ shows that ua is unit-regular. An analogous argument shows that au is unit-regular.

□

Theorem 2.1.28. ([86]) If a is unit-regular then a is SR1.

Proof. If a is unit-regular write $a = ve$, $e^2 = e$, v a unit. So it suffices to show that e is SR1. If $Re + Rb = R$, $b \in R$, we need a unit u such that $e - u \in R$. Let $1 - re \in Rb$ where $r \in R$, and define $u = 1 - (1 - e)re$. Then u is a unit, and $e - u = (e - 1) + (1 - e)re = (e - 1)(1 - re) \in Rb$. □

Note that the class of unit-regular rings is contained properly in the class of SR1 rings since the ring element $\bar{2} \in \mathbb{Z}_4$ is an SR1, but is not unit-regular.

Example 2.1.29. If R is a Boolean ring, then R is SR1.

Proof. Let R be Boolean ring, then if $e \in R$, we have that e is an idempotent, that is, $e = e^2$ and each idempotent is a unit-regular element since $e = e^2 = e \cdot 1 \cdot e$, thus, e is an SR1 element. Therefore, R is SR1. □

In view of unit-regularity, we can see that any division ring is SR1 since it consists of 0 and units, 0 is an idempotent, and so unit-regular, thus, SR1. Units are unit-regular and so SR1.

Theorem 2.1.30. (Bass)([44]) Any semilocal ring is SR1.

Proof. By definition, if R is semilocal, then $R/J(R)$ is semisimple, and so $R/(J)$ is unit-regular, thus, $R/(J)$ is SR1, this is equivalent to saying that R is SR1. □

[6] This result can also be concluded from Vasrstein's formula ([102, Theorem 3]): $\mathrm{sr}(\mathbb{M}_m(R)) = 1 + \lfloor \frac{\mathrm{sr}(R)-1}{m} \rfloor$ for any ring R and $m \geq 1$.

Example 2.1.31. Any One-sided artinian ring (hence artinian ring), semiprimary ring, left or right prefect ring or semiperfect ring is SR1 ring.

Proof. By Remark 1.1.3 and Theorem 2.1.30. $\square$

Corollary 2.1.32. Any ring with finitely many elements is SR1.

Proof. Clear since any ring with finitely many elements is artinian, thus, SR1 by Example 2.1.31. (e.g., the ring of integers $\mathbb{Z}_n$ modulo n)[7] $\square$

Recall that a ring R is casilocal if $R/J(R)$ is unit-regular.

Theorem 2.1.33. (Horoub)[63] If R is casilocal, then R is SR1.

Proof. By definition, if we assume R to be casilocal, then $R/J(R)$ is unit-regular, thus, SR1 by Theorem 2.1.28. It follows by Lemma 2.1.21 that R is SR1. $\square$

The converse of Horoub's Theorem 2.1.33 fails to be true in general; because the existence of an SR1 integral domain which is not a field[8] is guaranteed by Theorem [42, Theorem 4.4], and so, the class containment of casilocal rings in SR1 rings is proper.

The following theorem shows that, in fact, SR1 elements form a multiplicative submonoid of a ring R.

Theorem 2.1.34. ([32]) If R is any ring, the product of SR1 elements is again SR1.

Proof. Let a and a' be stable, and assume that $ra'a+b = 1$ in R. Since a is SR1, it follows that $a + tb = u \in U(R)$ for some $t \in R$. Hence $1 = ra'(u - tb) + b = ra'u + xb$, $x \in R$. Conjugating by u gives $1 = ura' + uxbu^{-1}$. As a' is SR1 we obtain $a' + t'bu^{-1} = u_1 \in U(R)$ where $t' \in R$.Hence $a'u + t'b = u_1u$ so, since $u = a + tb$, $a'a + (a't + t')b = u_1u \in U(R)$, proving that aa' is SR1. $\square$

The following theorem shows that SR1 condition passes to corner.

Theorem 2.1.35. ([103]) If R is SR1 ring and $p^2 = p \in R$, then pRp is also a SR1 ring.

Proof. Let a and b be in $pRp = R'$ and $R'a + R'b = R'$. Consider $a + 1 - p$ and b in R. We have $R'(1 - p) = 0$, so $R(a + 1 - p) + Rb \supseteq R'a + R'b \ni p$. On the other hand, $(1 - p)a = 0 = (1 - p)b$. So $R(a + 1 - p) + Rb \ni (1 - p)(a + 1 - p) + (1 - p)b = 1 - p$. Thus, $R(a + 1 - p) + Rb \ni p + 1 - p = 1$. Since R is SR1, there is t in R such that $R(a + tb + 1 - p) = R$. We have

$$(1 - (1 - p)tb)(1 + (1 - p)tb) = 1 = (1 + (1 - p)tb)(1 - (1 - p)tb)$$

So, $1 - (1 - p)tb$ is a unit of R, hence

$$R = R(a + tb + 1 - p)(1 - (1 - p)tb) = R(a + ptb + 1 - p)$$

Therefore, $R'(a + ptpb) = R'$.

$\square$

[7] Note that the class of SR2 rings (rings with stable range 2) is not closed under homomorphic images as $\mathbb{Z}$ is SR2 but its homomorphic image $\mathbb{Z}_n$ is SR1.

[8] Note that regular integral domains are always fields.

And so, SR1 is a Morita invariant property for rings.

Theorem 2.1.36. ([33]) A regular ring R is SR1 if and only if it is unit regular.

Proof. Assume that R is SR1 and let $a \in R$. Since R is regular,there exists $x \in R$ such that $axa = a$. Clearly, $ax + (1 - ax) = 1$. By the assumption on R, there exists $y \in R$ such that $u = a + (1 - ax)y$ is invertible. Therefore, $axu = ax(a + (1 - ax)y) = axa = a$. It follows that $ax = au^{-1}$ from which we have $au^{-1}a = axa = a$. Hence, R is unit-regular. The converse is already proved in Theorem 2.1.28. $\square$

Definition 2.1.37. ([88]) A module ${}_RM$ is said to satisfy **Fitting's lemma** (or, **fitting module**) if, for all $\alpha \in \mathrm{End}({}_RM)$, there exists an integer $n \geq 1$ such that $M = M\alpha^n \oplus \ker(\alpha^n)$.

Theorem 2.1.38. ([6]) An R-module M satisfies Fitting's Lemma if and only if $\mathrm{End}(M)$ is strongly π-regular.

Proof. The proof of this theorem is omitted —see [6, Proposition 2.3] $\square$

Example 2.1.39. ([5]) Every strongly π-regular ring is SR1.[9]

Proof. The proof is omitted —see [5, Theorem 4] or alternatively [100, Theorem 5.23]. $\square$

Corollary 2.1.40. If ${}_RM$ is a left R-module satisfying Fitting's lemma then ${}_RM$ substitutable.

Let A be a ring and E be an A-module. The **trivial ring extension** of A by E (also called the **idealization** of E over A) is the ring $R = A \ltimes E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. The jacobson radical of $A \ltimes E$ is $J(A \ltimes E) = J(A) \ltimes E$. Moreover, $(A \ltimes E)/(J(A) \ltimes E) \cong A/J(A)$.

Theorem 2.1.41. ([41]) Let A be a ring and, E be an A-module, and let $R = A \ltimes E$ be the trivial ring extension of A by E. Then, R is SR1 if and only if so does A.

Proof. Since $(A \ltimes E)/(J(A) \ltimes E) \cong A/J(A)$, it follows that by Lemma 2.1.21 that R is SR1 if and only if so does A. $\square$

Theorem 2.1.42. Let R be the polynomial ring $R = S[x]$ over the ring S. If R is SR1, then so is S.

Proof. If R is SR1, then so is its factor ring $R/\langle x\rangle$ using Theorem 2.1.24. Hence, S is SR1 because $S \cong R/\langle x\rangle$. $\square$

The converse of Theorem 2.1.42 fails as the following example shows

Example 2.1.43. The polynomial ring over SR1 ring need not be SR1 in general.

Proof. According to [102, Theorem 8], for any field $F \subseteq \mathbb{R}$, $\mathrm{sr}(F[x_1, x_2, \ldots, x_n]) = n + 1$. In particular, $\mathrm{sr}(F[x]) = 2 \neq 1$. $\square$

[9] This result can also be seen in [19]. In fact there is two more classes lie strictly between strongly π-regular rings and SR1 rings. [19, Corollary 4.1(1)] asserts that every strongly π-regular ring is feckly polar, but the converse is not. The localization $\mathbb{Z}_{(2)}$ of integers at prime 2 is a pseudopolar ring, but it is not strongly π-regular. While [19, Corollary 4.2] states that every feckly polar ring is SR1. More explicitly, $\{\text{strongly } \pi\text{-regular}\} \subseteq \{\text{pseudopolar rings}\} \subseteq \{\text{feckly polar rings}\} \subseteq \{\text{SR1}\}$

Definition 2.1.44. ([76]) A ring R is called **uniquely morphic** if for any element a in the ring R there exists a unique element b in the ring R such that $Ra = \mathtt{l}(b)$ and $\mathtt{l}(a) = Rb$.

Uniquely morphic rings are fully classified up to isomorphism according to the following theorem.

Theorem 2.1.45. ([76]) Any uniquely morphic ring R is one of the following five types:

1. R is a division ring.

2. R is a Boolean ring.

3. $R \cong \mathbb{Z}_2[x]/\langle x^2 \rangle$

4. $R \cong \mathbb{Z}_4$

5. $R \cong \mathbb{M}_2(\mathbb{Z}_2)$

Proof. The proof is omitted—see [76, Theorem 7]. □

So we observe that

Example 2.1.46. Any uniquely morphic ring is SR1.

Proof. By Theorem 2.1.45, If a ring R is division ring, then it is SR1 by Example 2.1.4. Else, if R is Boolean ring, then it is SR1 by Example 2.1.29. Else, if $R \cong \mathbb{Z}_2[x]/\langle x^2 \rangle = \left\{\bar{0}, \bar{1}, \bar{x}, \overline{x+1}\right\}$, then it is semilocal, thus, SR1 by Theorem 2.1.30. Alternatively, R is finite, thus, SR1 by Remark 2.1.32. Else, if $R \cong \mathbb{Z}_4$, then it has a unique maximal, namely, $\{\bar{0}, \bar{2}\}$, and so, local, thus, SR1 by Example 2.1.22. Alternatively, R is finite, thus, SR1 by Remark 2.1.32. Finally, SR1 condition passes to matrix ring, then since $\mathbb{Z}_2$ is field (and hence, SR1) we have that $\mathbb{M}_2(\mathbb{Z}_2)$ is again SR1. Alternatively, R is finite, thus, SR1 by Remark 2.1.32. □

Theorem 2.1.47. ([107]) Quasi-normal exchange rings are SR1.

Proof. Let R be a quasi-normal exchange ring. Then $R/J(R)$ is exchange with all idempotents central by Theorem 1.3.23, so by [110, Theorem 6], $R/J(R)$ has stable range 1. Therefore, R is SR1. □

It turns out that Theorem 2.1.47 is very generous because it implies directly that

Corollary 2.1.48. The following are true:

1. Semiabelian exchange rings are SR1.

2. Quasi-normal clean rings are SR1.

3. Quasi-normal π-regular rings are SR1.

4. Abelian exchange rings are SR1.

5. Abelian clean rings are SR1.

6. Commutative exchange rings are SR1.

7. Commutative clean rings are SR1.

8. Commutative π-regular rings are SR1.

Also, since strongly π-regular rings are precisely the reduced π-regular rings as seen in [80, Lemma 4], we have that strongly π-regular rings are Abelian π-regular rings, and so Quasi-normal exchange and so SR1 by Theorem 2.1.47, thus, another proof of Example 2.1.39

SR1 rings can be characterized in terms of unit lifting, before proving this, we need this lemma.

Lemma 2.1.49. ([54]) Let a, b, c be elements of a ring R, such that $ab + c = 1$. If there exists $x \in R$ such that $a + cx$ is invertible, then there exists $y \in R$ such that $b + yc$ is invertible.

Proof. Set $u = a + ex$, and set $v = b + (1 - bx)u^{-1}c$ and $w = a + x(1 - ba)$. Now, observe that:

$$va = ba + (1 - bx)u^{-1}ca \tag{1}$$

$$vx = bx + (1 - bx)u^{-1}(u - a) = 1 - (1 - bx)u^{-1}a \tag{2}$$

$$vx(1 - ba) = (1 - ba) - (1 - bx)u^{-1}(1 - ab)a = 1 - ba - (1 - bx)u^{-1}ca \tag{3}$$

Adding equations (1) and (3) yields $vw = 1$. Next, observe that:

$$wb = ab + xb(1 - ab) = ab + xbc \tag{4}$$

$$\begin{aligned} w(1 - bx) &= a + x(1 - ba) - abx - xbcx \\ &= a + (1 - ab)x - xb(a + cx) \\ &= a + cx - xbu = (1 - xb)u \end{aligned} \tag{5}$$

$$w(1 - bx)u^{-1}c = (1 - xb)c \tag{6}$$

Adding equations (4) and (6) yields $wv = ab + c = 1$. □

Theorem 2.1.50. ([95]) Let R be a ring. Then the following are equivalent:

1. R is SR1.

2. Every left unit lifts modulo every left principal ideal.

3. Every right unit lifts modulo every right principal ideal.

Proof. (1) $\implies$ (2) We assume R is SR1. Let $a, b, c \in R$ such that $ab - 1 \in Rc$ i.e b is a left unit modulo the left principal ideal Rc. We show that there exists a left unit $u \in R$ such that $b - u \in Rc$. Let $x \in R$ such that $ab - 1 = xc$. Then $ab - xc = 1$. Since R is SR1, from the above Lemma 2.1.49, there exists $t \in R$, $u \in U(R)$ such that $b - txc = u$. Therefore $b - u \in Rc$ where u is invertible (and hence left invertible) in R.

(2) $\implies$ (1) We show that R is SR1. Let $a, b, c \in R$ such that $ab + c = 1$. Then $ab - 1 \in Rc$. So by our hypothesis, there exists a left unit $u \in R$ such that $b - u \in Rc$. Then from Lemma 2.1.49 we have that in R every left unit is a right unit and hence invertible in R. Thus $b - u = xc$ for some $x \in R$, $u \in U(R)$ i.e $b + (-x)c = u \in U(R)$. Therefore from Lemma 2.1.49, R is SR1. □

2.2 Left UG Rings

In this section we shall discuss some basics about left uniquely generated rings (left UG Rings).[10]. These rings are invented by Irving Kaplanky [70].

We state the condition for rings in which a ring must be left UG.

Definition 2.2.1. ([73]) An element a in a ring R is called **left uniquely generated (left UG)** if $Ra = Rb$, $b \in R$, implies $b = ua$ for some $u \in U(R)$, and R is called a **left UG** ring if every element in R is left UG.

As an observation, Kaplansky observed that

Theorem 2.2.2. ([70]) Let R be a ring in which all right divisors of 0 are in the radical. Then $aR = bR$ implies that a, b are right associates.

Proof. We have $a = by$, $b = ax$, so $a = axy$. If $a, b = 0$ there is nothing to prove. Otherwise $a\,(1 - xy) = 0$ shows that $1-xy$ is in the radical, whence x and y are units. □

The following example is prototypical.

Example 2.2.3. The ring of integers $\mathbb{Z}$ is a left UG ring.

Proof. We know that $n\mathbb{Z} = m\mathbb{Z}$ iff $n = \pm m$. Now, since $\pm 1 \in U(\mathbb{Z})$, this implies that $\mathbb{Z}$ is a left UG ring. □

As a commutative non-example is

Example 2.2.4. ([13]) Let $R = \mathrm{C}([0, 3])$, the ring of continuous real-valued functions on the real interval $[0, 3]$. Certainly R is a commutative ring whose identity element is the constant function 1. Note that $R^{\times} = \{f \in R : f(t) \neq 0 \;\forall\, t \in [0, 3]\}$. Consider the following three functions in R:

$$a(t) = \begin{cases} 1-t & : t \in [0,1] \\ 0 & : t \in [1,2] \\ t-2 & : t \in [2,3] \end{cases}, b(t) = \begin{cases} 1-t & : t \in [0,1] \\ 0 & : t \in [1,2] \\ 2-t & : t \in [2,3] \end{cases} \text{ and } c(t) = \begin{cases} 1 & : t \in [0,1] \\ 3-2t & : t \in [1,2] \\ -1 & : t \in [2,3] \end{cases}$$

Clearly $(a) = (b)$ since $c(t)a(t) = b(t)$ and $c(t)b(t) = a(t)$. However, there is no unit $u(t) \in R$ with $a(t)u(t) = b(t)$. Indeed, if $a(t)u(t) = b(t)$, then it must be the case that $u(0) = 1$ and $u(3) = -1$. By the Intermediate Value Theorem, since $u \in R$ is continuous, $u(t_0) = 0$ for some $t_0 \in (0, 3)$, whence $u \notin R^{\times}$.

And so, a commutative ring need not be left UG.

[10]Further results on left UG rings can be found, for instance, in [86], [27], [28], [70], [13], [95],[111], [82], [9], [109]

Theorem 2.2.5. ([27]) **(Canfell's Theorem)**. For any ring R, the following are equivalent:

1. If $Ra + \mathtt{l}(a) = R$, $a, b \in R$, then $a - u \in l(b)$ for some unit $u \in R$.
2. R is left UG.
3. If $Ra = Rb$, $a, b \in R$, then $a = vb$ for some left unit $v \in R$.

Proof. The proof is omitted —see[27, Corollary 4.4] or alternatively, [86, Theorem 5] □

Also, noncommutative rings are not so far from being left UG as the following example shows.

Example 2.2.6. ([27]) Let $R = \mathbb{Z}[x, y]/\langle y^2, yx\rangle$. Then R is a noncommutative ring with zero-divisors, whose principal right ideals and principal left ideals are uniquely generated, and which is not SR1. In addition, R is left noetherian but not right noetherian.

Proof. Each element of R can be written as $f(x) + g(x)y$ where $f(x), g(x) \in \mathbb{Z}[x]$. The units of R have the form $\pm 1 + g(x)y$, and one-sided inverses in R are two-sided. We note that $\mathtt{r}(y) = \mathbb{Z}[x]x \oplus \mathbb{Z}[x]y$ and this is the only non-trivial right annihilator of an element of R. To show that principal right ideals are uniquely generated, we we apply Canfell's theorem. Suppose that $a, e \in R$ satisfy $aR + \mathtt{r}(e) = R$. Then $ab + j = 1$ for $h \in R$, $ej = 0$. The only nontrivial case is when $j \in \mathbb{Z}[x]x \oplus \mathbb{Z}[x]y$. Writing $a = f(x) + g(x)y, b = h(x) + k(x)y$, $j = xs(x) + t(x)y$, and substituting int $ab + j = 1$, we find that $f(x)h(x) = 1$. Hence, $f(x) = \pm 1$, and so $a = \pm 1 + g(x)y$ is a unit of R. Similarly, left principal ideals of R are uniquely generated. Finally, to see that R is not SR1, we note that R contains $\mathbb{Z}$ as a subring and then use an argument similar to that in Example 2.1.10. □

Unit-regular elements have the left UG property as the following result proves.

Theorem 2.2.7. ([73]) If $a, a' \in \mathrm{ureg}(R)$, then $aR = a'R$ iff $a' = au$ for some $u \in U(R)$.

Proof. Let $a = ev$ and $a' = e'v'$, where e, e' are idempotents, and $v, v' \in U(R)$. Since $aR = evR = eR$, and $a'R = e'v'R = e'R$, we have $aR = a'R$ iff $eR = e'R$. Thus, it suffices to handle the case where $a = e$ and $a' = e'$. We need only check the "only if" part, so assume that $eR = e'R$. Then $ee' = e'$, and $e'e = e$. Since $ee'(1-e)$ is an element of square zero, we have $u := 1+ee'(1-e) = 1+e'-e \in U(R)$. Now $eu = e(1+e'-e) = e+e'-e = e'$, as desired. □

And so we conclude that

Corollary 2.2.8. Unit-regular rings are left UG.

More generally, we have

Theorem 2.2.9. ([111]) If R is SR1, then R is left UG.

Proof. We have $a = by$, $b = ax$ so $a = axy$. If $a = b = 0$ there is nothing to prove. Otherwise $a(1 - xy) = 0$. Let $1 - xy = c$, then $xR + cR = R$, $ac = 0$. Since R is SR1, we have $x + cv = u \in U(R)$ for $v \in R$. Thus $ax + acv = au$. Then $ax - au = b$ and $bu^{-1} = a$. □

And so, we have the following irreversible implications

$$\text{unit-regular} \implies \text{SR1} \implies \text{left UG}$$

Recall that, the first implication fails to be reversed because the ring $\mathbb{Z}_4$ exists, and the second one fails because the ring $\mathbb{Z}$ exists.

As it is the case for SR1 rings, the left UG rings are closed under products.

Theorem 2.2.10. ([86]) $\Pi_{i\in I}R_i$ is a UG ring if and only if R_i is UG for each $i \in I$.

Proof. Coordinate-wise calsulations. □

Kaplansky's subring is a ring of the form $K_p = \{(n, \lambda) \in \mathbb{Z} \times \mathbb{Z}_p[x] \mid \lambda(\overline{0}) = \overline{n}\}$. In [86, Example 8], it is proven that if $p = 2$ or 3, then K_p is left UG, and it is not the case whenever $p \geq 5$. So for the smallet possible p, we have that:

Example 2.2.11. ([70]) Let $K_5 = \{(n, \lambda) \in \mathbb{Z} \times \mathbb{Z}_5[x] \mid \lambda(\overline{0}) = \overline{n}\}$, where $\overline{k} = k + 5\mathbb{Z}$ in $\mathbb{Z}_5$. Then $(0, \overline{x})$ and $(0, \overline{2}x)$ generate the same ideal of K_5 but are not unit multiples.

Recall that a regular ring has the property that every finitely generated right (left) ideal is generated by an idempotent. Regular rings are **left PP rings**[11], that is, principal left ideals are all projective (the left ideal Ra is projective if and only if $\mathtt{l}(a) = Re$ where e is an idempotent.). A commutative ring R is called **PP** ring if each element $x \in R$ can be written in the form $x = re$ where r is regular and e is idempotent. And so, a commutative regular ring is a PP ring.

So we have that

Theorem 2.2.12. ([4]) Every commutative PP ring is UG.[12]

Proof. Trivial. □

The converse of Theorem 2.2.12 fails because

Example 2.2.13. ([11]) $\mathbb{Z} \times \mathbb{Z}$ is a PP ring that is not regular.

Commutativity in Theorem 2.2.12 is not superfluous because

Example 2.2.14. ([63]) Not every left PP-ring is left UG.

Proof. If D is a division ring the ring $\mathbb{M}_\omega(D)$ is regular and so left PP but not left UG because it is not Dedekind finite.[13] □

A ring R is called **left quasi-morphic** if the collection of all left principal ideals coincides with the collection of all left annihilators in the ring.

Theorem 2.2.15. ([95]) Let R be a ring. If R is left quasi-morphic, then the following are equivalent:

1. R is left UG.
2. R is SR1.

[11]left PP rings are also known as **left Rickart** rings

[12]commutative UG rings have been called **strongly associate** rings in [4]

[13]left UG rings are known to be Dedekind finite (one-sided units are two sided).

Proof. (1) $\implies$ (2) In view of Theorem 2.1.50. It suffices to show that every left unit lifts modulo every left principal ideal in R. Let x be a left unit that lifts modulo the left principal ideal Ry i.e there exists $z \in R$ such that $zx - 1 \in Ry$. We would like to show that there exists a unit (and hence left invertible) $u \in U(R)$ such that $x - u \in Ry$. Since R is left quasi-morphic, there exists $a, b \in R$ such that $Ry = \mathtt{l}(a)$ and $R(xa) = \mathtt{l}(b)$. Since $zx - 1 \in Ry$ we have $Rx + Ry = R$. But for any $r \in R$, $rx(ab) = (rxa)b = 0$ since $rxa \in R(xa) = \mathtt{l}(b)$. Also $ry(ab) = ((ry)a)b = 0 \cdot b = 0$ since $ry \in Ry = \mathtt{l}(a)$. Therefore $Rx \subseteq \mathtt{l}(ab)$ and $Ry \subseteq \mathtt{l}(ab)$. Hence we have

$$R = Rx + Ry = \mathtt{l}(ab) \implies ab = 0 \implies a \in \mathtt{l}(b) \implies Ra \subseteq \mathtt{l}(b)$$

Also we have $\mathtt{l}(b) = R(xa) \subseteq Ra$ Therefore $\mathtt{l}(b) = R(xa) = Ra$. Now since R is left uniquely generated and $R(xa) = Ra$, there exists a unit $u \in R$ such that $xa = ua$. This implies that $(x - u)a = 0 \implies (x - u) \in \mathtt{l}(a) = Ry$. Thus, from Theorem 2.1.50, the ring R is .

(2) $\implies$ (1) Theorem 2.2.9 says that it is always the case. ☐

Recall that a topological space is **continuum** if it is both compact[14] and connected[15]. And $\mathrm{C}(X)$ denotes the **ring of all continuous real-valued functions on a completely regular Hausdorff space** X. For $f \in \mathrm{C}(X)$, the **zero set** of f is $Z(f) = \{x \in X : f(x) = 0\}$, the **support** of f is $\mathrm{Supp}(f) = \mathrm{cl}_X(X - Z(f))$. Moreover, If $Z(f)$ is a neighborhood of $Z(g)$, then f is a multiple of g, that is, $f = hg$ for some $h \in \mathrm{C}(X)$. $\mathrm{C}^*(X)$ is the **ring of bounded continuous functions** on X. A subspace $A \subseteq X$ is said to be C^***-embedded** (in X) if every $f \in \mathrm{C}^*(A)$ can be extended to some $g \in \mathrm{C}^*(X)$. (See [53] for further results and notations).

Theorem 2.2.16. ([9]) Let X be continuum and $f \in \mathrm{C}(X)$. Then f is UG if and only if $\mathrm{Supp}(f)$ is connected.

Proof. Let $\mathrm{Supp}(f)$ be connected and $(f) = (g)$ for some $g \in \mathrm{C}(X)$. Then there exist $s, t \in \mathrm{C}(X)$ such that $f = sg$ and $g = tf$. Take $x_0 \in \mathrm{Supp}(f)$. We claim that $s(x_0) = 0 = t(x_0)$. If $s(x_0) = 0$, we may take a net (x_λ) in $X - Z(f)$ such that $x_\lambda \longrightarrow 0$. Since $t = \frac{1}{s}$ on $X - Z(f)$, $t(x_\lambda) \longrightarrow \infty$ which means that t is discontinuous at x_0, a contradiction. Similarly, we have $t(x_0) = 0$. Hence $Z(s)$ and $Z(t)$ are disjoint from $\mathrm{Supp}(f)$. On the other hand, if s changes sign on $\mathrm{Supp}(f)$, then $\mathrm{Supp}(f)$ will be disconnected which is impossible by our hypothesis. Without loss of generality, let $s > 0$ on $\mathrm{Supp}(f)$. But $\mathrm{Supp}(f)$ is compact, so s has a minimum value on $\mathrm{Supp}(f)$, say $s(y_0) = \alpha$, $y_0 \in \mathrm{Supp}(f)$. We have $\alpha > 0$, for otherwise if $\alpha = 0$, as in the above argument, t will be discontinuous at y_0. Now take $u = s \vee \alpha$. Clearly u is unit and $f = ug$. Conversely, suppose that $\mathrm{Supp}(f)$ is disconnected. We show that f is not UG. Let U and V be two disjoint open sets in X such that $U \cap \mathrm{Supp}(f) \neq \emptyset \neq V \cap \mathrm{Supp}(f)$ and $\mathrm{Supp}(f) \subseteq U \cup V$. Therefore, we have also $U \cap (X - Z(f)) \neq \emptyset \neq V \cap (X - Z(f))$. Now define

$$g(x) = \begin{cases} f(x) & : x \in U \cap (X - Z(f)) \\ 0 & : x \in Z(f) \\ -f(x) & : x \in V \cap (X - Z(f)) \end{cases}, \quad s(x) = \begin{cases} 1 & : x \in U \cap Supp(f) \\ -1 & : x \in V \cap Supp(f) \end{cases}$$

[14] A topological space is **compact** if each open cover (collection of open sets in which their union is a superset or equal the the whole space) admits a finite subcover.

[15] A topological space is **connected** if it can not be expressed a union of two proper clopen sets.

then $g \in \mathrm{C}(X)$ and $s \in \mathrm{C}(\mathrm{Supp}(f))$ for $U \cap \mathrm{Supp}(f)$ and $V \cap \mathrm{Supp}(f)$ are disjoint clopen sets in $\mathrm{Supp}(f)$ whose union is $\mathrm{Supp}(f)$. Since $\mathrm{Supp}(f)$ is compact, it is C^*-embedded . Hence s has an extension s^* in $\mathrm{C}(X)$. Clearly, $f = s^*g$ and $g = s^*f$, i.e. $(f) = (g)$. Now if there exists a unit $u \in \mathrm{C}(X)$ such that $f = ug$, then $u = 1$ on the nonemptyset $U \cap (X - Z(f))$ and $u = -1$ on the nonempty set $V \cap (X - Z(f))$, i.e. $\mathrm{pos}\, u \neq \emptyset \neq \mathrm{neg}\, u$. But $X = \mathrm{pos}\, u \cup \mathrm{neg}\, u$ implies that X is disconnected, a contradiction. Therefore, f is not UG. $\square$

Example 2.2.17. ([9]) The product of two UG elements need not be UG in general.

Proof. Let $X = [-1, 1] \times [-1, 1]$. Let $A = \left\{(x, y) : -\frac{1}{2} \leq x \leq \frac{1}{2}, y \geq 0\right\}$ and $B = \left\{(x, y) : -\frac{1}{2} \leq x \leq \frac{1}{2}, y \leq 0\right\}$, we choose A and B to be such that $A = Z(f)$ and $B = Z(g)$ where $f, g \in \mathrm{C}(X)$. By Theorem 2.2.16 f and g are both UG as both $\mathrm{cl}_X(X - A)$ and $\mathrm{cl}_X(X - B)$ are obviously connected and X is continuum. But $\mathrm{cl}_X(X - (A \cup B)) = \mathrm{cl}_X(X - Z(fg))$ is disconnected, hence, again by Theorem 2.2.16, the product fg is not UG. $\square$

Theorem 2.2.18. ([82]) Let R be a regular ring. Then R is unit-regular if and only if every principal right ideal is uniquely generated.

Proof. Suppose every principal right ideal of the regular ring R is uniquely generated. For any $x \in R$, choose $y \in R$ such that $x = xyx$, then $xR = xyR$ implies $xy = xu$ for some unit u, whence $x = xux$. So R is unit-regular.

Conversely, suppose R is unit-regular. Let $a, b \in R$ satisfy $aR = bR$. Choose units $u, v \in R$ such that $a = aua$ and $b = bvb$. Now, $a = bs$ and $b = at$ for some $s, t \in R$, thus, $1 - st \in r(b) = (1 - vb)R$. Consequently, $sR + (1 - vb)R = R$. Since R is SR1, there exists some $r \in R$ such that $s + (1 - vb)r$ is a unit of R. Since $a = b(s + (1 - vb)r)$, we conclude that a and b are right associates. Thus, every principal right ideal of R is uniquely generated. $\square$

And so, for a regular ring R, we have the following equivalence:

$$\text{unit-regular} \implies \text{SR1} \implies \text{left UG} \implies \text{unit-regular}$$

In 2017, Nicholson [86] defined the left annihilator-stabilty conditions as follows:

Definition 2.2.19. ([86]) An element a in a ring R is called **left annihilator-stable** (left AS element) if $Ra + \mathtt{l}(b) = R$, $b \in R$, then $a - u \in \mathtt{l}(b)$ for some unit $u \in R$. A ring R is called a **left annihilator-stable** ring (a left AS ring) if every element of R is left AS.

It is observe that

Theorem 2.2.20. ([86]) A ring R is left AS if and only if R is left UG.

Proof. Obvious, by Canfell's Theorem 2.2.5. $\square$

Theorem 2.2.21. ([9]) If $f \in \mathrm{C}(X)$, then f is SR1 if and only if f is AS.

Proof. Let $f \in \mathrm{C}(X)$ be AS and $(f)+(g) = \mathrm{C}(X)$ for some $g \in \mathrm{C}(X)$. Then $Z(f) \cap Z(g) = \emptyset$ and hence there is $t \in \mathrm{C}(X)$ such that $t(Z(f)) = 0$ and $t(Z(g)) = 1$. Take $Z(h) = \{x \in X : t(x)\}$ and $Z(k) = \{x \in X : t(x)\}$. Since $k \in \mathrm{ann}(h)$ and $Z(f) \cap Z(k) = \emptyset$, we have $(f) + \mathrm{ann}(h) = \mathrm{C}(X)$. But f is AS, so $(f - u)h = 0$ for some unit $u \in \mathrm{C}(X)$. This implies that $X - Z(h) \subseteq Z(f - u)$ whence $X - Z(h) \subseteq \mathrm{int}_X Z(f - u)$. On the other hand, $Z(g)X - Z(h)$, so $Z(g) = \mathrm{int}_X Z(f - u)$. Hence $f - u$ is a multiple of g, i.e. $f - u \in (g)$. This means that f is SR1. $\square$

Lemma 2.2.22. ([86]) Left AS condition is translation invariant.

Proof. Let a be left AS. If $R(ua) + \mathtt{l}(b) = R$ then $Ra + \mathtt{l}(b) = R$ so $a - x \in \mathtt{l}(b)$ where $x \in R$ is a unit. Hence $ua - ux \in \mathtt{l}(b)$, and ux is a unit. This shows that ua is left AS. Turning to au, let $R(au) + \mathtt{l}(b) = R$ so $Ra + \mathtt{l}(b)u^{-1} = R$. But $\mathtt{l}(b)u^{-1} = \mathtt{l}(ub)$, and we obtain $Ra + \mathtt{l}(ub) = R$. Hence $a - z \in \mathtt{l}(ub) = \mathtt{l}(b)u^{-1}$, z a unit, and so $au - zu \in \mathtt{l}(b)$. Thus, au is left AS. $\square$

Lemma 2.2.23. ([86]) Let R be a ring. The following are equivalent for an element a in R :

1. a is left AS.

2. If $Rab = Rb$, $b \in R$, then $ab = ub$ for some unit $u \in R$.

Proof. (1) $\implies$ (2). Given (1), suppose $Rab = Rb$, $b \in R$. If $b = rab$, $r \in R$, we have $1 - ra \in \mathtt{l}(b)$ so $Ra + \mathtt{l}(b) = R$. Then $a - u \in \mathtt{l}(b)$ for some $u \in U(R)$ by (1). Hence $ab = ub$, proving (2). (2) $\implies$ (1). Assume (2) and let $Ra + \mathtt{l}(b) = R$, $b \in R$, say $1 = ra + m$, $r \in R$, $m \in \mathtt{l}(b)$. Hence $b = rab$, so $Rab = Rb$. But then (2) implies that $ab = ub$, where $u \in U(R)$, so $a - u \in \mathtt{l}(b)$, proving (1). $\square$

Proposition 2.2.24. ([86]) For any ring R, if $a \in J(R)$, then a is left AS.

Proof. If $a \in J(R)$ let $Ra + \mathtt{l}(b) = R$. Then $l(b) = R$ as $Ra \in J(R)$, so $a - u \in \mathtt{l}(b)$ for any unit u. $\square$

Theorem 2.2.25. ([86]) If a is regular and left AS then a is unit-regular.

Proof. Let $axa = a$ where $x \in R$. We may assume that $xax = x$ too (via $x \mapsto xax$). It follows that $1 - xa \in \mathtt{l}(x) = \mathtt{l}(xa)$ so $R = Ra + \mathtt{l}(xa)$. As a is left AS, let $a - u \in \mathtt{l}(xa) = \mathtt{l}(x)$ for some unit u in R. Hence $ax = ux$, so $a = axa = uxa$. Thus $u^{-1}a = xa$, and so $au^{-1}a = axa = a$. $\square$

So, assuming regularity, we have the equivalence for rings:

$$\text{unit-regular} \implies \text{SR1} \implies \text{left AS} \implies \text{unit-regular}$$

Nicholson observed that

Theorem 2.2.26. ([86]) If either $R[x]$ or $R[[x]]$ is left AS then R is left AS.

Dealing with elements is, in fact, more sensitive than dealing with rings. As shown in Example 2.2.17, the product of two UG elements need not be UG. However, this is not the case when elements are AS.

Theorem 2.2.27. ([109]) If $a, b \in R$ are left AS, then ab is left AS.

Proof. Assume that $Rab + \mathtt{l}(c) = R$ with $c \in R$. Then $1 = rab + x$ where $r \in R$ and $x \in \mathtt{l}(c)$, so $c = rabc$. From $Rab + \mathtt{l}(c) = R$, it follows that $Rb + \mathtt{l}(c) = R$. Since b is left AS, $b - u \in \mathtt{l}(c)$ for some unit $u \in R$. Thus, $bc = uc$, and so $abc = auc$ and $c = rabc = rauc$. Hence, $1 - rau \in \mathtt{l}(c)$, so $Rau + \mathtt{l}(c) = R$. Since a is left AS and u is a unit, au is left AS by Lemma 2.2.22. It follows that $au - v \in \mathtt{l}(c)$ for a unit v in R. Thus, $auc = vc$. As $auc = abc$, we obtain that $abc = vc$, i.e., $ab - v \in \mathtt{l}(c)$. Hence, ab is left AS. $\square$

Even more sensitive. The conditions UG and AS are skew (none implies the other) for elements. Before showing this, we need the following couple of lemmas.

Lemma 2.2.28. ([9]) $f \in \mathrm{C}(X)$ is UG if and only if f^2 is UG. (Inductively, $f \in C(X)$ is UG if and only if f^n is UG where $n \in \mathbb{N}$)

Proof. Let f be UG. First we show that f^3 is UG. Let $(f^3) = (h)$, $h \in \mathrm{C}(X)$. Clearly $(f) = (h^{\frac{1}{3}})$ and since f is UG, there is a unit $u \in \mathrm{C}(X)$ such that $f = uh^{\frac{1}{3}}$. Therefore, $f^3 = u^3h$, i.e. f^3 is UG. Next we show that f^2 is UG. Let $(f^2) = (h)$, $h \in \mathrm{C}(X)$. Hence we have $(f^3) = (fh)$ and since f^3 is UG, $f^3 = ufh$, where $u \in \mathrm{C}(X)$ is unit. So $f(f^2 - uh) = 0$ implies that $f^2 - uh = 0$ on $X - Z(f)$ and since $Z(f) = Z(h)$, we have also $f^2 - uh = 0$ on $Z(f)$ and therefore $f^2 - uh = 0$. Conversely, suppose that f^2 is UG and $(f) = (h)$, $h \in \mathrm{C}(X)$. Clearly $(f^2) = (fh)$ and hence $f^2 = ufh$ for some unit $u \in \mathrm{C}(X)$. Hence $f(f - uh) = 0$ and by a similar argument as above, we have $f - uh = 0$, i.e. f is UG. $\square$

Lemma 2.2.29. ([9]) If $f \in \mathrm{C}(X)$ and $0 \geq f$ (or $f \geq 0$), then $sr(f) = 1$

Proof. Suppose there exists $g \in \mathrm{C}(X)$ such that $(f) + (g) = \mathrm{C}(X)$, then $Z(f) \cap Z(g) = \emptyset$ implies that $f + g^2 = u$ for some unit u in $\mathrm{C}(X)$. Hence $f - u \in (g)$ and this means that $\mathrm{sr}(f) = 1$. $\square$

The following lemma serves as a criterion for an element in $\mathrm{C}(\mathbb{R})$ to be or not to be UG.

Lemma 2.2.30. ([9]) Let $f, g \in \mathrm{C}(\mathbb{R})$.

1. Let $Z(f) = Z(g) = [a, \infty)$ $(Z(f) = (-\infty, a])$. Then $(f) = (g)$ if and only if $\lim_{x\to a^-} \frac{f(x)}{g(x)}$ $(\lim_{x\to a^+} \frac{f(x)}{g(x)})$ exists and it is nonzero. Furthermore, if $Z(f) = [a, \infty)$ or $Z(f) = (-\infty, a]$ then f is UG.

2. Let $Z(f) = Z(g) = [a, b]$. Then $(f) = (g)$ if and only if $\lim_{x\to a^-} \frac{f(x)}{g(x)}$ and $\lim_{x\to a^+} \frac{f(x)}{g(x)}$ exist and both are nonzero. Furthermore, if $f \in \mathrm{C}(\mathbb{R})$ and $Z(f) = [a, b]$, then f is never UG.

Proof.

1. Whenever $(f) = (g)$, then $f = tg$ and $g = sf$ for some $s, t \in \mathrm{C}(\mathbb{R})$. Hence $t = fg$ and $s = gf$ on $(-\infty, a)$, so $\lim_{x\to a^-} t(x)$ and $\lim_{x\to a^-} s(x)$ exist and clearly $\lim_{x\to a^-} t(x) = 0$ (otherwise $\lim_{x\to a^-} s(x) = \infty$). Conversely, suppose that $\lim_{x\to a^-} \frac{f(x)}{g(x)} = \alpha \neq 0$. We define $u \in \mathrm{C}(\mathbb{R})$ such that $u = fg$ on $(-\infty, a)$ and $u = \alpha$ on $[a, \infty)$. Clearly u is unit and $f = ug$. This implies that $(f) = (g)$ and since u is a unit, this also shows that f is UG. In case $Z(f) = (-\infty, a]$, the proof is similar.

2. If $(f) = (g)$, then $f = tg$ and $g = sf$ for some $s, t \in \mathrm{C}(\mathbb{R})$. As in the above argument, we observe that $\lim_{x\to a^-} t(x)$, $\lim_{x\to a^+} t(x)$, $\lim_{x\to a^-} s(x)$ and $\lim_{x\to a^+} s(x)$ exist and all are nonzero. Conversely, let $\lim_{x\to a^-} \frac{f(x)}{g(x)} = \alpha \neq 0$ and $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \beta \neq 0$. Define $h \in \mathrm{C}(\mathbb{R})$ such that $h = fg$ on $R - [a, b]$ and $h(x) = \alpha + \frac{\beta-\alpha}{b-a}(x - a)$ for each $x \in [a, b]$. Clearly $h \in \mathrm{C}(X)$ and $f = hg$. Similarly, there is $k \in \mathrm{C}(X)$ such that $g = kf$ and hence $(f) = (g)$. Finally, suppose that $f \in \mathrm{C}(R)$ and $Z(f) = [a, b]$. Consider $g \in \mathrm{C}(R)$ such that $Z(g) = [a, b]$, $g = f$ on (b, ∞) and $g = -f$ on $(-\infty, a)$. In this case, $\lim_{x\to a^-} \frac{f(x)}{g(x)} = -1$ and $\lim_{x\to b^+} \frac{f(x)}{g(x)} = 1$. Hence using the first part of

(2), $(f) = (g)$. Now if there exists a unit $u \in \mathrm{C}(R)$ such that $f = ug$, then $u = 1$ on (b, ∞) and $u = -1$ on $(-\infty, a)$. But u is unit and it does not take the value zero, so $R = \operatorname{pos} u \cup \operatorname{neg} u$, i.e. R is disconnected, a contradiction.

□

The following couple of examples show that neither of the conditions AS and UG for elements of $\mathrm{C}(X)$ necessarily implies the other.

Example 2.2.31. ([9]) Define functions $f, g \in \mathrm{C}(\mathbb{R})$ as follows:

$$f(t) = \begin{cases} t-1 & : t \geq 1 \\ 0 & : -1 \leq t \leq 1 \\ -t-1 & : t \leq -1 \end{cases}, g(t) = \begin{cases} -t+1 & : t \geq 1 \\ 0 & : -1 \leq t \leq 1 \\ -t-1 & : t \leq -1 \end{cases}$$

Since $f \geq 0$, f is SR1 by Lemma 2.2.29 and hence it is AS by Theorem 2.2.21. By Lemma 2.2.30, f is not UG.

Example 2.2.32. ([9]) Define functions $g, h \in \mathrm{C}(\mathbb{R})$ as follows:

$$g(t) = \begin{cases} 0 & : |x| \leq 1 \\ x^2-1 & : |x| \geq 1 \end{cases}, h(t) = \begin{cases} x^2-1 & : |x| \leq 1 \\ 0 & : |x| \geq 1 \end{cases}$$

Let $i \in \mathrm{C}(\mathbb{R})$ be identity function. Since $hg = 0$, $h \in \operatorname{ann}(g)$. On the other hand, $Z(i) \cap Z(h) = \emptyset$ implies that $(i) + \operatorname{ann}(g) = \mathrm{C}(\mathbb{R})$. But if there exists a unit $u \in \mathrm{C}(\mathbb{R})$ with $(i-u)g = 0$, then $i = u$ on $\operatorname{coz}(g)$. Hence $u(x)$ is positive for $x > 1$ and it is negative for $x < -1$. Since u is continuous, it must take the value zero, a contradiction. It is clear that every non-zero-divisor is a UG element. Hence i is a UG element but it is not AS.

2.3 IC Rings

Recall that a module ${}_RM$ is said to have internal cancellation (or M is internally cancellable) if it satifies the condition: If $M = N \oplus K = N_1 \oplus K_1$ and $N \cong N_1$, then $K \cong K_1$. From this perspective, a ring R is called an IC ring if ${}_RR$ has internal cancellation. It is well-known that IC is an ER-property. We start this section with the following definition.

Definition 2.3.1. ([73]) A ring R said to have **internal cancellation (IC)** if it satisfies the following equivalent conditions:[16]

1. ${}_RR$ has internal cancellation.
2. Isomorphic idempotents in R have isomorphic complementary idempotents.
3. Any regular element in R is also a unit-regular.[17]
4. For any idempotent $e \in R$, $aR + eR = R$ (or alternatively, $ar + e = 1$) implies that $a + ex \in U(R)$ for some $x \in R$.

Moreover, if $\mathbb{M}_n(R)$ is IC whenever R is IC, then R is called **stably IC**.

We now shall mention some examples.

Example 2.3.2. Any unit-regular is an IC ring.

Proof. Regular elements is such ring would be exactly the unit-regular ones. □

Example 2.3.3. Every RS[18] ring is IC.

Proof. If R is RS, $a \in R$ is regular implies that a is strogly regular, thus, a unit-regular by Theorem 1.2.33. Therefore, R is IC. □

Example 2.3.4. Any commutative ring is IC.

Proof. Let R be commutative ring. If $a \in R$ is regular, then a is strongly regular by Remark 1.2.32, thus, unit-regular by Theorem 1.2.33. Henceforth, R is IC. □

More generally, we have

Example 2.3.5. Any Abelian ring is IC

Proof. Any Abelian regular ring must be strongly regular. Hence, in an Abelian ring, regular elements must be strongly regular ones, thus, unir-regular. □

A module is **indecomposable** if it is non-zero and cannot be written as a direct sum of two non-zero submodules.

[16]In any ring R, we say that a is **similar to** b (or that a and b are **similar**) if $a = u^{-1}bu$ for some $u \in U(R)$. And a is said to be **pseudo-similar to** b (or that a and b are **pseudo-similar**) if there exist $x, y, z, w \in R$ such that $a = zbx$, $b = xaw$, and $x = xzx = xwx$. Note that it is always the case that if a, b are similar in R, then a is pseudo-similar to b by taking $x = u$ and $w = z = u^{-1}$. Moreover, IC rings are precisely the rings in which pseudo-similarity implies similarity.

[17]Because of this characterization, IC rings have also been called **partially unit-regular** rings (p.u.r for short) in [57].

[18]These rings have been called **strongly IC** in [75]

Example 2.3.6. ([73]) Any right artinian ring is IC.

Proof. Assume that $e, e' \in R$ are idempotents in R such that $eR \cong e'R$. Applying the **classical Krull–Schmidt Theorem**[19] to R_R implies that $(1-e)R = (1-e')R$. Therefore, R is IC. $\square$

IC rings have been characterized by many authers, one of nice characterizations of IC rings is the following.

Theorem 2.3.7. ([73]) For any ring R, the following are equivalent:

1. R is IC.

2. Every regular element in R has right UG.

3. Every unit-regular element in R has right UG.

4. Every idempotent in R has right UG.

Proof. (4) $\Longrightarrow$ (1): we verify the condition $\mathrm{reg}(R) = \mathrm{ureg}(R)$. Given $x \in \mathrm{reg}(R)$, write $x = xyx$ (for some $y \in R$). Then xy is an idempotent, and $xR = xyR$. By (4), we have therefore $xy = xv$ for some $v \in U(R)$, and hence $x = xyx = xvx \in \mathrm{ureg}(R)$.

(1) $\Longrightarrow$ (2): Suppose $xR = zR$, where $x \in \mathrm{reg}(R)$. We can write $x = xyx$ for some $y \in U(R)$. As in the above, $xR = eR$, where $e := xy$ is an idempotent. Since $ex = x, z \in xR$ implies that $ez = z$ also. Now, $zR + (1-e)R = xR + (1-e)R = R$, and $1-e$ is an idempotent. Thus, there is a unit $u = z + (1-e)r$ for some $r \in R$. Leftmultiplying this equation by e, we get $eu = ez = z$, and thus $z = x(yu)$ with $yu \in U(R)$, as desired. $\square$

So we conclude that

Corollary 2.3.8. Any left or right UG ring is IC.

As another characterizations of IC rings we have.

Theorem 2.3.9. ([49]) For a ring R, the following are equivalent:

1. R is an IC ring.

2. If $erse = e$ for some $e^2 = e, r, s \in R$, then there exists $u \in U(R)$ such that $erue = e$.

3. If $erse = e$ for some $e^2 = e, r, s \in R$, then there exists $v \in U(R)$ such that $evse = e$.

Proof. (1) $\Longrightarrow$ (2): Suppose $erse = e$ for some $e^2 = e, r, s \in R$. As $erR = eR$, by Theorem 2.3.7, there exists $u \in U(R)$ such that $e = eru$, then $e = erue$.

(2) $\Longrightarrow$ (1): Let a be a regular element in R, so that $a = axa$ for some $x \in R$. Then $e = ax$ is an idempotent in R and $e = eaxe$. By (2), there exists a unit u in R such that $e = eaue$, so $ea = eauea$. As $ea = axa = a$, we have $a = aua$, that is, a is unit-regular.

By the left–right symmetry of internal cancellation of R, we have (1) $\Longleftrightarrow$ (3). $\square$

[19] Given a chain of submodules of M of the form $M_0 \subset M_1 \subset \cdots \subset M_n = M$, we say that n is the length of the chain. The length of M is defined to be the largest length of any of its chains. If no such largest length exists, we say that M has infinite length. The classical Krull–Schmidt Theorem [45] asserts that if M is a module of finite length, then any two direct sum decompositions $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n \cong N_1 \oplus N_2 \oplus \cdots \oplus N_m$, of M into indecomposable summands M_i, N_j are isomorphic.

Theorem 2.3.10. ([73]) If R is an IC ring, then so is the corner ring eRe (for any idempotent $e \in R$).

Proof. R being IC means that the module R_R is internally cancellable. Since $R_R = eR \oplus (1-e)R$, we see easily that its direct summand $(eR)_R$ is also internally cancellable. Since internal cancellation is an ER-property, it follows that the endomorphism ring $\mathrm{End}_R(eR) \cong eRe$ is an IC ring. □

Example 2.3.11. ([73]) There exists a stably IC (hence, IC) ring R such that the polynomial ring $R[x]$ is not IC.

Proof. The proof is omitted —see[73, Proposition 5.10] □

Theorem 2.3.12. ([73]) Let S be a (unital) subring in an IC ring R. If $R = S \oplus I$ for some ideal $I \subseteq R$, then S is also IC.

Proof. Let e, e' be a pair of isomorphic idempotents in S. Then, e, e' are also isomorphic in R, and so $1-e$, $1-e'$ are isomorphic in R. Applying the natural ring homomorphism from R to $R/I \cong S$, we see that $1-e, 1-e'$ are also isomorphic in S. This checks that S is an IC ring. □

Theorem 2.3.13. ([74]) If R is an IC ring and isomorphic idempotents lift (in particular, if regular elements lift) modulo an ideal $I \leq R$, then R/I is also an IC ring.

Proof. Given a pair of isomorphic idempotents of R/I, any isomorphic lifts to R will be conjugate from the IC hypothesis. Conjugate idempotents in R push down to conjugate idempotents in R/I. Thus, all isomorphic idempotents of R/I are conjugate. □

Lemma 2.3.14. ([79, Ex. 21.20]) Let I be an ideal in R which contains no nonzero idempotents (e.g. $I \subseteq J(R)$). Let e, f be commuting idempotents in R. If $e = f$ in R/I, then $e = f$ in R. Moreover, If e, f are orthogonal in R/I, then e, f are orthogonal in R.[20]

Proof. Since $ef = fe$, we have $(e-ef)^2 = e^2(1-f)^2 = e(1-f)$, so $e-ef$ is an idempotent. On the other hand, $e-f \in J$ implies that $e-ef = e(e-f) \in I$, so $e-ef = 0$. Similarly, $f-ef = 0$, so $f = ef = e$. For the last statement, assume that $ef = 0 \in R/I$. Then $ef \in I$. Since $(ef)^2 = e^2f^2 = ef$, we have $ef = 0$. □

Theorem 2.3.15. ([73]) Let I be an ideal in a ring R, and let $\overline{R} = R/I$.

1. If $I \subseteq J(R)$ and $\overline{R}$ is IC, then R is IC.
2. Assume that either $I \subseteq \mathrm{reg}(R)$, or $I \subseteq J(R)$ and idempotents of $\overline{R}$ can be lifted to R. If R is IC, then so is $\overline{R}$.

Proof. 1. Suppose e, e' are isomorphic idempotents in R. Then $\overline{e}$ and $\overline{e'}$ are isomorphic in $\overline{R}$, and so by assuming that $\overline{R}$ is IC, we have $\overline{1-e}, \overline{1-e'}$ are isomorphic in $\overline{R}$. Since $I \subseteq J(R)$, Lemma 2.3.14 implies that $1-e$ and $1-e'$ are isomorphic in R. This proves that R is IC.

[20]Commutativity assumption is not superfluous because considering the ring $R = \mathbb{M}_2(\mathbb{Z})$, the ideal $\mathbb{M}_2(2\mathbb{Z})$, and the two (noncommuting) idempotents $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ implies $f - e = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in I$ so $e = f \in R/I$, but $e \neq f$. Similarly, for f as above and $e' = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = e'^2$ we have we have $e'f = 0, fe' = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in I$, so e', f are orthogonal in R/I, but not in R.

2. Now assume R is IC. If $I \subseteq J(R)$ and idempotents in $\overline{R}$ can be lifted to R, the same argument as in (1) shows that R is IC. Next, assume that $I \subseteq \text{reg}(R)$. To see that R is IC, it suffices to check the equation $\text{reg}(R) = \text{ureg}(R)$. Let $a \in R$ be such that $a \in \text{reg}(R)$, say $a = axa$, for some $x \in R$. Then $a - axa \in I \subseteq \text{reg}(R)$, so there exists $y \in R$ such that

$$a - axa = (a - axa)y(a - axa) = a(1 - xa)y(1 - ax)a \in aRa.$$

This gives $a \in aRa$, so $a \in \text{reg}(R)$. Since R is IC, we have $a = aua$ for some $u \in U(R)$. Then $a = aua$ with $u \in U(R)$, so we have $a \in \text{ureg}(R)$, as desired.

□

Khurana and Tsit-Yuen Lam deduced the following result

Theorem 2.3.16. ([73]) The following statements hold:

1. A ring S is IC iff the power series ring $R = S[[x]]$ is IC.

2. S is IC if the polynomial ring $S[x]$ is IC.

As an element-wise version of [73, Theorem 6.5], we have:

Theorem 2.3.17. ([62]) If a is a left exchange element in a ring R, then the following statements are equivalent:

1. a is left SR1.

2. a is left UG.

3. a is left IC.

Proof. (1) $\implies$ (2) $\implies$ (3) are automatic implications.

(3) $\implies$ (1). Assume that a is both left IC and left exchange, and let $Ra + L = R$ where L is a left ideal of R. Since a is left exchange, we choose $e^2 = e$ in R with $e \in Ra$ and $1 - e \in L$. Now, as $R = Re + R(1-e)$ and $Re \subseteq Ra$, it follows that $Ra + R(1-e) = R$. Hence, $a - u \in R(1-e) \subseteq L$ for some $u \in U(R)$ because a is left IC by assumption, and so a is left SR1, proving (1). □

Beside exchange rings, this also holds for any **left pseudo-morphic** ring R (R is called **(left) pseudo-morphic** if $\{Ra : a \in R\} \subseteq \{\mathtt{l}(b) : b \in R\}$, that is, every (left) principal ideal is a left annihilator ideal). In fact, every regular ring is pseudo-morphic.

Theorem 2.3.18. ([86]) If a ring R is left pseudo-morphic, the following are equivalent:

1. R is SR1.

2. R is left UG.

3. R is right UG.

Proof. Since "SR1" is left-right symmetric, we only prove (1) $\implies$ (3). Assume (1) and let $Ra + Rb = R$. As R is left pseudo-morphic write $Rb = \mathtt{l}(c)$ where $c \in R$. Because R is left AS by Canfell's theorem, we have $a - u \in \mathtt{l}(c) = Rb$ for some unit $u \in R$. This shows that R is SR1. □

Following S.Garg and H.K.Grover [49], modules in which any two isomorphic summands have a common complement are said to be **perspective** modules. Two summands A, B of a module M will be denoted by $A \sim B$, if they have a common complement, i.e., there exists a submodule C such that $M = A \oplus C = B \oplus C$. It is clear that $A \sim B$ implies $A \cong B$. A module M is perspective when $A \cong B$ implies $A \sim B$ for any two summands A, B of M. It is clear that perspective modules satisfy the internal cancellation property in the sense that complements of isomorphic summands are isomorphic. Moreover, a module having the substitution property is a perspective module. Perspectivity is an ER-property.

Definition 2.3.19. ([49]) A ring R is said to be **perspective** if it satisfies any of the followng equivalent conditions:

1. If $Ra + Rb = R$ for some $a, b \in R$ and if $aR \oplus X = R$ for some right ideal X of R, then $b\mathtt{r}(a)$ and X have a common complement.
2. If $Ra + Rb = R$ for some $a, b \in R$ and $aR \oplus X = R$ for some right ideal X of R, then there exists $e \in I(R)$, such that $eR = X$ and $a + eb$ is a unit.
3. If $aR \oplus X = R$ for some $a \in R$, then $\mathtt{r}(a)$ and X have a common complement.

Example 2.3.20. ([49]) Any SR1 ring is perspective.

Proof. Since any substitutable module is perspective. □

Example 2.3.21. ([49]) Every Abelian ring is a perspective ring.

Proof. If e and f are idempotents in an abelian ring R such that $eR \cong fR$,then $e = f$, implying that $(1 - e)R = (1 - f)R$ is a common complement of eR and fR. □

Example 2.3.22. ([49]) Any perspective ring is an IC ring.

Proof. Clearly, since any perspective module satisfies the internal cancellation. □

So, for modules, we have that:

$$\text{substitution} \implies \text{perspectivity} \implies \text{internal cancellation}$$

And for rings, we have:

$$\text{SR1} \implies \text{perspective} \implies \text{IC}$$

Regular elements in IC rings and arbitrary elements in SR1 rings both are left UG. Also an exchange IC ring is SR1. So one may wonder if suitable elements in a perspective ring have the left UG property. The following example shows that this is not the case even in commutative rings!

Example 2.3.23. ([49]) Let $R = \{(n, f(x)) \in \mathbb{Z} \times \mathbb{Z}_{16}[x] : n \equiv f(0) \mod 16\}$. Then $a = (0, 2x) \in R$ is a nilpotent and therefore, a suitable element. If $b = (0, 6x)$, then $b = (0, 6x) = (0, 2x)(3, 3) \in aR$. Also $a = (0, 2x) = (0, 6x)(11, 11) \in bR$, implying that $aR = bR$. If a and b are associates, then there exists $u \in U(R)$ such that $a = bu$. If $u = (n, f(x))$, then $n = \pm 1$ and $f(x) = \pm 1 + a_1 x + a_2 x^2 + \cdots + a_k x^k \mod 16$. So $(0, 2x) = (1, f(x))(0, 6x)$ implies that $2x = 6xf(x) \mod 16$. This is not possible. Thus a and b are not associates.

Internal cancellation is, in fact, a weaker property than cancellation.

Theorem 2.3.24. ([77]) If a module A is cancellable, then A is internally cancellable.

Proof. Say $A = N \oplus K = N' \oplus K'$, with $N \cong N'$. Since N is a direct summand of A, N is also cancellable. Thus, from $N \oplus K = N' \oplus K' \cong N \oplus K'$, we get $K \cong K'$. $\square$

So, we have for an R-module:

$$\text{substitution} \implies \text{cancellation} \implies \text{internal cancellation}$$

Definition 2.3.25. ([77]) A module A_R over a ring R is said to have the n-**exchange property** (or A is an n-exchange module) if, whenever (a copy of) A is a direct summand in any module $M = M_1 \oplus \cdots \oplus M_n$, A has a complement in M of the form $M_1' \oplus \cdots \oplus M_n'$ for suitable submodules $M_i' \subseteq M_i$.

Observe from the definition that, each M_i' is a direct summand of M, and hence of M_i. Thus, we can write $M_i = M_i' \oplus M_i''$ for suitable submodules $M_i'' \subseteq M_i$.

The notion of exhcange of modules coincide with that of rings.

Theorem 2.3.26. ([87]) If R is a ring, the following conditions are equivalent for a left R-module M:

1. $\mathrm{End}_R({}_RM)$ is right exchange.
2. M has the finite exchange property.
3. $\mathrm{End}_R({}_RM)$ is left exchange.

Proof. The proof of this theorem is omitted —see [87, Theorem 2.1] $\square$

And so for a ring R and a left R-module M we always have that:

$$R \text{ is exchange} \iff \mathrm{End}_R(M) \text{ is exchange} \iff {}_RR \text{ is exchange} \iff R_R \text{ is exchange}$$

Theorem 2.3.27. ([77]) Let A be a module with the finite exchange property. Then the following conditions on A are equivalent:

1. A has the substitution property.
2. A is cancellable.
3. A is internally cancellable.

Proof. (1) $\implies$ (2) $\implies$ (3) by Theorem 2.1.17 and Theorem 2.3.24.

(3) $\implies$ (2). Assume that A is internally cancellable, and consider a module $M = A \oplus B = A' \oplus C$, where $A' \cong A$. Since A is assumed to have 2-exchange, we can write $M = A' \oplus X \oplus Y$ for suitable submodules $X \subseteq A$ and $Y \subseteq B$. Write $A = U \oplus X$and $B = V \oplus Y$. Then $A' \cong MX \oplus Y = A \oplus BX \oplus Y \cong U \oplus V$. Since $A' \cong A = U \oplus X$ has internal cancellation, we have $X \cong V$. Therefore, $B = V \oplus Y \cong X \oplus Y \cong M/A' \cong C$, so we have proved the cancellation property for A.

(3) $\implies$ (1) For a ring-theoretical approach, see [73, Theorem 6.5].[21] $\square$

[21] The module-theoretic approach is of our interests.

2.4 DF Rings

In this last section of current chapter, we finally discuss direct finiteness condition - a condition that characterizes the question "When left unit is right unit?". We say that a ring R has **IBN**[22] if and only if for any pair of matrices $A \in \mathbb{M}_{n\times m}(R), B \in \mathbb{M}_{m\times n}(R)$ such that $AB = I_n$, $BA = I_m$, one can infer that $n = m$. This reveals the left-right symmetry of the IBN notion. In fact, we are more interested the subclass of the directly finite rings. We start with the following definition:

Definition 2.4.1. A ring R is called **directly finite** (DF) if for all $a, b \in R$, $ab = 1$, implies $ba = 1$ (equivalently, R is DF if and only if $Ra = R$, $a \in R$, implies $aR = R$). And called **direclty infinite** if it is not directly finite. Moreover, a ring R is called **stably finite** if for all $A, B \in \mathbb{M}_n(R)$, $AB = I$ implies $BA = I$. (that is, $\mathbb{M}_n(R)$ is directly finite for any $n \in \mathbb{N}$).

So now, it is clear by Definition 2.4.1 that saying $ba = 1$ implies $ab = 1$ is redundant, that is, the notion of direct finiteness is left-right symmetric.

Remark 2.4.2. For a ring R we have the implications:

$$\text{stably finite} \implies \text{directly finite} \implies \text{IBN}$$

Example 2.4.3. Any commutative ring is stably finite.

Proof. Let R be commutative ring, since $ab = ba$ for any $a, b \in R$, it follows that R is directly finite. Now, Let $AB = I$ for $A, B \in \mathbb{M}_n(R)$, $I = AB$ implies $1 = det(AB) = det(A)det(B)$. Commutativity of R implies that $det(A) \in R$ is a two sided unit, the same for $det(B)$. Hence, $BA = I$. Therefore, R is stably finite. □

Example 2.4.4. Any domain is directly finite

Proof. Assume that R is a domain, then $ab = 1$ implies that $ab-1 = 0$, and so, $(ab-1)a = a(ba-1) = 0$, thus, $ba = 1$ as a cancels from left. Therefore, R is directly finite. □

The following example is, in fact, a nostalgic recall of linear algebra.

Example 2.4.5. The ring of complex numbers $\mathbb{C}$ is stably finite.

Theorem 2.4.6. ([77]) Any SR1 ring is DF.

Proof. Let R be SR1 where $ac = 1$. Then $Ra + R(1-ca) = R$ implies that some $u := a+s(1-ca)$ is left-invertible. Right-multiplying by c, we get $uc = ac+s(c-cac) = 1$. Thus, $u \in U(R)$, and hence $c \in U(R)$. Therefore, R is DF. □

Theorem 2.4.7. Any SR1 ring is stably finite.

Proof. Let R be SR1 ring, then by Theorem 2.4.6, R is DF. Moreover, it follows by 2.1.26 that any full matrix ring over R is DF. Henceforth, R is stably finite. □

[22] There exist rings R in which $R \cong R \oplus R$ as left modules. Hence ${}_RR$ may have bases of one and two elements. In fact, $R^n \cong R$ for each $n \geq 1$, so R contains a basis of n elements for each $n \geq 1$

Example 2.4.8. ([1]) Every left Noetherian ring R is Dedekind-finite.

Proof. Suppose that $ab = 1$ for some $a, b \in R$. Define the map $f : R \mapsto R$ by $f(r) = rb$. Clearly f is an R-module homomorphism and f is onto because $f(ra) = (ra)b = r(ab) = r$, for all $r \in R$. Now we have an ascending chain of left ideals of R

$$\ker f \subseteq \ker f^2 \subseteq \ker f^3 \subseteq \cdots .$$

Since R is left Noetherian, this chain stabilizes at some point, i.e. there exists some n such that $\ker f^n = \ker f^{n+1}$. Clearly f^n is onto because f is onto. Thus $f^n(c) = ba - 1$ for some $c \in R$. Then

$$f^{n+1}(c) = f(ba - 1) = (ba - 1)b = b(ab) - b = 0$$

Hence $c \in \ker f^{n+1} = \ker f^n$ and therefore $ba - 1 = f^n(c) = 0$.

□

Theorem 2.4.9. ([86]) If a left AS element $a \in R$ is either left or right invertible, then a is a unit. In particular every left AS ring is DF.

Proof. If $ba = 1$, $b \in R$, then $Ra + 1(1) = R$ so, as a is left AS, $a - u \in l(1) = 0$ for some unit u. It follows that $ab = 1$. In the other case, suppose $ac = 1$. Then $Rac = R = Rc$ so, by Lemma 2.2.23, $ac = vc$ where v is a unit. As $ac = 1$ we have $c = v^{-1}$. Hence $a = v$ is a unit in this case too, so $ca = 1$. Now the last statement is clear. □

So it follows by Theorem 2.4.9 and Theorem 2.2.20 that

Remark 2.4.10. Left UG rings are DF.

The following example shows that direct finiteness property of a ring R does not pass to full matrix rings $\mathbb{M}_n(R)$, even when $n = 2$!

Example 2.4.11. ([94])**(Shepherdson)** If R is directly finite, then the full matrix ring $\mathbb{M}_n(R)$ need not be directly finite.

Proof. Let $S = \mathbb{Z}[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$ be the polynomial ring in noncommuting indeterminants x_{ij} and y_{ij} , and let A denote the ideal of S generated by the following four polynomials:

$$x_{11}y_{11} + x_{12}y_{21} - 1,\ x_{11}y_{12} + x_{12}y_{22},\ x_{21}y_{11} + x_{22}y_{21},\ x_{21}y_{12} + x_{22}y_{22} - 1.$$

Define $R = S/A$, and write $a_{ij} = x_{ij} + A$ and $b_{ij} = y_{ij} + A$ for all i and j. Then, the matrices $a = [a_{ij}]$ and $b = [b_{ij}]$ in $\mathbb{M}_n(R)$ satisfy $ab = 1$, but $ba \neq 1$. □

Note that R constructed in Example 2.4.11 is a domain, thus, left UG and IC. From which it follows that also property of a ring R being left UG, IC, or DF does not pass to full matrix rings $\mathbb{M}_n(R)$ in general.

Remark 2.4.12. Not every R is left UG, IC, or DF is stably IC or stably DF.

The following example shows that DF condition does not pass to factor rings in general.

Example 2.4.13. ([73]) Let $R = \mathbb{Q}\langle x, y\rangle$, then R is a domain and so, directly finite, but the factor ring using the relation $xy = 1$ is not directly finite (Hence, neither IC nor left UG).

And so, it turns out that left UG, IC and DF properties do not pass to factor rings in general.

Theorem 2.4.14. ([60]) A ring R is DF if and only if $R/J(R)$ is DF.

Proof. ($\Longrightarrow$) For ease of use, $\overline{R} = R/J(R)$. Let $\overline{a}\overline{b} = 1$, then $1 - ab \in J(R)$, which implies that $1 - (1 - ab)1 = ab$ is a unit. That is $abc = 1 = cab$ for some c. Since R is DF, it follows that $bca = 1$ and so, $(\overline{b}\overline{c})\overline{a} = \overline{1}$. Hence, $\overline{a}$ has left and right inverses, thus, a unit. ($\Longleftarrow$) Conversely, suppose that $\overline{R}$ is DF and that $ab = 1$. Then, $\overline{a}\overline{b} = \overline{1} = \overline{b}\overline{a}$ implies that $1 - ba \in J(R)$. And so $1 - (1 - ba)1 = ba$ is a unit in R implying that $bac = 1 = cba$ for some $c \in R$. Hence a has left and right inverses and thus is a unit. $\square$

Theorem 2.4.15. ([17]) Let $e^2 = e \in R$. If R is DF, then so is the corner ring eRe.

Proof. If $ab = e$ for $a, b \in eRe$ and $f = 1 - e$. It follows that $(a + f)(b + f) = ab + f = e + f = 1$. So, $(b + f)(a + f) = 1$. Henceforth, $ba = 1 - f = e$, thus, eRe is DF. $\square$

Theorem 2.4.16. ([60]) If R is a DF ring and S is a subring with unity, then S is DF.

Proof. Let R be a DF ring and let S be a subring of R with unity e. Suppose that $x, y \in S$ and $xy = e$. If $f = yx = f^2$, then $(fxf)(fyf) = f$ and thus since fRf is DF by Theorem 2.4.15, it follows that $y^2x^2 = fyf(fxf) = yx$. On premultiplication by x and postmultiplication by y this yields $yx = eyxe = e$. $\square$

Example 2.4.17. ([62]) Consider the ring $S = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \cdots$, and let R be the subring of S consisting of sequences of the form $(x_1, x_2, \cdots, x_n, m, m, \cdots)$ where $n \geq 1$, $m \in \mathbb{Z}$ and $x_i \in \mathbb{Q}$. R is DF and IC, but it is not SR1.

Proof. R is IC and DF using because it is commutative. In addition, R is not SR1 because R has an epimorphic image that is isomorphic to $\mathbb{Z}$, where the latter ring is not SR1. $\square$

The following example shows that DF condition is not closed under homomorphic image in general.

Example 2.4.18. ([17]) If R is DF, then a homomorphisc image of R need not be DF.

Proof. Let R have no zero divisors and let $R[x, y]$ be the polynomial ring over R in noncommuting indeterminates x and y. Let I be the ideal of $R[x, y]$ generated by $xy - 1$. Then $x + I$ is right invertible but not invertible in the quotient ring R/I. $\square$

Theorem 2.4.19. ([17]) Let I be a nilpotent ideal in a ring R. Then R is DF if and only if R/I is DF.

Proof. Suppose R is DF and let $(a + I)(b + I) = ab + I = 1 + I \in R/I$. Then $ab \in 1 + I \subseteq U(R)$, so that a is left invertible and hence invertible. Thus $a + I$ is invertible in R/I so that R/I is DF. Conversely, let R/I be DF and suppose $ab = 1$. Then $(a + I)(b + I) = 1 + I = (b + I)(a + I)$ so $ba \in U(R)$. Hence a is left invertible and R is DF. $\square$

Theorem 2.4.20. ([17]) Let R be a ring and $\mathbb{T}_n(R)$ be the ring of upper triangular matrices over R. Then R is DF if and only if $\mathbb{T}_n(R)$ is DF.

Proof. The ideal I of R consisting of the strictly upper triangular matrices is nilpotent and R/I is isomorphic to the direct sum of n copies of R. $\square$

The following example shows that DF ring need not be regular in general.

Example 2.4.21. ([60]) Let $R = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix}$. Then, clearly R is not regular, since

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

However, R is DF by Theorem 2.4.20. (In particular, regularity, unit-regularity and strong regularity properties do not pass to the ring of upper triangular matrices.)

In 1999, Cohn [36] defined a new class of rings, the class reversible rings. A ring R is said to be **reversible** if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. Clear that in the class of reversible rings every left zero-divisor is a right zero-divisor and, of course, conversly. Lately in 2017, Ghashghaei and Koşan [50] defined the class of rings which characterizes the answer of the question "when is every left zero-divisor a right zero-divisor and conversely?" and the class is called so, the class of eversible rings. Interesting results related to DF rings have been found. A ring R is called **eversible** if every left zero-divisor in R is also a right zero-divisor and conversely.

We shall show that the class of reversible rings is contained properly in the class of eversible rings.

Example 2.4.22. Any reversible ring is eversible. The converse need not be true.

Proof. Clear that every revesible ring is eversible. To deny the converse, consider the eversible ring $R = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{bmatrix}, a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $ab = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, while $ba = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. $\square$

Next we mention an example of a ring in which every right zero-divisor in R is a left zero-divisor. while the converse is not.

Example 2.4.23. ([50]) There exists a ring that is not eversible.

Proof. Consider the upper triangular matrix ring $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Obviously, every right zero-divisor in R is a left zero-divisor while $R = \begin{bmatrix} 2 & \bar{0} \\ 0 & \bar{1} \end{bmatrix}$ is a left zero-divisor which is not a right zero-divisor. $\square$

Example 2.4.24. Every domain is eversible.

Proof. Trivial, as there is no disagreement with definition. $\square$

Theorem 2.4.25. ([50]) Every eversible ring is DF.

Proof. Let R be an eversible ring and $ab = 1$. Thus $a = aba$ and $a(1 - ba) = 0$. We are proceeding to show that $ba = 1$. If $1 - ba \neq 0$, then a is a left zero-divisor. Since R is eversible, we obtain that a is a right zero-divisor. Thus there exists $c \neq 0$ such that $ca = 0$. Hence, $c = c1 = cab = (ca)b = 0b = 0$ that is a contradiction. This means $1 - ba = 0$, thus, $ba = 1$. Therefore, R is DF. $\square$

Let R be an Artinian ring. It is clear that any injective homomorphism $\varphi : R \mapsto R$ is surjective. Hence every left zero-divisor is a right zero-divisor and conversely. This means that every Artinian ring is eversible. In particular, every finite ring is eversible.[23]

Theorem 2.4.26. ([50]) A regular ring is DF if and only if it is eversible.

Proof. Assume that R is DF. Suppose a is a left zero-divisor in R. Since R is assumed to be regular then there exists b such that $aba = a$. Therefore, we have $a(1 - ba) = 0$. If $1 - ba = 0$ then a is would not be a left zero-divisor which leads to a contradiction. Hence, we conclude that $1 - ba \neq 0$. Since R is DF then $1 - ba \neq 0$. Henceforth, $(1 - ab)a = 0$ and a is a right zero-divisor. The converse is trivial. $\square$

As a result we have

Corollary 2.4.27. Every unit-regular ring is eversible.

Proof. Clear, since every unit-regular ring is DF and every regular DF is eversible by Theorem 2.4.26. $\square$

Theorem 2.4.28. Every IC ring is DF.

Proof. Since any R-module R with internal cancellation is directly finite by Theorem 2.4.34, left-right symmetry of direct finiteness condition implies that $R \cong \mathrm{End}({}_R R)$, Hence, by Theorem 2.4.35, R is DF. $\square$

Corollary 2.4.29. Every stably IC ring is stably DF.

The converse of Theorem 2.4.28 fails as the following example exhibits.

Example 2.4.30. ([55]) Choose a field F , let $T = F[[t]]$ be the ring of formal power series over F in an indeterminate t, and let K denote the quotient field of T. Let $S = \{x \in \mathrm{End}_F(T) \mid (x - a)(t^n T) = 0, \text{for some } a \in K \text{ and } n > 0\}$. By [55, Example 4.26], for each $x \in S$ there is an unique element$\varphi x \in K$ such that $(x - \varphi x)(t^n T) = 0$ for some $n > 0$. Since K is commutative, the map $\varphi : S \mapsto K$ also defines a ring map $\varphi : S^{op} \mapsto K$. Consequently, the set $R = \{(x, y) \in S \times S^{op} \mid \varphi x = \varphi y\}$ is a subring of $S \times S^{op}$ and R is regular, stably finite but not unit-regular.

Proof. For even more details —see [8, Examples 3.13] and [65, Example 2.7], or alternatively, [55, Example 5.10, Example 5.12]. $\square$

[23]Note that the existence of identity in a finite ring is not superfluous condition, for example the ring $\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ is not eversible. Moreover, Ganesan [48] showed that if a ring has only finitely many zero-divisors, then it is finite. Hence every ring with finitely many zero-divisors is eversible.

Every SR1 ring is left UG by Theorem 2.2.9, the converse fails by Example 2.1.10. Every left UG ring is IC by 2.3.7, the converse fails Example 2.2.11. Finally, every IC ring is DF by 2.4.28, the converse fails by Example 2.4.30. So now we have the following irreversible inclusions:

$$\text{SR1} \implies \text{left UG} \implies \text{IC} \implies \text{DF}$$

Let X be a set of indeterminates of arbitrary cardinality. Let $R[X]$ and $R[\tilde{X}]$ denote the rings of polynomials in commuting elements of X and polynomials in noncommuting elements of X respectively. Let $R[[X]]$ be the power series ring in X.

Theorem 2.4.31. ([17]) The following are equivalent:

1. R is DF.
2. $R[X]$ is DF.
3. $R[\tilde{X}]$ is DF.
4. $R[[X]]$ is DF.

Proof. Since R is a subring, each of (2), (3) and (4) implies (1), so it remains to show that (1) implies each of (2), (3) and (4). In each case, suppose $f(X)g(X) = 1$, and let f_0 and g_0 be the corresponding terms of degree 0. Then $f_0g_0 = 1$ so $g_0f_0 = 1$. In cases (2) and (3), this implies that f_0 and g_0 are not zero divisors, so that $f(X)$ and $g(X)$ are invertible. In case (4), $g_0f_0 = 1$ implies immediately that $f(X)$ and $g(X)$ are invertible. $\square$

Theorem 2.4.32. ([74]) Let $I \leq R$, and suppose that I contains no nonzero idempotents (such as with the Jacobson radical). If isomorphic idempotents lift modulo I (e.g. if regular elements lift), then R is Dedekind-finite if and only if R/I is Dedekind-finite.

Proof. Suppose first that R is not Dedekind-finite. We can then fix $x, y \in R$ with $xy = 1$ but $yx \neq 1$. It is easy to check that $1 - yx$ is an idempotent. Since I contains no nonzero idempotents, we have $1 - yx \notin I$ and so R/I is also not Dedekind-finite. Next assume R is Dedekind-finite. The Dedekind-finite property is equivalent to saying that the only idempotent isomorphic to 1 is 1 itself. So assume there is an idempotent $e + I \in I(R/I)$ with $e + I \cong_{R/I} 1 + I$, it suffices to show that $e \equiv 1 \mod I$. By hypothesis, there exist two isomorphic idempotents $g, h \in I(R)$ such that $g - e, h - 1 \in I$. As $1 - h \in I$ and I does not contain any nonzero idempotents, $h = 1$. So we have $g \cong_R 1$, and as R is Dedekind-finite, we get that $g = 1$. But that means $e \equiv g = 1 \mod I$ as needed. $\square$

Definition 2.4.33. An R-module M is called **Dedekind-finite** if $M \cong M \oplus N$ for some module N, then $N = 0$. Otherwise, M is called **Dedekind-infinite**.

Theorem 2.4.34. If an R-module M is internally cancellable, then M is Dedekind-finite.

Proof. Let M be an internally cancellable R-module and consider the isomorphism, $M \cong M \oplus N$. Now, since it is always true that $M \cong M \oplus 0$, we get $M \cong M \oplus 0 \cong M \oplus N$. Internal cancellability of M implies that $N = 0$. Therefore, M is Dedekind-finite. $\square$

Theorem 2.4.35. ([55]) A right R-module A is directly finite if and only if $xy = 1$ implies $yx = 1$, for all $x, y \in \mathrm{End}_R(A)$.

Proof. If A is directly infinite, then $A = B \oplus C$ with $B \cong A$ and $C \neq 0$. Choose an isomorphism $y : A \mapsto B$, and define $x \in \mathrm{End}_R(A)$ such that $xC = 0$ and x restricts to $y^{-1} : B \mapsto A$. Then $xy = 1$ and $yx \neq 1$. Conversely, suppose that $x, y \in \mathrm{End}_R(A)$ with $xy = 1$ and $yx \neq 1$. Since yx is idempotent and $yxy = y$, we obtain $A = yA \oplus (1 - yx)A$. Observing that $yA = A$ and $(1 - yx)A \neq 0$, we conclude that A is directly infinite. $\square$

So, Dedekind-finiteness is an ER-property.

Gathering results of Theorem 2.1.17, Theorem 2.3.24 and Theorem 2.4.34, we have the following hierarchy of module-theoretic properties on an R-Module M.

Substitution $\Longrightarrow$ Cancellation $\Longrightarrow$ Internal Cancellation $\Longrightarrow$ Dedekind-Finite

Theorem 2.4.36. ([17]) Let M be an R-module. If M is a DF module, then so is any direct summand of M.

Proof. Let $M = N \oplus K$. If L is a proper direct summand of N isomorphic to N, then $L \oplus K$ is a direct summand of M isomorphic to M, a contradiction. $\square$

Theorem 2.4.37. ([17]) There is a monomorphism $f \in \mathrm{End}(M)$ with $\mathrm{Im}\, f$ a proper direct summand if and only if there is an epimorphism $g \in \mathrm{End}(M)$ with $\ker g$ a proper direct summand.

Proof. Suppose such an f exists and let $M = K \oplus \mathrm{Im}\, f$. Let $h : \mathrm{Im}\, f \mapsto M$ be any isomorphism and define g to be zero on K and h on $\mathrm{Im}\, f$. Conversely, let $M = \ker g \oplus N$. Note that $M \cong M/\ker g = N$ and let $f : M \mapsto N$ be any isomorphism, regarded as an endomorphism of M. $\square$

A module A_R over a ring R is called **quasi-injective** if, for any submodule $B \subseteq A$, any $f \in \mathrm{Hom}_R(B, A)$ can be extended to an endomorphism of A. Quasi-injective modules are defined by a weakening of the well-known notion of **injectivity** because if A_R is an injective module, then for any two modules $B \subseteq C$, any $f \in \mathrm{Hom}_R(B, A)$ can be extended to some $g \in \mathrm{Hom}_R(C, A)$. Since we can, in particular, take C to be A, we see that an injective module A is always quasi-injective. However, the convers fails (For more details, see [77]).

Theorem 2.4.38. ([77]) Any quasi-injective module A is an exchange module.

Proof. The proof is omitted —see [77, Theorem 7.8] $\square$

Theorem 2.4.39. ([77]) If a direct sum $A \oplus D$ is quasi-injective, then so are A and D.

Proof. The proof is omitted —see [77, Corollary 7.4]. $\square$

Note also, $R \cong R^{op}$ if R is an exchange ring. Furthermore, as a summing up, for modules we always have that:

injective $\Longrightarrow$ quasi-injective $\Longrightarrow$ exchange

Definition 2.4.40. ([17]) A right (left) R-module M is called **right (left) Hopfian** if every surjective R-endomorphism is invertible, it is **co-Hopfian** if every injective R-endomorphism is invertible.

Theorem 2.4.41. ([17]) If an R-module M is Hopfian or co-Hopfian, then it is DF.

Proof. Suppose M is not DF, so M it has a proper isomorphic summand N. If M is Hopfian, then the canonical projection of M on N, composed with an isomorphism of N onto M is an epimorphism in $\mathrm{End}(M)$ containing a non-trivial kernel, a contradiction. If M is co-Hopfian, then any isomorphism of M onto N is a monomorphism in $\mathrm{End}(M)$ which is not surjective, again a contradiction. $\square$

Theorem 2.4.42 (Suzuki). ([98]) Let A be a quasi-injective module. Then A is Dedekind-finite iff any isomorphism $f : N \mapsto N'$ from one submodule of A to another extends to an automorphism of A. In particular, if A is Dedekind-finite, then for any two isomorphic submodules $N \cong N'$ in A, we have $A/N \cong A/N'$.

Proof. The proof is omitted —see [98]. $\square$

Theorem 2.4.43. ([77]) For any quasi-injective module A, the following are equivalent:

1. A is Dedekind-Finite.
2. A is internally cancellable.
3. A is cancellable.
4. A is substitutable.
5. A is co-Hopfian.

Proof. (1) $\implies$ (2). Follows from Suzuki's Theorem 2.4.42

(2) $\iff$ (3) $\iff$ (4). Follows from Theorem 2.3.27 since quasi-injective modules has the exchange property by Theorem 2.4.38

(3) $\implies$ (5) Assume that A is cancellable, and consider any injection $f : A \mapsto A$. Then $B := f(A) \cong A$, so we can take an isomorphism $B \mapsto A$, and extend it to an endomorphism $g : A \mapsto A$ (using the quasi-injectivity of A). For $K = \ker(g)$, we then have $A = K \oplus B$. Since $B \cong A$, cancellation of A yields $K = 0$, so $B = A$. This proves that f is an automorphism, and hence A is cohopfian.

(5) $\implies$ (1) By Theorem 2.4.41 $\square$

A ring R is **left self-injective** if the module ${}_RR$ is an injective module. While rings with unity are always projective as modules, they are not always injective as modules.

Theorem 2.4.44. ([77]) For a right self-injective ring R, the following are equivalent:

1. R is SR1.
2. R is left UG.
3. R is stably IC.
4. R is IC.
5. R is stably DF.
6. R is DF.

Theorem 2.4.45. ([90]) Let R be a ring. Every left non-zero-divisor of R is a unit if and only if R_R is cohopfian.

Proof. We have a natural isomorphism $\mathrm{End}(R_R) \cong R$, and injective endomorphisms correspond to left non-zero-divisors. $\square$

As a ring-theoretic conclusion of Theorem 2.4.43, we have the following.

Corollary 2.4.46. ([77]) For a left self-injective ring R, the following are equivalent:

1. R is SR1.
2. R is left UG.
3. R is right UG.
4. R is IC.
5. R is DF.
6. Every left non-zero-divisor of R is a unit.

The classes of modules satifying substitution, cancellation, internal cancellation or Dedekind-Finiteness can coincide under another module-theoretic condition. But before reaching this result, we need some definitions.

A left R-module M is called a **Utumi-module** (U-module for short) if for any two non-zero submodules A and B of M with $A \cong B$ and $A \cap B = 0$, there exist two summands K and L of M such that $A \subseteq^{ess} K$, $B \subseteq^{ess} L$ and $K \oplus L \subseteq^{\oplus} M$. A module M is called **square-free** if it contains no non-zero submodules isomorphic to a square $A \oplus A$ (note that every square-free module is a U-module). A module M is said to satisfy the **C1-condition** if every submodule of M is essential in a direct summand. Morover, it satisfies the **C3-condition** if the sum of any two summands of M with zero intersection is a summand of M. And M is called **quasi-continuous** if it satisfies both the C1- and C3-conditions. Finally,a module N is said to be **M-injective** if for every submodule K of M, any homomorphism $\varphi : K \mapsto N$ can be extended to a homomorphism $\psi : M \mapsto N$.

The following implications always hold for modules:[24]

$$\text{injective} \implies \text{quasi-injective} \implies \text{quasi-continuous} \implies U\text{-module}$$

We also need the following triple of lemmas

Lemma 2.4.47. ([66]) If M is a U-module, then $M = Q \oplus T$ where:

1. Q is a quasi-injective module.
2. $Q = A \oplus B \oplus D$, where $A \cong B$ and D is isomorphic to a summand of $A \oplus B$.
3. T is a square-free module.
4. T is Q-injective.

Proof. This proof is omitted —See [66, Theorem 3.13] $\square$

[24] For more results on these properties, the reader is referred to [101],[66],[84].

Lemma 2.4.48. ([66]) Every square-free right R-module M with the finite internal exchange property satisfies the internal cancellation property. In particular, every square-free right R-module M with the finite exchange property has the substitution, and hence the cancellation, property.

Proof. This proof is omitted —See [66, Lemma 5.7] □

Lemma 2.4.49. ([84]) In a quasi-continuous module M, isomorphic directly finite submodules have isomorphic complements. In particular M has the internal cancellation property if and only if M is directly finite.

Proof. This proof is omitted —See [66, Theorem 2.33] □

Theorem 2.4.50. ([66]) If M is a right U-module with the (finite) exchange property, then the following are equivalent:

1. M has the substitution property.
2. M has the cancellation property.
3. M has the internal cancellation property.
4. M is Dedekind-finite.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) automatically, (3) implies (1) since M has exchange. It suffices to show that (4) $\implies$ (2). By Lemma 2.4.47, M can be decomposed as $M = Q \oplus K$ with Q quasi-injective and K square-free, and by Lemma 2.4.48, K has the cancellation property. Since summands of Dedekind-finite modules are again Dedekind finite, we infer from Lemma 2.4.49 that Q has the internal cancellation property. Since Q has also the finite exchange property, we conclude that Q has the cancellation property. Now, by 2.1.15, M has the cancellation property, completing the proof. □

Chapter 3

$\mathcal{L}$-stability

Now that the four major key classes in Chapert 2 are fully discussed. We are ready to introduce the concept that is common between them. The concept of $\mathcal{L}$-stability, it was first declared in 2018 by Ayman Horoub in his seminal work [62] influenced by H. Bass the one who invented the concept of stable range in [14], I. Kaplansky, who invented the concept of left UG rings in [70] and WK Nicholson who defined and characterized left AS rings [86]. Also with D. khurana and TY Lam by their generous survey about IC rings in [73].

3.1 Idealtors and Affordability

We start with the following definition:

Definition 3.1.1. ([63]) A **left-ideal-map** $\mathcal{L}$ is a function that associates to every ring R a well-defined non-empty set $\mathcal{L}(R)$ of left ideals of R.

Two important properties of left-ideal-maps are also defined as follows:

Definition 3.1.2. ([63]) Let $\mathcal{L}$ be a left-ideal-map, and let $\theta : R \mapsto S$ be an onto ring morphism.

1. θ is called **$\mathcal{L}$-fit** if $L \in \mathcal{L}(R)$ implies $\theta(L) \in \mathcal{L}(S)$.

2. θ is called **$\mathcal{L}$-full** if $X \in \mathcal{L}(S)$ implies $X = \theta(L)$ for some $L \in \mathcal{L}(R)$.

The following lemma is a key one:

Lemma 3.1.3. ([63]) Let $\mathcal{L}$ be any left-ideal-map. The following are equivalent.

1. Every ring isomorphism is $\mathcal{L}$-fit.

2. Every ring isomorphism is $\mathcal{L}$-full.

Proof. Let $\sigma : R \mapsto S$ be a ring isomorphism.

(1) $\implies$ (2). If $X \in L(S)$ then $\sigma^{-1}(X) \in \mathcal{L}(R)$ by (1). So $X = \sigma(L)$ where $L = \sigma^{-1}(X) \in \mathcal{L}(R)$.

(2) $\implies$ (1). Let $L \in \mathcal{L}(R)$: By (2), $L = \sigma^{-1}(X)$ for some $X \in L(S)$, so $\sigma(L) = X \in \mathcal{L}(S)$. $\square$

The following example is to prove the existence of isomorphisms that are not necessarily $\mathcal{L}$-fit or $\mathcal{L}$-full for a given left idealtor $\mathcal{L}$.

Example 3.1.4. ([62]) Fix a ring R_0 with $J(R_0) \neq 0$. Define, the left idealtor $\mathcal{L}$ for each ring R as follows:

$$\mathcal{L}(R) = \begin{cases} \{J(R_0)\} & \text{if } R = R_0 \\ \{0\} & \text{if } R \neq R_0 \end{cases}$$

Write $S = R_0 \times \{0\}$, and define the isomorphism $\phi : S \mapsto R_0$ by $(r, 0)\phi = r$ for any $r \in R_0$. Then, ϕ is neither $\mathcal{L}$-fit nor $\mathcal{L}$-full.

Proof. If $L \in \mathcal{L}(S)$, then by definition we have $L = 0$. Thus, $\phi(L) = \phi(0) = 0 \notin \mathcal{L}(R_0)$, and hence ϕ is not $\mathcal{L}$-fit. In addition, if $X \in L(R_0)$, then by definition we have $X = J(R_0) \neq 0$. On the other hand, $L = 0$ is the only element in $\mathcal{L}(S)$ and $\phi(L) = \phi(0) = 0 \neq J(R_0) = X$ which implies that ϕ is not $\mathcal{L}$-full, as required. $\square$

From the previous Example 3.1.4, we coclude that, isomorphisms do not preserve $\mathcal{L}$-stability in general. However, if the left idealtor $\mathcal{L}$ enjoys the followng property, then isomorphisms *do* preserve $\mathcal{L}$-stability.

Definition 3.1.5. ([63]) A left-ideal-map $\mathcal{L}$ is **natural** if every ring isomorphism is $\mathcal{L}$-fit. (equivalently, $\mathcal{L}$-full). In this case we shall call $\mathcal{L}$ a **left idealtor**.

The following will be an example of left idealtors that will be used for the rest of this context.

Example 3.1.6. ([62]) Let R be any ring.

1. The set of all left ideals of R will be denoted by

$$\mathcal{B}(R) = \{L \mid L \text{ is a left ideal of } R\}$$ [1]

2. The set of all left principal ideals of R generated by a will be denoted by

$$\mathcal{P}(R) = \{Ra : a \in R\}$$

3. The set of left annihilators of a in R will be denoted by

$$\mathcal{K}(R) = \{\mathtt{l}(a) : a \in R\}$$ [2]

4. The set of all left principal ideals of R generated by idempotent e will be denoted by

$$\mathcal{E}(R) = \{Re : e^2 = e \in R\}$$

5. The set of all left ideals of R contained in its Jacobson's radical will be denoted by

$$\mathcal{J}(R) = \{L \leq R : L \subseteq J(R)\}$$

[1] The calli letter "$\mathcal{B}$" stands for Bass

[2] The calli letter "$\mathcal{K}$" stands for Kaplanski

Example 3.1.7. ([62]) For the ring of integers $\mathbb{Z}$, we have:

1. $\mathcal{B}(\mathbb{Z}) = \{n\mathbb{Z} \mid n \in \mathbb{Z}\}$
2. $\mathcal{P}(\mathbb{Z}) = \{n\mathbb{Z} \mid n \in \mathbb{Z}\}$
3. $\mathcal{K}(\mathbb{Z}) = \{0, \mathbb{Z}\}$
4. $\mathcal{E}(\mathbb{Z}) = \{0, \mathbb{Z}\}$
5. $\mathcal{J}(\mathbb{Z}) = \{0\}$

Definition 3.1.8. ([63]) Let $\mathcal{L}$ be a left idealtor. An element a in a ring R is called **$\mathcal{L}$-stable**[3] if $Ra + L = R$ where $L \in \mathcal{L}(R)$ implies $a - u \in L$ for some unit u in R. And a ring R is called **$\mathcal{L}$-stable** if every element a in R is $\mathcal{L}$-stable.

Again, we are insisting that every idealtor $\mathcal{L}$ is natural, that is $\mathcal{L}$ has the property that all ring isomorphisms are both $\mathcal{L}$-fit and $\mathcal{L}$-full. The reason for this is because otherwise $\mathcal{L}$-stability may not be preserved under ring isomorphisms!

Example 3.1.9. ([63]) Given a division ring D, let $E = M_\omega(D)$ and let $S = E \times 0$ where 0 is the zero ring. With this define a left-ideal-map $\mathcal{L}$ such that $\mathcal{L}(E) = \{E\}$ and $\mathcal{L}(R) = \{0\}$ for any ring $R \neq E$. Then $E \cong S$ as rings, E is $\mathcal{L}$-stable, but S is not $\mathcal{L}$-stable.

Proof. First $E \cong S$ as rings via $\alpha \mapsto (\alpha, 0)$ for $\alpha \in E$. To see that E is $\mathcal{L}$-stable, assume that $E\alpha + L = E$, $\alpha \in E$, $L \in \mathcal{L}(E)$. Since $\mathcal{L}(E) = \{E\}$ we have $L = E$ so $\alpha - 1 \in L$, as required. To see that S is not $\mathcal{L}$-stable, we show that if S is $\mathcal{L}$-stable then S is Dedekind finite (a contradiction as $S \cong E$). So let $ba = 1$ in S. Then $Sa + 0 = S$ and $0 \in \mathcal{L}(S)$ as $S \neq E$. If S is $\mathcal{L}$-stable this implies $a - u \in 0$ where $u \in U(R)$. Thus a is a unit so, as $ab = 1$, we get $ba = 1$. □

Note that the left-ideal-map $\mathcal{L}$ in Example 3.1.9 is not natural because $E \mapsto R \notin \mathcal{L}(R)$. Hence $\mathcal{L}$ is not a left idealtor. However, this is not the case when the left-ideal-map is an idealtor.

Theorem 3.1.10. ([63]) Let $\mathcal{L}$ be any left idealtor. If $\sigma : R \mapsto S$ is a ring isomorphism, then R is $\mathcal{L}$-stable if and only if S is $\mathcal{L}$-stable.

Proof. Let R be $\mathcal{L}$-stable. To show that S is $\mathcal{L}$-stable, let $Sb + X = S$, $X \in \mathcal{L}(S)$, $b \in S$. Apply σ^{-1} to get $R\sigma^{-1}(b) + \sigma^{-1}(X) = R$. But $\sigma^{-1}(X) \in \mathcal{L}(S)$ because σ^{-1} is $\mathcal{L}$-fit by hypothesis, so the fact that R is $\mathcal{L}$-stable shows that $\sigma^{-1}(b) - u \in \sigma^{-1}(X)$ where $u \in U(R)$. Applying σ shows that $b - \sigma(u) \in X$. Since $\sigma(u) \in U(S)$, this proves that S is $\mathcal{L}$-stable. The converse is analogous. □

Remark 3.1.11. If $\mathcal{L}$ is a left idealtor and $\theta : R \mapsto S$ is any onto ring morphism, we regard θ as a map i.e., $\theta : \mathcal{L}(R) \mapsto \mathcal{L}(S)$ where $L \mapsto \theta(L)$. Observe that:

1. θ is $\mathcal{L}$-fit if and only if $\theta[\mathcal{L}(R)] \subseteq \mathcal{L}(S)$.
2. θ is $\mathcal{L}$-full if and only if $\mathcal{L}(S) \subseteq \theta[\mathcal{L}(R)]$.

[3]Note that we did not say that the element a is "left $\mathcal{L}$-stable" here; because "sidedness" of an $\mathcal{L}$-stable element stands for the "sidedness" of the unit u in the definition.

3. θ is $\mathcal{L}$-fit and $\mathcal{L}$-full if and only if $\theta[\mathcal{L}(R)] = \mathcal{L}(S)$.

Proposition 3.1.12. ([63]) Let $\mathcal{L}$ be a left idealtor, and let $\sigma : R \mapsto S$ be a ring isomorphism. Then:

1. $|\mathcal{L}(R)| = |\mathcal{L}(S)|$ via the bijection $L \mapsto \sigma(L)$ from $\mathcal{L}(R) \mapsto \mathcal{L}(S)$.

2. $\mathcal{L}(S) = \{\sigma(L) \mid L \in \mathcal{L}(R)\}$.

Proof. Because $\mathcal{L}$ is natural, σ is $\mathcal{L}$-fit so $L \mapsto \sigma(L)$ defines a map $L(R) \mapsto L(S)$. Similarly $X \mapsto \sigma^{-1}(X)$ carries $L(S) \mapsto L(R)$. As these maps are mutually inverse, (1) and (2) follow. □

Here we list some useful facts about when onto ring morphisms are full or fit.

Remark 3.1.13. Let $\mathcal{L}$ be a left idealtor, and let φ and θ denote onto ring morphisms.

1. If φ and θ are $\mathcal{L}$-fit ($\mathcal{L}$-full), then so is the composition $\varphi \circ \theta$.

2. If σ, τ are ring isomorphisms then $\theta \circ \sigma$ (respectively $\tau \circ \theta$) is $\mathcal{L}$-fit ($\mathcal{L}$-full) if and only if the same is true for θ.

3. $\theta : R \mapsto S$ is $\mathcal{L}$-fit ($\mathcal{L}$-full) if and only if the same is true of the coset map $R \mapsto R/\ker(\theta)$.

Definition 3.1.14. ([62]) We say that a class $\mathfrak{C}$ of rings is **afforded** by a left idealtor $\mathcal{L}$ if $\mathfrak{C}$ is the class of all $\mathcal{L}$-stable rings. We say that a class $\mathfrak{C}$ of rings is **affordable** if it is afforded by some left idealtor $\mathcal{L}$ (or, that the left idealtor $\mathcal{L}$ **affords** the class of rings $\mathfrak{C}$).

Now, the following example seems familiar.

Example 3.1.15. ([62]) The class of SR1 rings is afforded the left idealtor $\mathcal{B}(R) = \{L \mid L$ is a left ideal of $R\}$

Proof. By definition, a ring R is SR1 if $Ra + L = R$, $a \in R$ and L is a left ideal of R, implies that $a - u \in L$ for some $u \in U(R)$. So, a would be $\mathcal{B}$-stable, thus, the class of SR1 rings is precisely the class of $\mathcal{B}$-stable rings, i.e., $\{\text{SR1}\} = \{\text{left } \mathcal{B}\text{-stable}\}$ □

Lemma 3.1.16. ([62]) Let $\mathcal{M}$ and $\mathcal{L}$ be two left idealtors. If $\mathcal{M}(R) \supseteq \mathcal{L}(R)$ for each ring R, then $\{\text{left } \mathcal{M}\text{-stable}\} \subseteq \{\text{left } \mathcal{L}\text{-stable}\}$.

Proof. Let R be left $\mathcal{M}$-stable. Suppose that $Ra + L = R$ with $L \in \mathcal{L}(R)$ and $a \in R$. Then, by assumption, $L \in \mathcal{M}(R)$ and hence $a - u \in L$ for some unit u in R because R is left $\mathcal{M}$-stable. Hence, a is left $\mathcal{L}$-stable, and so R is left $\mathcal{L}$-stable, as required. □

The following example proves that the left idealtor that affords a class of rings need not be unique.

Example 3.1.17. ([62]) The class of SR1 rings is afforded by the left idealtor $\mathcal{P}(R) = \{Ra \mid a \in R\}$.

Proof. Since $\{\text{left } \mathcal{B}\text{-stable}\} = \{\text{SR1}\}$, we show that $\{\text{left } \mathcal{P}\text{-stable}\} = \{\text{left } \mathcal{B}\text{-stable}\}$. Assume that R is left $\mathcal{P}$-stable. To see that R is left $\mathcal{B}$-stable, let $Ra + L = R$ with $a \in R$ and $L \in \mathcal{B}(R)$, say $ra + b = 1$ where $r \in R$ and $b \in L$. Thus, $Ra + Rb = R$ which implies that $a - u \in Rb \subseteq L$ for some unit u in R because a is left $\mathcal{P}$-stable . Hence, $\{\text{left } \mathcal{P}\text{-stable}\} \subseteq \{\text{left } \mathcal{B}\text{-stable}\}$. Now, because $\mathcal{P}(R) \subseteq \mathcal{B}(R)$ for every ring R, we have $\{\text{left } \mathcal{B}\text{-stable}\} \subseteq \{\text{left } \mathcal{P}\text{-stable}\}$ by Lemma 3.1.16, as required. □

Example 3.1.18. ([63]) The class of all left UG rings is afforded by the left idealtor $\mathcal{K}(R) = \{\mathtt{l}(b) \mid b \in R\}$.

Proof. Let R be left UG. If $Ra + \mathtt{l}(b) = R$ then $Rab = Rb$ so, as R is left UG, $ab = ub$ with $u \in U(R)$. Hence $a - u \in \mathtt{l}(b)$, so R is $\mathcal{K}$-stable. Conversely, if R is $\mathcal{K}$-stable and $Ra = Rb$, write $a = pb$ and $b = qa$ where $p, q \in R$. Then $b = qpb$, so $1 - qp \in \mathtt{l}(b)$ and we have $Rp + \mathtt{l}(b) = R$. Since p is $\mathcal{K}$-stable we have $p - u \in \mathtt{l}(b)$ for some $u \in U(R)$, so $pb = ub$, that is, $a = ub$. $\square$

Example 3.1.19. ([63]) The class of IC rings are afforded by the left idealtor $\mathcal{E}(R) = \{Re \mid e^2 = e \in R\}$.

Proof. Suppose R is left $\mathcal{E}$-stable. To see that R is IC, let $a \in R$ be regular, say $a = aba$, and write $e = ba$. Then $e^2 = e$ and $Ra = Re$, so $Ra + R(1 - e) = R$. As a is $\mathcal{E}$-stable we have $a - u \in R(1 - e)$ for some $u \in U(R)$. Hence $ue = ae = a$, so $a(u^{-1}a) = ae = a$, as required. Conversely, if R is IC, let$Ra + Re = R$, $e^2 = e$, say $ra + se = 1$. We must show that $a - u \in Re$ for some $u \in U(R)$. Write $f = 1 - e$. As $ra + se = 1$ we have $raf = f$; so $af(raf) = af^2 = af$. Thus af is regular, hence unit-regular (by hypothesis), whence SR1 (by Theorem 2.1.28). But $raf = f = 1 - e$, so $Raf + Re = R$. As af is SR1 it follows that $af - u \in Re$ for some $u \in U(R)$. Finally $af = a - ae$ so $a - u = (af + ae)u = (af - u) + ae \in Re$, as required. Finally, $\mathcal{E}$ is clearly natural. $\square$

Theorem 3.1.20. ([63]) Let $\mathcal{L}$ be a left idealtor such that for each ring R, there exists $I \in \mathcal{L}(R)$ such that $I \subseteq J(R)$, then Then $\{\mathcal{L}\text{-stable}\} \subseteq \{\text{DF}\}$.

Proof. Let R be $\mathcal{L}$-stable. Choose $I \in \mathcal{L}(R)$ where $I \subseteq J(R)$. To prove R is DF, let $Ra = R$. Then certainly $Ra + I = R$ so, as R is $\mathcal{L}$-stable, let $a - u \in I$, $u \in U(R)$. But if we write $a - u = b$ then $a = u + b$ is a unit too as $b \in I \subseteq J(R)$. Hence $aR = R$, proving that R is DF. $\square$

Example 3.1.21. ([63]) The class of DF rings is afforded by the left idealtor $\mathcal{D}(R) = \{L \mid L \subseteq J(R)\}$.

Proof. We have {left $\mathcal{D}$-stable} $\subseteq$ {DF} by Theorem 3.1.20. Conversely, assume R is DF and let $Ra + L = R$, $L \in D(R)$. Thus $L \subseteq J(R)$, and so $Ra = R$: But then a is a unit (R is DF), so $a - u \in L$ where $u = a \in U(R)$. So R is $\mathcal{D}$-stable. $\square$

We now show that the class of DF rings is afforded by some left idealtors, which gives a new perspective on these rings, as follows

Example 3.1.22. ([62]) The class {DF} is afforded by the left idealtor $\mathcal{T}(R)$.

Proof. Since {DF} is afforded by $\mathcal{J}(R)$ which in turn, contains $\mathcal{T}(R)$. It follows that {DF} = {left $\mathcal{J}$-stable} $\subseteq$ {left $\mathcal{T}$-stable}. So it suffices to show that {left $\mathcal{T}$-stable} $\subseteq$ {DF}. To finish this, Let $ba = 1$ in R, we need to show that a is a unit in R. Now, since $ba = 1$ we have $Ra + 0 = R$. As $0 \in \mathcal{T}(R)$, it follows that $a - u \in 0$ for some $u \in U(R)$ because R is left $\mathcal{T}$-stable. Therefore, $a = u$ is a unit in R which implies that R is a DF ring, as required. $\square$

Remark 3.1.23. Let $\mathfrak{C}$ be an affordable class of rings by the left idealtors $\mathcal{L}(R)$ and $\mathcal{N}(R)$, where R is a ring in $\mathfrak{C}$. If $\mathcal{M}(R)$ is a left idealtor such that $\mathcal{L}(R) \subseteq \mathcal{M}(R) \subseteq \mathcal{N}(R)$, then $\mathfrak{C}$ is afforded by $\mathcal{M}(R)$ too.

Proof. Since $\mathfrak{C}$ is afforded by the left idealtors $\mathcal{L}(R)$ and $\mathcal{N}(R)$, then by Lemma 3.1.16 we have that $\mathfrak{C} = \{\text{left } \mathcal{L}\text{stable}\} \supseteq \{\text{left } \mathcal{M}\text{-stable}\} \supseteq \{\text{left } \mathcal{N}\text{-stable}\} = \mathfrak{C}$, which implies that $\mathfrak{C} = \{\text{left } \mathcal{M}\text{-stable}\}$, i.e., $\mathfrak{C}$ is afforded by $\mathcal{M}(R)$. □

Corollary 3.1.24. The class $\{\text{DF}\}$ is afforded by each one of the following left idealtors:

1. $\mathcal{D}_{Tnilpotent}(R) = \{L \mid L \text{ is a left } T\text{-nilpotent ideal of } R\}$.
2. $\mathcal{D}_{locally}(R) = \{L \mid L \text{ is a locally nilpotent ideal of } R\}$.
3. $\mathcal{D}_{nilbdd}(R) = \{L \mid L \text{ is nil of bounded index ideal of } R\}$.
4. $\mathcal{D}_{Wedderburn}(R) = \{W(R)\}$
5. $\mathcal{D}_{Levitsky}(R) = \{\text{Levi}(R)\}$.
6. $\mathcal{D}_{lower}(R) = \{\text{Nil}_*(R)\}$
7. $\mathcal{D}_{upper}(R) = \{\text{Nil}^*(R)\}$
8. $\mathcal{D}(R) = \{L \mid L \subseteq J(R)\}$.
9. $\mathcal{D}_{Jacobson}(R) = \{J(R)\}$.
10. $\mathcal{D}_{nil}(R) = \{L \mid L \text{ is a nil left ideal of } R\}$.
11. $\mathcal{D}_{nilpotent}(R) = \{L \mid L \text{ is a nilpotent ideal of } R\}$.
12. $\mathcal{D}_{qreg}(R) = \{L \mid L \text{ is a left quasi-regular ideal of } R\}$.
13. $\mathcal{T}(R) = \{0\}$.

Proof. All of the listed idealtors lie between $\mathcal{T}(R)$ and $\mathcal{D}(R)$. And the result follows by applying remark 3.1.23. □

It is quite remarkable that the class of all SR1 rings acts like a "lower bound" among all affordable classes. We have the following relatable theorem.

Theorem 3.1.25. ([63]) If $\mathfrak{C}$ is an affordable class of rings, then $\{\text{SR1}\} \subseteq \mathfrak{C}$.

Proof. Since the left idealtor $\mathcal{B}(R)$ consists of all left ideals of R, then for any left idealtor $\mathcal{L}(R)$, it would be always the case that $\mathcal{B}(R) \supseteq \mathcal{L}(R)$. It follows by Lemma 3.1.16 that $\{\mathcal{B}\text{-stable}\} \subseteq \{\mathcal{L}\text{-stable}\}$. Henceforth, $\{\text{SR1}\} \subseteq \mathfrak{C}$. □

Corollary 3.1.26. Let $\mathfrak{C} \subseteq \{\text{SR1}\}$ where $\mathfrak{C}$ is affordable. By Theorem 3.1.25, $\{\text{SR1}\} \subseteq \mathfrak{C}$

Thus, the contrapositive of Theorem 3.1.25 gives an explicit statement which make us decide when a class of rings is not affordable.

Remark 3.1.27. (Non-affordability Criterion) A class of rings $\mathfrak{C}$ is not affordable if there exists an SR1 ring that is not in $\mathfrak{C}$.

A ring R is called an **SBI** ring or **Lift/rad** ring if all idempotents of R lift modulo $J(R)$.[4]. The class of SBI rings includes: regular rings, π-regular rings, exchange rings, Zorn rings and potent rings.

Corollary 3.1.28. ([63]) Any class $\mathfrak{C}$ of rings in which $J(R)$ is lifting for each $R \in \mathfrak{C}$ is not affordable (In particular, {exchange} is not affordable).

Proof. Consider the ring $\mathbb{Z}_{(2,3)} = \left\{\frac{a}{b} \in \mathbb{Q} \mid 2 \nmid b, 3 \nmid b\right\}$, then $\mathbb{Z}_{(2,3)}/J(\mathbb{Z}_{(2,3)}) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, so $\mathbb{Z}_{(2,3)}$ is SR1 but $J(\mathbb{Z}_{(2,3)})$ is not lifting, thus, not exchange. Hence, in particular, {exchange} is not affordable class of rings and the result follows. $\square$

Proposition 3.1.29. If $\mathfrak{C}$ is not affordable class of rings and $\mathfrak{D}$ is a class of rings such that $\mathfrak{D} \subseteq \mathfrak{C}$, then $\mathfrak{D}$ is not affordable as well.

Proof. Let $\mathfrak{C}$ be not affordable, then by Remark 3.1.27 we have an SR1 ring R such that $R \notin \mathfrak{C}$ and since $\mathfrak{D} \subseteq \mathfrak{C}$, we have that $R \notin \mathfrak{D}$ too. Hence, $\mathfrak{D}$ is not affordable. $\square$

However, If $\mathfrak{C} \subseteq \mathfrak{D}$ are classes of rings and $\mathfrak{D}$ is affordable then $\mathfrak{C}$ need not be affordable. We have plenty of denials, for instance

- {eversible} $\subseteq$ {DF}. {DF} is affordable but {eversible} is not.
- {Abelian} $\subseteq$ {IC}. {IC} is affordable but {Abelian} is not.
- {domain} $\subseteq$ {left UG}. {left UG} is affordable but {domain} is not.
- {semilocal} $\subseteq$ {SR1}. {SR1} is affordable but {semilocal} is not.

A ring is said to be **right duo** if all right ideals are two-sided ideals. Left duo rings are defined similarly, and a ring is called **duo** if it is both left and right duo [24]. A ring R is **quasi-Frobenius** if R is Noetherian on one side and self-injective on one side. A ring R is **left P-injective** (**left mininjective**) if every R-linear map $L \mapsto_R R$, $L \subseteq_R R$, extends to R where L is any principal (respectively simple) left ideal. (Clearly left self-injective rings are left P-injective) [89]. A ring R is called left quasi-morphic if, for every $a \in R$, we have $Ra = \mathtt{l}(b)$ and $\mathtt{l}(a) = Rc$ for some b and c in R and it is left morphic if $b = c$ for each $a \in R$. R is called **(left) pseudo-morphic** if $\{Ra : a \in R\} \subseteq \{\mathtt{l}(b) : b \in R\}$, that is, every (left) principal ideal is a left annihilator ideal. Obviuosly, left quasi-morphic rings are left pseudo-morphic [23]. A ring R is called **left special** if R is left morphic, local and $J(R)$ is nilpotent. A ring R is called **(left) generalized morphic** if $\{\mathtt{l}(b) : b \in R\} \subseteq \{Ra : a \in R\}$, that is, every left annihilator ideal is a left principal ideal[5]. A ring is called left PP if principal left ideals are all projective (equivalently, $\mathtt{l}(a)$ is a direct summand of ${}_RR$). Left PP are left generalized morphic [112]. A ring R is called **Baer** if the left annihilator of every nonempty subset of R is generated by an idempotent. Every Baer ring is left PP [93]. A ring R is said to satisfy the **IFP** (insertion of factors property) if $\mathtt{l}(a)$ is an ideal of R for all $a \in R$ (equivalently, $\mathtt{l}(X)$ is an ideal of R for all nonempty subsets X of R). IFP rings are Abelian [15]. A ring R is called **left fusible** if every nonzero element is **left fusible**, that is, the sum of a left zero-divisor and a non-left zero-divisor. [51]. An element a in

[4] The abbreviation "SBI" was introduced by Irving Kaplansky and stands for "suitable for building idempotent elements". For further results on SBI rings, the reader is referred to [68, Chapter 3]

[5] The left ideal Ra is projective if and only if $\mathtt{l}(a) = Re$ where e is an idempotent.

R is said to be **left G-morphic** if there exists $n > 0$ with $a^n = 0$ such that a^n is left morphic, equivalently, there exist $n > 0$ with $a^n \neq 0$ and $b \in R$ such that $\mathtt{l}(a^n) = Rb$ and $\mathtt{l}(b) = Ra^n$. The ring itself is called **left G-morphic** if every element is left G-morphic. A ring is **SSP** if the sum of two direct summands is a direct summand. R is SSP and IC if and only if The product of two regular elements is unit-regular. [29]. A ring R is called a **left-max** ring if every nonzero right R-module has a maximal submodule. A ring R (equivalently, if $R/J(R)$ is a left-max ring, and the ideal $J(A)$ is left T-nilpotent). We call R a **right complemented** ring, if for each $a \in R$, there is $ab \in R$ such that $ab = 0$ and $a + b$ is regular. Clear that if R is right (left) complemented, then R is right (left) fusible because if we let $0 \neq a \in R$ and choose $b \in R$ such that $ab = 0$ and $a+b$ is regular, then $a = (a + b) - b$ is a right fusible representation. it is also reduced for $a \in R$ such that $a^2 = 0$, choose d regular such that $ad = a^2$. This forces $a = 0$. A ring is called an **idempotent-fine** ring (briefly, an **IF** ring) if all its nonzero idempotents are fine, that is, a sum of a nilpotent and a unit. A nonzero element in a ring is called fine if it is a sum of a unit and a nilpotent and a ring is a **fine** ring if every nonzero element is fine . Rings whose all nonzero nilpotents are fine are be called **nilpotent-fine** (briefly, **NF**) [21]. We call a ring **left soclin** if every simple left ideal is contained in the Jacobson radical [64]

Example 3.1.30. None of the following classes of rings are affordable:

1. {commutative},{Abelian}, {reversible}, {unit-central}.
2. {exchange} , {clean}, {strongly clean}, {special clean} , {semiperfect},

   {commutative regular} , {regular}, {π-regular} , {semiregular}, {unit-regular}, {SUR}, {0-dimensional commutative}, {strongly regular}, {local}, {semisimple} , {Euler}, {exact Euler}, {Boolean} , {division ring}, {field}.
3. {artinian} , {one-sided artinian}, {semiprimary} , {left perfect}, {perfect}, {semilocal} , {casilocal}.
4. {field}, {Euclidean domain}, {PID}, {UFD}, {integral domain}, {domain}.
5. {quasi-normal}.
6. {Zorn}, {potent}.
7. {one-sided Noetherian}, {Noetherian}, {quasi-Frobenius}
8. {prime}, {semiprime}, {left mininjective}, {left P-injective}, {left self-injective}, {left Kasch}.
9. {left generalized morphic}, {left PP} ,{Baer}, {left morphic}, {left quasi-morphic}, {left pseudo-morphic}, {left special}.
10. {left fusible}, {left complemented}
11. {left G-morphic}.
12. {eversible}.
13. {SSP}
14. {idempotent-fine}, {nilpotent-fine}, {fine}.

15. {J-Abelian}, {J-quasipolar}, {J-clean}, {J-Armendariz}.

16. {left soclin}.

17. {RS}

Proof. 1. Because $\begin{bmatrix} \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \end{bmatrix}$ is SR1 but not Abelian. Hence, {Abelian} is not affordable, and the result follows since all are subclasses of {Abelian}.[6]

2. All are subclasses of exchange rings. And the fact that they are not affordable follows from Corollary 3.1.28

3. All are proper subclasses of SR1 rings.

4. All are connected, thus, Abelian.

5. Let D be a division ring and $R = \begin{bmatrix} D & D & D \\ 0 & D & D \\ 0 & 0 & D \end{bmatrix}$. Then R is SR1. Consider the idempotent $e = e_{11} + e_{33}$; by computing, we can see that $eR(1-e)Re = \begin{bmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$ so R is not quasi-normal.

6. A Zorn ring must satisfy that $J(R)$ is nil, so $J(R)$ is lifting. A potent ring must satisfy that $J(R)$ is lifting.

7. The ring of all algebraic integers $\overline{\mathbb{Z}}$ is SR1 by Example 2.1.13 but not left-Noetherian. For example, it contains the infinite ascending chain of principal ideals.
$$\langle 2 \rangle, \langle 2^{\frac{1}{2}} \rangle, \langle 2^{\frac{1}{4}} \rangle, \langle 2^{\frac{1}{8}} \rangle, \cdots$$

8. $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ is SR1 but enjoys none of the properties.

9. Let $R = (\Pi_{i=1}^{\infty} \mathbb{Z}_2) \,/\, (\oplus_{i=1}^{\infty} \mathbb{Z}_2)$. It is obvious that R is a Boolean ring, thus, SR1, hence $\mathbb{T}_n(R)$ is SR1 for any $n \in \mathbb{N}$ but not left generalized morphic for some $n \geq 1$ (See [81, Example 3.13]). Furthermore, $S = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ is SR1 but not left pseudo-morphic.[7]

[6] In [91, Figure 1], relations between well-known large classes of rings (e.g. {reduced} , {symmetric} , {reversible} , {semi-commutative}, {Guassian},{Armendariz}) have been studied. It is illustrated that all of prementioned classes of rings are Abelian. Moreover, the classes of duo rings, and left duo rings are known to be semi-commutative [24], thus, Abelian. As an extra piece of information, every reduced ring is Armendariz and the following two inclusion chains are known to be irreversible:
$$\{\text{reduced}\} \subseteq \{\text{symmetric}\} \subseteq \{\text{reversible}\} \subseteq \{\text{semi-commutative}\} \subseteq \{\text{Abelian}\}$$
$$\{\text{commutative}\} \subseteq \{\text{duo}\} \subseteq \{\text{one-sided duo}\} \subseteq \{\text{semi-commutative}\}$$

[7] In fact[23, Proposition 2.6] asserts that no upper triangular matrix ring is left or right pseudo morphic.

10. $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ is SR1 but not left fusible.[8]. Alternatively, for any prime integer p and $n \geq 2$, $\mathbb{Z}_{p^n}$ is is SR1 but not fusible.[9] (See [51, Example 2.3])

11. $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ is SR1 but not left G-morphic. (See [65, Example 2.10]).

12. There exists a local ring, thus, SR1 ring which is not eversible (See [50, Example 2.3]).

13. If $R = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$, then R is SR1 (hence IC) and $e = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $f = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ are idempotents, thus, regular but their product $ef = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ is not regular, as $ef \notin (ef)R(ef) = 0$, hence not unit-regular and so R is not SSP. (See [29, Example 2.11])

14. $\mathbb{Z}_6$ is SR1 being finite; but the idempotent $\overline{4} = \overline{0} + \overline{4} = \overline{1} + \overline{3} = \overline{2} + \overline{2}$ is not fine. Also, $\mathbb{Z}_4$ is SR1 being finite; but the nilpotent $\overline{2} = \overline{2} + \overline{0} = \overline{1} + \overline{1}$ is not fine.

15. The ring $R = \mathbb{H}(\mathbb{Z}_{(3)})/J(\mathbb{H}(\mathbb{Z}_{(3)})) \cong \mathbb{M}(\mathbb{Z}_3)$ is clearly SR1, and also clearly not Abelian. Moreover, it is not J-Abelian because $J(R) = 0$. (See [56, Example 2.9]).[10]

16. The ring $R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix} | a, b, c, d \in D \right\}$ where D is a division ring, is semilocal, thus, SR1 but not right soclin. With same reasoning, there exists SR1 rings that are not left soclin. (See [64, Example 4.9] for more details).

17. The existence of Example 1.2.52 adapts.

$\square$

Definition 3.1.31. ([62]) Two left idealtors $\mathcal{M}$ and $\mathcal{N}$ will be called **equivalent** (written $\mathcal{M} \equiv \mathcal{N}$) if $\mathcal{M}$ and $\mathcal{N}$ afford the same class of rings, that is if $\{\mathcal{M}\text{-stable}\} = \{\mathcal{N}\text{-stable}\}$.

As the name suggests, it is obvious that "$\equiv$" in Definition 3.1.31 is an equivalence relation on the class of left idealtors. The following example is familiar.

Example 3.1.32. A list

1. The class {SR1} is afforded by both of the left idealtors $\mathcal{B}$ and $\mathcal{P}$. Therefore, $\mathcal{B} \equiv \mathcal{P}$.

2. The class {DF} is afforded by each of the left idealtors $\mathcal{J}, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$ and $\mathcal{T}$. Therefore, $\mathcal{J} \equiv \mathcal{J}_1 \equiv \mathcal{J}_2 \equiv \mathcal{J}_3 \equiv \mathcal{J}_4 \equiv \mathcal{T}$.

[8] [51, Corollary 2.14] tells that if R is a ring and $n \geq 2$, then ring of upper triangular matrices $\mathbb{T}_n(R)$ is never a left fusible ring.

[9] In fact, $\mathbb{Z}_n$ is fusible if and only if n is square free

[10] In fact, the statements [56, Lemma 2.3, Lemma 2.4, Proposition 2.5, Lemma 2.8, Theorem 2.11]) assert that if R is J-Armendariz, J-clean or J-quasipolar ring, then R is J-Abelian. Also, J-Abelian rings are DF. Furthermore, J-Abelian exhachange rings are clean. For more on these rings, see [56].

Example 3.1.33. ([62]) Each of the following five classes of rings is afforded by the corresponding left idealtor

1. The class of all rings is afforded by $\mathcal{C}(R) = \{R\}$.
2. The class of SR1 rings is afforded by $\mathcal{B}(R) = \{L \mid L \text{ is a left ideal of } R\}$ and $\mathcal{P}(R) = \{Ra : a \in R\}$.
3. The class of left UG rings is afforded by $\mathcal{K}(R) = \{l(a) : a \in R\}$.
4. The class of IC rings is afforded by $\mathcal{E}(R) = \{Re : e^2 = e \in R\}$.
5. The DF rings are afforded by the following equivalent left idealtors:
   (a) $\mathcal{J}(R) = \{L \mid L \subseteq J(R)\}$.
   (b) $\mathcal{J}_1(R) = \{J(R)\}$.
   (c) $\mathcal{J}_2(R) = \{L \mid L \text{ is a nil left ideal of } R\}$.
   (d) $\mathcal{J}_3(R) = \{L \mid L \text{ is a nilpotent ideal of } R\}$.
   (e) $\mathcal{J}_4(R) = \{L \mid L \text{ is a left quasi-regular ideal of } R\}$.
   (f) $\mathcal{T}(R) = \{0\}$.

Definition 3.1.34. ([63] If $\mathcal{M}$ and $\mathcal{L}$ are left idealtors, we say that $\mathcal{M}$ **covers** $\mathcal{L}$ and write $\mathcal{M} \geq^c \mathcal{L}$ if for each ring R : $b \in L \subseteq \mathcal{L}(R)$ implies that $b \in M \subseteq L$ for some $M \in \mathcal{M}(R)$.

Proposition 3.1.35. ([63]) Let $\mathcal{M}$ and $\mathcal{L}$ be any left idealtors, and let R denote a ring. Then

1. If $\mathcal{M}(R) \supseteq \mathcal{L}(R)$ for each ring R, then $\mathcal{M} \geq^c \mathcal{L}$.
2. If $\mathcal{M} \geq^c \mathcal{L}$, then $\{\mathcal{M} - \text{stable}\} \subseteq \{\mathcal{L}\text{stable}\}$.

Proof. 1. Assume that $\mathcal{M}(R) \supseteq \mathcal{L}(R)$ for each ring R. If $b \in L \in \mathcal{L}(R)$, then $b \in M \subseteq L$ where $M = L \in \mathcal{M}(R)$. This proves that $\mathcal{M} \geq^c \mathcal{L}$.

2. Let R be a $\mathcal{M}$-stable ring. If $Ra+L = R$ where $a \in R$ and $L \in \mathcal{L}(R)$, then$ra+b = 1$ for some $r \in R$ and $b \in L$. Because $\mathcal{M} \geq^c \mathcal{L}$, we have $b \in M \subseteq L$ for some $M \in \mathcal{M}(R)$. Hence, $1 = ra+b \in Ra+M$, and so $Ra+M = R$. Now, as R is $\mathcal{M}$-stable, we have $a - u \in M$ for some $u \in U(R)$. Since $M \subseteq L$, it follows that a is left $\mathcal{L}$-stable, and hence R is an $\mathcal{L}$-stable ring. Therefore, we have $\{\mathcal{M} - \text{stable}\} \subseteq \{\mathcal{L} - \text{stable}\}$, as required.

□

Recall that for a left idealtor $\mathcal{L}$, an element a in a ring R is $\mathcal{L}$-stable if $Ra + L = R$, $L \in \mathcal{L}(R)$, implies that $a - u \in L$ for some unit u. We now investigate the situation where u is only required to be left invertible, that is $Ru = R$. Our starting point is Vaserstein's proof [103, Theorem 2.6] that left units in an SR1 ring are right units, i.e., SR1 rings are DF.

Definition 3.1.36. ([63]) For a ring R and a left idealtor $\mathcal{L}$, an element $a \in R$ will be called **$\mathcal{L}$-Vaserstein** if $axa = a$, $x \in R$ implies $R(1 - xa) \in \mathcal{L}(R)$.

Here's a characterization for $\mathcal{L}$-Vaserstein elements.

Theorem 3.1.37. ([63]) Fix a left idealtor $\mathcal{L}$ and a ring R. If $a \in R$, the following conditions are equivalent:

1. a is $\mathcal{L}$-Vaserstein.

2. If $f^2 = f \in \mathtt{r}(a)$ and $1 - f \in Ra$, then $Rf \in \mathcal{L}(R)$.

Proof. (1) $\implies$ (2). If $f^2 = f \in \mathtt{r}(a)$ and $1 - f \in Ra$, write $1 - f = xa$, $x \in R$. Then $axa = a$, so (1) applies.

(2) $\implies$ (1) If $axa = a$ write $f = 1 - xa$. Then the hypotheses in (2) are satisfied. $\square$

Definition 3.1.38. ([63]) If $\mathcal{L}$ is a left idealtor call $a \in R$ **left $\mathcal{L}$-stable** if $Ra + L = R$, $L \in \mathcal{L}(R)$, implies $a - x \in R$ for some $x \in R$ with $Rx = R$.

Theorem 3.1.39. ([63]) Let $a \in R$ be $\mathcal{L}$-Vaserstein, and let $\mathcal{L}$ be any left idealtor. If a is left $\mathcal{L}$-stable, then $ab = 1$, $b \in R$ implies $ba = 1$.

Proof. If $ab = 1$, write $f = 1 - ba$. Then $f = f^2$, $1 - f = ba \in Ra$ and $af = a - aba = 0$. As a is $\mathcal{L}$-Vaserstein, $Rf \in \mathcal{L}(R)$. But $ba + f = 1$ so $Ra + Rf = R$. As a is left $\mathcal{L}$-stable, let $a - x \in Rf$ where $Rx = R$. Now observe that $fb = b - bab = 0$, so $(a-x)b \in Rfb = 0$. Thus $xb = ab = 1$, so x is right invertible too, and hence is a unit. But then b is also a unit (because $xb = 1$), whence a is a unit (because $ab = 1$). It follows that $ba = 1$, as required. Moreover, $a = b^{-1} = x$. $\square$

Theorem 3.1.40. ([63]) Let $\mathcal{L}$ be a left idealtor, and let R be an $\mathcal{L}$-Vaserstein ring. Then:

1. If R is left $\mathcal{L}$-stable then R is Dedekind finite.

2. R is $\mathcal{L}$-stable if and only if R is left $\mathcal{L}$-stable.

Proof. Each $a \in R$ is $\mathcal{L}$-Vaserstein by hypothesis, so (1) holds by Theorem 3.1.39. But then $Rx = R$ implies x is a unit, and (2) follows. $\square$

3.2 Morphisms and Basic Properties

In this section we study various interesting results involving elements in $\mathcal{L}$-stable rings.

We begin this section with the following lemma

Lemma 3.2.1. ([63]) Let $\mathcal{L}$ be a left idealtor, and let $\theta : R \mapsto S$ be an onto ring morphism. Then for any element $a \in R$ we have:

1. If θ is $\mathcal{L}$-fit, then $\theta(a)$ is $\mathcal{L}$-stable in S, then a is $\mathcal{L}$-stable in R provided either (a) $\ker\theta \subseteq J(R)$ or (b) units lift modulo $\ker\theta$ and $\ker\theta \subseteq L$ for all $L \in \mathcal{L}(R)$.

2. If θ is $\mathcal{L}$-full, then a is $\mathcal{L}$-stable in R, then $\theta(a)$ is $\mathcal{L}$-stable in S provided either (a) $\ker\theta \subseteq J(R)$ or (c) $L + \ker\theta \in \mathcal{L}(R)$ for all $L \in \mathcal{L}(R)$.

Proof. For ease of use, write $\theta(r) = \overline{r}$.

(1) Assume $\overline{a}$ is $\mathcal{L}$-stable in $S = \overline{R}$. Let $Ra + L = R$, $L \in \mathcal{L}(R)$, say $ra + l = 1$ where $r \in R$ and $l \in L$. Then $\overline{ra} + \overline{l} = \overline{1}$, so $\overline{Ra} + \overline{L} = \overline{R}$. Here $\overline{L} \in \mathcal{L}(S)$ because is $\mathcal{L}$-fit, and $\overline{a}$ is $\mathcal{L}$-stable in S by hypothesis. So we have $\overline{a} = \overline{u} \in \overline{L}$ for some $\overline{u} \in U(S)$. Hence,

$$a - u \in L + \ker\theta \text{ where } \overline{u} \in U(S) \quad \ldots \ (\star)$$

(a) By $(\star)$ let $a - u - l \in A$ where $l \in L$. Writing $c = a - u - l$ we have $a - (u + c) = l \in L$. Moreover, $u + c \in U(R)$ because $c \in A \subseteq J(R)$ by (a). Hence a is $\mathcal{L}$-stable, proving (1) in this case.

(b) Now $(\star)$ gives $a - u \in L + A = L$ and, as $\overline{u} \in U(S)$, we may assume $u \in U(R)$ again by (b). This proves (1) in this case.

(2) Assume a is $\mathcal{L}$-stable in R: Let $S\overline{a} + X = S$, $X \in \mathcal{L}(S)$. As θ is $\mathcal{L}$-full write X $= \overline{L}$ where $L \in \mathcal{L}(R)$. Then $S\overline{a} + \overline{L} = S$, say $ra + l - 1 \in A$, $r \in R$, $l \in L$. It follows that $Ra + L + A = R$. This implies $Ra + L = R$ in both cases (a) and (c). But then, as a is $\mathcal{L}$-stable in R, we have $a - u \in L$ where $u \in U(R)$. Hence $\overline{a} - \overline{u} \in \overline{L} = X$ and $\overline{u} \in U(S)$, proving (2).

□

As Theorem 3.2.1 deals with elements. The result for rings follows.

Corollary 3.2.2. ([63]) Let $\theta : R \mapsto S$ be an onto ring morphism and let $\mathcal{L}$ be a left idealtor.

1. If S is $\mathcal{L}$-stable; then R is $\mathcal{L}$-stable if θ is $\mathcal{L}$-fit and either $\ker(\theta) \subseteq J(R)$ or units lift modulo $\ker(\theta)$, and $\ker(\theta) \subseteq L$ for all $L \in \mathcal{L}(R)$.

2. If R is $\mathcal{L}$-stable, then S is $\mathcal{L}$-stable if θ is $\mathcal{L}$-full and either $\ker(\theta) \subseteq J(R)$ or $L + \ker(\theta) \subseteq L(R)$ for all $L \in L(R)$.

For SR1 rings it is clear that every onto ring morphism $\theta : R \mapsto S$ is $\mathcal{B}$-fit and $\mathcal{B}$-full. So if R is SR1 then S is SR1 by Corollary 3.2.2 (2). However, the converse can fail (consider $\mathbb{Z} \mapsto \mathbb{Z}_2$).

Lemma 3.2.3. ([63]) If $\mathcal{L}$ is any left idealtor, the following hold for each ring R.

1. $u^{-1}Lu \in \mathcal{L}(R)$ for any $L \in L(R)$ and any unit u of R.

2. $Lu \in L(R)$ for any $L \in \mathcal{L}(R)$ and any unit u of R.

Proof. If $u \in U(R)$, consider the conjugation isomorphism $\sigma_u : R \mapsto R$ where $\sigma_u(r) = uru^{-1}$ for all $r \in R$: Since $\mathcal{L}$ is natural, σ_u is $\mathcal{L}$-fit, which proves (1). Then (2) follows because $vL = L$ for any unit v and any left ideal L. □

Next theorem says that the subset of all left $\mathcal{L}$-stable elements of a ring R admits an algebraic structute, more procisely, a multiplicative submonoid of R as long as $\mathcal{L}(R)$ is left idealtor.

Let $\mathcal{S}_{\mathcal{L}}(R)$ denote the set of all left $\mathcal{L}$-stable elements of a ring R. We have the following nice result.

Theorem 3.2.4. ([63])**(Product Theorem)** For any left idealtor $\mathcal{L}$, $\mathcal{S}_{\mathcal{L}}(R)$ is closed under multiplication.

Proof. If a and d are $\mathcal{L}$-stable we show that da is also $\mathcal{L}$-stable. So let $Rda + L = R$, $L \in \mathcal{L}(R)$, say $rda + b = 1$, $r \in R, b \in L$. Thus $Ra + L = R$ so (as a is $\mathcal{L}$-stable) let $a - u \in L$, for some unit u. Write $c = a - u \in L$. Then $1 = rd(c + u) + b$, so $rdu + (rdc + b) = 1$. Thus $rdu + g = 1$, where $g = rdc + b \in L$ (because $c, b \in L$). Multiply on the left by u, and then on the right by u^{-1}, to obtain $urd + ugu^{-1} = 1$, from which $Rd + uLu = 1 = R$. But $uLu^{-1} \in L(R)$ by Lemma 3.2.3. So, as d is $\mathcal{L}$-stable, let $d - v \in uLu^{-1}$ where $v \in U(R)$, say $d - v = uhu^{-1}$ where $h \in L$. Thus $du - vu = uh$ so (since $u = a - c$) we obtain $da - vu = d(c + u) - (du - uh) = dc + uh \in L$ because $c, h \in L$. As vu is a unit, this shows da is $\mathcal{L}$-stable, as required. □

Example 3.2.5. For the left idealtor $\mathcal{B}$, if a and b are in $\mathcal{S}_{\mathcal{B}}(R)$, is their sum $a + b$ need not be in $\mathcal{S}_{\mathcal{B}}(R)$ in general.[11]

Proof. Considering $1 \in \mathbb{Z}$, then 1 is clearly SR1 being a unit. While $1 + 1 = 2$ is not SR1 (already explained in Example 2.1.10). □

However, it is not futile to think about the algebraic structute $\mathcal{S}_{\mathcal{L}}(R)$; because

Lemma 3.2.6. ([63]) If $\mathcal{L}$ is a left idealtor, then $\mathcal{S}_{\mathcal{L}}(R) + J(R) \subseteq \mathcal{S}_{\mathcal{L}}(R)$.

Proof. Let $R(r + c) + L = R$, where $r \in R$ is $\mathcal{L}$-stable, $c \in J(R)$ and $L \in L(R)$. It follows that $Rr + J(R) + L = R$, whence $Rr + L = R$. By hypothesis, let $r - u \in L$ where $u \in U(R)$. Thus $(r + c) - (u + c) \in L$, and $u + c \in U(R)$ because $c \in J(R)$. □

Example 3.2.7. ([62]) For the ring of integers $\mathbb{Z}$, we have $\mathcal{S}_{\mathcal{B}}(\mathbb{Z}) = \{-1, 0, 1\}$.

Proof. Obviously, we have $\mathcal{S}_{\mathcal{B}}(\mathbb{Z}) \supseteq \{-1, 0, 1\}$. Suppose $k \in \mathcal{S}_{\mathcal{B}}(\mathbb{Z}) \setminus \{-1, 0, 1\}$. If p is any prime with $p \nmid k$, then $\mathbb{Z}k + \mathbb{Z}p = \mathbb{Z}$. As $k \in \mathcal{S}_{\mathcal{B}}(\mathbb{Z})$, we have $p \mid (k - 1)$ or $p \mid (k + 1)$. It follows that $p \mid (k^2 - 1)$, a contradiction because there are infinitely many such primes p. Hence, $\mathcal{S}_{\mathcal{B}}(\mathbb{Z}) = \{-1, 0, 1\}$. □

Which motivates the following generalization

Theorem 3.2.8. ([63]) Let R be a PID with infinitely many primes but having a finite unit group. Then $\mathcal{S}_{\mathcal{B}}(R) = \{0\} \cup U(R)$

Proof. Clearly $S_{\mathcal{B}}(R) \supseteq \{0\} \cup U(R)$. Suppose $a \in S_{\mathcal{B}}(R) \setminus (\{0\} \cup U(R))$. Let p be any prime not dividing a. Then $Ra + Rp = R$ as Rp is maximal. As $a \in \mathcal{S}_{\mathcal{B}}(R)$, $a - u \in Rp$ for some $u \in U(R)$, that is $p \mid (a - u)$ for some $u \in U(R)$. If we write $U(R) = \{u_1, u_2, \ldots, u_n\}$ this means that $p \mid \Pi_{i=1}^{n}(a - u_i)$, a contradiction as there are infinitely many primes p not dividing a. □

The fact that every image of an SR1 ring is again SR1 is a special case of the following theorem.

Theorem 3.2.9. ([63]) Let $\mathcal{L}$ be any left idealtor, let $\theta : R \mapsto S$ be an onto ring morphism, and assume $\theta^{-1}(X) + ker(\theta) \in \mathcal{L}(R)$ for all $X \in \mathcal{L}(S)$. Then S is $\mathcal{L}$-stable if R is $\mathcal{L}$-stable.

[11] The case "when is $\mathcal{S}_{\mathcal{L}}(R)$ closed under addition?" remains an open question. Note that it is the case whenever R is left $\mathcal{L}$-stable in sense that a ring R is $\mathcal{L}$-stable if and only if $\mathcal{S}_{\mathcal{L}}(R) = R$.

Proof. As before write $\theta(r) = \overline{r} \in \overline{R} = S$. Suppose $\overline{R\overline{a}} + X = \overline{R}$, $X \in \mathcal{L}(S)$. As θ is onto, we have $X = \theta[\theta^{-1}(X)] = \overline{\theta^{-1}(X)}$. It follows that $Ra + \theta^{-1}(X) + \ker(\theta) = R$. By hypothesis, there exists $u \in U(R)$ where $a - u \in \theta^{-1}(X) + \ker(\theta)$. Thus $\overline{a} - \overline{u} \in X$ and $\overline{u} \in U(S)$, as required. $\square$

Lemma 3.2.10. ([62])**(Full Lemma)**. Let $\theta : R \mapsto S$ be an onto ring morphism. For a left idealtor $\mathcal{L}$.

1. If $\theta^{-1}(X) \in \mathcal{L}(R)$ for every $X \in \mathcal{L}(S)$, then θ is $\mathcal{L}$-full.
2. The converse of (1) holds if $\ker(\theta) \subseteq L$ for all $L \in \mathcal{L}(R)$.

Proof.
1. As θ is onto, we have $\theta[\theta^{-1}(X)] = X$ for any left ideal X of S.
2. Assume θ is $\mathcal{L}$-full. If $X \in \mathcal{L}(S)$, write $X = \theta(L)$ for some $L \in \mathcal{L}(R)$. If $r \in \theta^{-1}(X)$ then $\theta(r) \in X = \theta(L)$, say $\theta(r) = \theta(l)$ for some $l \in L$. This means that $r - l \in ker(\theta)$, and it follows that $\theta^{-1}(X) \subseteq L + ker(\theta)$. But $ker(\theta) \subseteq \theta^{-1}(X)$ always holds, and $L \subseteq \theta^{-1}(X)$ because $X \subseteq \theta(L)$, proving that $\theta^{-1}(X) = L + ker(\theta) = L \in \mathcal{L}(R)$ by hypothesis, as promised.

$\square$

Let $Z_r(R) = \{z \in R : \mathtt{r}(z) \subseteq^{ess} R_R\}$ denote the **right singular ideal** of R.

Proposition 3.2.11. ([63]) Let $\mathcal{L}$ be a left idealtor. The following hold for any ring R.

1. $J(R) \subseteq S_{\mathcal{L}}(R)$.
2. ureg$(R) \subseteq S_{\mathcal{L}}(R)$.
3. $Z_r(R) \subseteq S_{\mathcal{L}}(R)$. provided $r(L) \neq 0$ whenever $R \neq L \in \mathcal{L}(R)$ (say R is **left Kasch**).

Proof.
1. Let $Ra + L = R$ where $L \in L(R)$ and $a \in J(R)$. Then $L = R$ so $a - 1 \in L$.
2. is clear.
3. Suppose $Rz + L = R$ where $z \in Z_r(R)$ and $L \in \mathcal{L}(R)$. Taking right annihilators we obtain $\mathtt{r}(z) \cap \mathtt{r}(L) = \mathtt{r}(R) = 0$, so $\mathtt{r}(L) = 0$ as $z \in Z_r(R)$. By hypothesis $L = R$, so $a - u \in L$ for any $u \in U(R)$.

$\square$

Lemma 3.2.12. ([62]) Let $\mathcal{L}$ be any left idealtor.

1. Let $R \overset{\rho}{\mapsto} S \overset{\tau}{\mapsto} R$ be ring morphisms with $\tau \circ \rho = 1_R$. Then, we have:
   (a) ρ is $\mathcal{L}$-fit implies τ is $\mathcal{L}$-full.
   (b) ρ is $\mathcal{L}$-full implies τ is $\mathcal{L}$-fit.
2. If $R \overset{\sigma}{\mapsto} S$ is a ring isomorphism, then the following statements hold:
   (a) σ is $\mathcal{L}$-fit if and only if σ^{-1} is $\mathcal{L}$-full.
   (b) σ is $\mathcal{L}$-full if and only if σ^{-1} is $\mathcal{L}$-fit.

Proof. 1. Assume that $L \in \mathcal{L}(R)$. Then, we have $L = \tau[\rho(L)]$, and $\rho(L) \in \mathcal{L}(S)$ because ρ is $\mathcal{L}$-fit, proving (a). For (b), let $X \in \mathcal{L}(S)$. As ρ is $\mathcal{L}$-full we have $X = \rho(L)$ for some $L \in \mathcal{L}(R)$.Then, we have $\tau(X) = \tau[\rho(L)] = L \in \mathcal{L}(R)$, as required.

2. First, we notice that (a) implies (b) by $\sigma \mapsto \sigma^{-1}$. But, (a) follows using (1) because:

$$\sigma \text{ is } \mathcal{L}\text{-fit} \overset{1(a)}{\Longrightarrow} \sigma^{-1} \text{ is } \mathcal{L}\text{-full} \overset{1(b)}{\Longrightarrow} \sigma \text{ is } \mathcal{L}\text{-fit}.$$

□

3.3 Closed Left Idealtors

In this last section, we shall discuss one last proerty of left idealtors, which is, the "closedness". Also, we mention the left-max idealtor which is defined in respect of the maximal left ideals for an arbitrary ring R.

We go straightforward with the following definition

Definition 3.3.1. The **closure** of the left idealtor $\mathcal{L}$ for any ring R, is denoted and defined as follows

$$\overline{\mathcal{L}}(R) = \{M \mid M \text{ is a left ideal of } R \text{ and } M \cong L \text{ for some } L \in \mathcal{L}(R)\}.$$

And $\mathcal{L}$ is said to be **closed** if $\mathcal{L} = \overline{\mathcal{L}}$.

The notion of closedness of left idealtors meet with the notion of closedness of sets in topology in sense following lemma

Lemma 3.3.2. ([62]) The following statements are true for any left idealtor $\mathcal{L}$:

1. $\mathcal{L}(R) \subseteq \overline{\mathcal{L}}(R)$ for any ring R.

2. $\overline{\mathcal{L}} = \overline{\overline{\mathcal{L}}}$.

Proof. (1). For any $L \in \mathcal{L}(R)$, we have $L \cong L$ and so $L \in \mathcal{L}(R)$, proving (1).

(2). Applying (1) to $\overline{\mathcal{L}}$ implies $\overline{\mathcal{L}}(R) \subseteq \overline{\overline{\mathcal{L}}}(R)$ for each ring R. Now, let $X \in \overline{\overline{\mathcal{L}}}(R)$, so there exists $M \in \overline{\mathcal{L}}(R)$ such that $X \cong M$. Then, in turn, let $M \cong L \in \mathcal{L}(R)$. Thus, $X \cong M \cong L \in \mathcal{L}(R)$ which implies $X \in \overline{\mathcal{L}}(R)$. Therefore, we also have $\overline{\overline{\mathcal{L}}}(R) \subseteq \overline{\mathcal{L}}(R)$, so $\overline{\overline{\mathcal{L}}}(R) = \overline{\mathcal{L}}(R)$ for each ring R. □

Obviously, some left idealtors have the closedness property.

Example 3.3.3. ([62]) Each of the following left idealtors is closed:

1. $\mathcal{B}(R) = \{L \mid L \text{ is a left ideal of } R\}$.

2. $\mathcal{P}(R) = \{Rb : b \in R\}$.

3. $\mathcal{T}(R) = \{0\}$

Proof. 1. Is trivial.

2. If N is any left ideal of R such that $N \cong Rb$ for some $b \in R$, then $N = R\phi(b)$ where $\phi : Rb \mapsto N$ is an isomorphism.

3. Let N be any left ideal of a ring R such that $N \cong L \in \mathcal{T}(R)$. Then, we have $N \cong 0$, and so $N = 0 \in \mathcal{T}(R)$ which implies that $\mathcal{T}$ is closed, as desired.

□

And of course, some do not have it

Example 3.3.4. ([62]) None of the following left idealtors is closed:

1. $\mathcal{K}(R) = \{\mathtt{l}(a) : a \in R\}$

2. $\mathcal{E}(R) = \{Re : e^2 = e \in R\}$

Proof. Let $R = \mathbb{Z}$ and $M = 2\mathbb{Z}$. Then:

1. $M \cong \mathbb{Z} = \mathtt{l}(0) \in \mathcal{K}(\mathbb{Z})$, but $M \neq \mathtt{l}(k)$ for all $k \in \mathbb{Z}$, so $\mathcal{K}$ is not closed.

2. $M \cong \mathbb{Z} = R1 \in \mathcal{E}(\mathbb{Z})$, but $M \neq Re$ for all $e^2 = e \in \mathbb{Z}$, so $\mathcal{E}$ is not closed.

□

Definition 3.3.5. ([89]) A ring R is called **left C2** ring if every left ideal isomorphic to a summand of ${}_RR$ is itself a summand.

Under some certain "non-trivial" conditions, the non-closed left idealtor $\mathcal{E}(R)$ becomes closed. In fact, we have

Theorem 3.3.6. ([63]) If R is a ring, then $\mathcal{E}(R)$ is closed if and only if R is left C2 ring.

Proof. Note that $\overline{\mathcal{E}}(R) = \{L \leq R \mid L \cong Re \text{ for some } e = e^2 \in R\}$. Assume $\overline{\mathcal{E}}(R) = \mathcal{E}(R)$. If L is a left ideal of R and $L \cong Re$, $e^2 = e$, then $L \in \overline{\mathcal{E}}(R) = \mathcal{E}(R)$, so $L = Rf$ for some $f^2 = f$. This shows that R is left C2. The converse is proved in similar manner. □

As another example of a non-affordable class of rings we have

Example 3.3.7. ([63]) The class of left C2 rings is not affordable.

Proof. The triangular matrix ring $R = \mathbb{T}_2(D)$ is an SR1 ring. Now, since $J(R) = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, hence the ideal $J(R)$ is not a direct summand of ${}_RR$. It follows that R is not left C2. Therefore, the class of left C2 rings is not affordable. □

The following lemma is key to define a new left idealtor.

Lemma 3.3.8. ([62]) Let L be left maximal ideal of R. Write $K = R/L$, and abbreviate $U = U(R)$, $K^\star = K \setminus \{0\}$. Then, the following statements are equivalent:

1. $Ra + L = R$ with $a \in R$, implies that $a - u \in L$ for some $u \in U$.

2. If $a, b \in R \setminus L$, then $u^{-1}a = v^{-1}b$ for some $u, v \in U$.

3. $K^\star = U\overline{a}$ for any $\overline{a} \in K^\star$.

Proof. (1) $\implies$ (2). If $a \in R \setminus L$, then $Ra + L = R$ because L is a maximal left ideal of R. So, using (1), let $a - u^{-1} \in L$ with $u \in U$. Hence, $ua - 1 \in L$, which implies that $u\overline{a} = \overline{1}$. Similarly, if $b \in R \setminus L$, then $v\overline{b} = \overline{1}$. Therefore, $u\overline{a} = v\overline{b}$ for some $u, v \in U$, as required.

(2) $\implies$ (3). We always have $U\overline{a} \subseteq K^\star$ for any $\overline{a} \neq \overline{0}$. Now, if $\overline{b} \in K^\star$, (2) gives $u\overline{a} = v\overline{b}$ for some $u, v \in U$. Hence, $\overline{b} = (v^{-1}u)\overline{a} \in U\overline{a}$, proving (3).

(3) $\implies$ (1). If $Ra + L = R$ with $a \in R$, then $a \in L$. Hence, $K^\star = U\overline{a}$ using (3). Thus, $u\overline{a} = \overline{1}$, which implies that $ua - 1 \in L$. Therefore, $a - u^{-1} \in L$, proving (1). □

Which enables us to state the following

Definition 3.3.9. ([62]) The left idealtor $\mathcal{X}$ defined by

$$\mathcal{X}(R) = \{L \mid L \text{ is a maximal ideal of } R\}$$

for each ring R will be called the **left-max idealtor**. Moreover, a maximal left ideal L of a ring R is said to be a **left-max stable ideal** if the conditions in Lemma 3.3.8. Furthermore, call a ring R **left-max stable** if it is left $\mathcal{X}$-stable.

As a prototypical example, we have.

Example 3.3.10. ([62]) Any SR1 ring is left-max stable ring.

Proof. Immeditate consequence of Theorem 3.1.25. □

On the other hand, there exists rings which are not left-max stable as the following example exhibits.

Example 3.3.11. ([62]) The (left) max stable ideals of the ring of integers $\mathbb{Z}$ are $2\mathbb{Z}$ and $3\mathbb{Z}$. Hence, $\mathbb{Z}$ is not a (left) max stable ring.

Proof. Observe first that the maximal (left) ideals of $\mathbb{Z}$ are of the form $p\mathbb{Z}$ where p is a prime number. Now, if $\overline{0} \neq \overline{a} \in \mathbb{Z} = p\mathbb{Z} \cong \mathbb{Z}_p$, then $U\overline{a} = \{\overline{a}, -\overline{a}\}$. Hence, $p\mathbb{Z}$ is a left-max stable if $\mathbb{Z}_p^\star = U\overline{a} = \{\overline{a}, -\overline{a}\}$, and so $\mathbb{Z}_p = \left\{\overline{0}, \overline{a}, -\overline{a}\right\}$. But, $|\mathbb{Z}_p| = p$, so we must have $p = 2$ if $\overline{a} = -\overline{a}$, and $p = 3$ if $\overline{a} \neq -\overline{a}$, as required. The last statement is clear. □

Finally, we enclose this chapter by the following unfortunate fact.

Proposition 3.3.12. ([62]) The left-max idealtor $\mathcal{X}$ is not a closed left idealtor.

Proof. Consider the ring of integers $\mathbb{Z}$. Then, clearly $4\mathbb{Z} \supsetneqq 2\mathbb{Z}$. But, we have $2\mathbb{Z} \in \mathcal{X}(\mathbb{Z})$ and $4\mathbb{Z} \notin \mathcal{X}(\mathbb{Z})$ by Example 3.3.11. Therefore, $\mathcal{X}$ is not a closed left idealtor. □

Chapter 4

Related Ring-theoretic Constructions

This chapter is set in order to discuss when the constructions of an $\mathcal{L}$-stable ring attain $\mathcal{L}$-stability and vice versa. The constructions we shall discuss are: Corners, direct products, factor rings, ideal extensions, polynomial rings and matrix rings.

4.1 Corners

We go ahead and begin with the following result.

Theorem 4.1.1. ([62]) Let $\mathcal{L}$ be any left idealtor, and let $e \in I(R)$. If R is left $\mathcal{L}$-stable, then so is eRe provided the following conditions hold:

1. If $X \in \mathcal{L}(eRe)$, then $RX \in \mathcal{L}(R)$.
2. One of the following two statements holds:
   (a) Every left $\mathcal{L}$-stable ring is DF.
   (b) The map $\theta : R \mapsto eRe$ defined by $\theta(r) = ere$ is a ring morphism.[1]

Proof. Let R be left $\mathcal{L}$-stable, write $S = eRe$, and let $Sa + X = S$ where $a \in S$ and $X \in \mathcal{L}(S)$, we want $a - w \in X$ for some unit w of S. Write $sa + x = e$, $s \in S$, $x \in X$. Then,

$$(s + 1 - e)(a + 1 - e) + x = (sa + 1 - e) + x = 1$$

Hence, $R(a + 1 - e) + RX = R$. Using (1), we have $(a + 1 - e) - v := b \in RX$ for some $v \in U(R)$ because R is left $\mathcal{L}$-stable by assumption. Thus, we have:

$$(a + 1 - e - b)u = 1 \text{ where } u := v^{-1} \in U(R) \tag{4.1}$$

Multiply both sides by e to get $(a - eb)ue = e$. But, we have $eb \in e(RX) = eR(eX) \subseteq X$ because X is a left ideal of S. In particular, $b = be$ and it follows that

$$(a - eb)eue = e, \; eb \in X \tag{4.2}$$

Write $w = a - eb$, so w has a right inverse in S. If (a) holds, it follows that w is a unit in S because S is DF whenever R is. But, as we have $a - w = eb \in X$, then a is left $\mathcal{L}$-stable

[1] This is equivalent to saying that the idempotent $e \in R$ is **quasi-normal**, that is, $eR(1 - e)Re = 0$. (See [107])

in S, as required. Now assume (b). We show that $eue \in U(S)$, and hence $a - eb \in U(S)$ by 4.2. As in 4.1 we have $u(a + 1 - e - b) = 1$, whence $eu(a - be) = e$. Now condition (b) shows that $eue(a - be) = e$. This with 4.2 shows that eue is a unit in S, and we are done as before. $\square$

Recall that Abelian rings are quasinormal, with this in mind, we have

Corollary 4.1.2. ([63]) Let $\mathcal{L}$ be any left idealtor. If R is $\mathcal{L}$-stable then eRe is $\mathcal{L}$-stable if $e^2 = e \in R$ is central and $\mathcal{L}(eRe) \subseteq \mathcal{L}(R)$.

Proof. Clearly (b2) holds. For (a): If $X \in \mathcal{L}(eRe)$ then $RX = R(eX) = eReX = X$. It follows by hypothesis that $RX \in \mathcal{L}(eRe) \subseteq \mathcal{L}(R)$. $\square$

Finally, we enclose this section by the following well-known result.

Corollary 4.1.3. ([63]) Each of the ring properties SR1, left UG, IC and DF passes to corners.

Proof. First consider SR1, IC and DF. Then (b1) holds. To verify (a) use, respectively, the left idealtors $\mathcal{B}(R), \mathcal{E}(R), \mathcal{T}(R)$. Then (a) is clear for $\mathcal{B}$ and $\mathcal{T}$, and it holds for $\mathcal{E}$ because $RSf = Rf$ whenever $f^2 = f \in S = eRe$. The fact that left UG passes to corners comes from [86, Theorem 30] where it is shown that if the Morita context ring $C = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is left UG, then R is left UG. $\square$

4.2 Direct Products

As a start, we have the following result.

Theorem 4.2.1. ([63]) Let $R = \Pi_{i\in I} R_i$ denote a direct product of rings R_i with canonical projections $\pi_k : R \mapsto R_k$ for each $k \in I$. Let L denote a left idealtor. Then

1. R is $\mathcal{L}$-stable $\implies$ each R_i is $\mathcal{L}$-stable provided $L_i \in \mathcal{L}(R_i)$ for each i implies $\Pi_{i\in I} L_i \in \mathcal{L}(R)$.
2. Each R_i is $\mathcal{L}$-stable $\implies$ R is $\mathcal{L}$-stable provided $L \in \mathcal{L}(R)$ implies $L = \Pi_{i\in I} L_i$ for $L_i \in \mathcal{L}(R_i)$.

Proof.

1. Assume that R is left $\mathcal{L}$-stable. Suppose $R_i a_i + L_i = R_i$ with $L_i \in \mathcal{L}(R_i)$ and $a_i \in R_i$, say $r_i a_i + x_i = 1_{R_i}$ where $x_i \in L_i$. Then, $\langle r_i\rangle\langle a_i\rangle + \langle x_i\rangle = 1_R$, and $\langle x_i\rangle \in \Pi_{i\in I} L_i \in \mathcal{L}(R)$ by the proviso. By hypothesis $\langle a_i\rangle - \langle u_i\rangle \in \langle x_i\rangle$ where $\langle u_i\rangle$ is a unit in R. Thus $a_i - u_i = x_i \in L_i$ for each i, and each u_i is a unit in R_i.
2. Now assume that each R_i is $\mathcal{L}$-stable. Suppose $R\langle a_i\rangle + L = R$ where $L \in \mathcal{L}(R)$. By the proviso, $L = \Pi_{i\in I} L_i$ where $L_i \in \mathcal{L}(R_i)$ for each i. Hence $\langle r_i\rangle\langle a_i\rangle + \langle x_i\rangle = \langle 1_{R_i}\rangle$ where $r_i \in R_i$ and $x_i \in L_i$ for each i. It follows that $R_i a_i + L_i = R_i$ so, by hypothesis, $a_i - u_i \in L_i$ for some unit u_i in R_i. Finally $\langle a_i\rangle - \langle u_i\rangle \in \Pi_{i\in I} L_i = L$ where $\langle u_i\rangle$ is a unit in R.

$\square$

From which it follows that

Corollary 4.2.2. ([63]) Let $R = \Pi_{i\in I} R_i$ denote a direct product of rings R_i. Then, R is SR1, left UG, IC or DF if and only if the same is true for each R_i.[2]

We conclude this Section with a result about a finite direct product R, viewed internally: $R = S_1 \oplus \cdots \oplus S_n$ where $S_i \lhd R$ for each i. Then $S_i = e_i R e_i$ where $e_i^2 = e_i$ is central for each i, the e_i are orthogonal, and $1 = e_1 + \cdots + e_n$.

Theorem 4.2.3. ([63]) Let L be any left idealtor and let $R = S_1 \oplus \cdots \oplus S_n$ where $S_i \lhd R$ for each i. Then

1. R is $\mathcal{L}$-stable $\implies$ every S_i is $\mathcal{L}$-stable provided $\mathcal{L}(S_i) \subseteq \mathcal{L}(R)$ for each i.
2. Every S_i is $\mathcal{L}$-stable $\implies$ R is $\mathcal{L}$-stable provided $\{S_i \cap L \mid L \in \mathcal{L}(R)\} \subseteq \mathcal{L}(S_i)$ for each i.

Proof. Write $S_i = e_i R e_i$ where $e_i^2 = e_i$ is central, $e_1 + \cdots + e_n = 1$, and $\{e_1, \cdots, e_n\}$ is orthogonal.

1. This follows from Theorem 4.1.1. Condition (b2) is satisfied because e_i is central; and condition (a) holds because if $X \in \mathcal{L}(S_i)$ then $RX = R(e_i X) = S_i X = X \in L(R)$ by the proviso.
2. Let $Ra + L = R$, $a \in R$, $L \in \mathcal{L}(R)$. Multiplying by e_i gives $S_i a e_i + L e_i = S_i$. Observe that $Le_i = S_i \cap L \in \mathcal{L}(S_i)$ by the proviso. Since S_i is $\mathcal{L}$-stable, there exists $u_i \in U(S_i)$ such that $ae_i - u_i \in Le_i$: Write $u = \sum_{i=1}^n u_i$ so u is a unit in R (with inverse $\sum_{i=1}^n v_i$ where $u_i v_i = e_i = v_i u_i$ for each i). Finally, we obtain $a - u = \sum_{i=1}^n (ae_i - u_i) \in \sum_{i=1}^n Le_i = \sum_{i=1}^n e_i L \subseteq L$, as required.

□

Corollary 4.2.4. ([63]) Let $R = S_1 \oplus \cdots \oplus S_n$ where $S_i \lhd R$ for each i. Then R enjoys each of the ring properties SR1, left UG, IC and DF if and only if the same is true of each S_i.

Proof. As in Theorem 4.2.3, write $S_i = e_i R e_i$ where $e_i^2 = e_i$ is central in R. Each property passes to every S_i by Corollary 4.1.3 because $S_i = e_i R e_i$ is a corner of R. So it remains to check proviso (2) of Theorem 4.2.3 in each case. It is clear that it holds for SR1 and DF using the left idealtors $\mathcal{B}(R)$ and $\mathcal{T}(R)$. For left UG, using $\mathcal{K}(R)$, the proviso in (2) also holds because $S_i \cap \mathtt{l}_R(b) = \mathtt{l}_{S_i}(b)$. Finally for IC, using $\mathcal{E}(R)$ the proviso in (2) holds because $Re_i \cap Rf = Re_i f$ for any idempotent $f \in R$ (e_i is central in R). □

4.3 Factor Rings

The left UG, IC and DF properties do not pass to factor rings (equivalently, homomorphic images) in general.

Example 4.3.1. ([62]) The free algebra $R = \mathbb{Q}\langle x, y\rangle$.[73] Then, R is a left UG ring being a domain, and so it is an IC ring and a DF. But, the factor ring of R obtained by using the relation $xy = 1$ is not a DF ring, and so it is neither IC nor left UG.

[2]The fact that the ring $\Pi_{i\in I} R_i$ is SR1 if and only if so is each R_i is a straightforward result of [102, Lemma 2] which asserts that $sr(R) = \max_{i\in I} sr(R_i)$.

Since left UG, IC and DF properties do not pass to factor rings in general, we conclude that also $\mathcal{L}$-stability does not pass to factor rings in general. Of course, this is not the case for the SR1 condition as Theorem 2.1.24 asserts since by Proposition 1.1.5, homomorphic images and quotients of a ring are the same up to isomorphism.

4.4 Subrings and Ideal Extensions

In this section, we show that $\mathcal{L}$-stability does not pass to subrings in general. However, in special cases we prove that $\mathcal{L}$-stability passes to subrings assuming certain conditions. We also get some results for the particular classes of rings: SR1 rings, left UG rings, IC rings and DF rings.

Remember that

Definition 4.4.1. ([63]) If S is a (unital) subring of a ring R, then R is said to be an **extension** of S. A ring R is called an **ideal extension**[3] of a (unital) subring S if $R = S \oplus A$ where $A \lhd R$ and $A \subseteq J(R)$. If the requirement that $A \subseteq J(R)$ is dropped then R is called a **Dorroh extension**[4] of R.

Example 4.4.2. An example of some extensions:

1. The formal power series ring $R = S[[x]]$ is an ideal extension of S.

2. The polynomial ring $R = S[x]$ is a Dorroh extension of S.

Proof. 1. Let $R = S[[x]]$ denote the ring of formal power series over a ring S. As usual, we identify S with the subring of constant series, and write $\langle x\rangle$ for the ideal of series with zero constant term. It is well known that $U(R) = U(S)$, and that $J(R) = J(S) \oplus \langle x\rangle$. Hence $R = S \oplus \langle x\rangle$ is an ideal extension.

2. Same reasoning.

$\square$

Theorem 4.4.3. ([63]) Let $R = S \oplus A$ be an ideal extension, and let $\mathcal{L}$ be a left idealtor. Define $\theta : R \mapsto S$ by $\theta(s + a) = s$ for all $s \in S$ and $a \in A$. Then

1. If R is $\mathcal{L}$-stable then S is $\mathcal{L}$-stable provided θ is $\mathcal{L}$-full.

2. If S is $\mathcal{L}$-stable then R is $\mathcal{L}$-stable provided θ is $\mathcal{L}$-fit.

Proof. For clarity write $\overline{r} = \theta(r)$ and $\overline{L} = \theta(L)$ for any $r \in R$ and any left ideal $L \subseteq R$. Note that θ is an onto ring morphism with kernel A. and that $\overline{s} = s$ for all $s \in S$: Clearly $U(S) \subseteq U(R)$, in fact $U(R) = U(S) \oplus A$ because $A \subseteq J(R)$.

1. If R is $\mathcal{L}$-stable, let $Sb+X = S$, $b \in S$, $X \in \mathcal{L}(S)$, say $1 = sb+x$; $s \in S$, $x2X$. As θ is $\mathcal{L}$-full, $X = \overline{L}$ where $L \in \mathcal{L}(R)$. Write $x = \overline{l}$, $l \in L$. Then $\overline{x} = x$ because $x \in S$, so $\overline{1 - sb - l} = \overline{x - l} = \overline{x} - \overline{l} = x - \overline{l} = 0$. Hence $1 - sb - l \in A$, so $Rb + L + A = R$. As $A \subseteq J(R)$ we obtain $Rb + L = R$. Since $L \in \mathcal{L}(R)$ and R is $\mathcal{L}$-stable, let $b - u \in L$ where $u \in U(R)$. But $\overline{b} = b$ so it follows that $b - \overline{u} = \overline{b} - \overline{u} = \overline{b - u} \in \overline{L} = X$. Since $\overline{u} \in U(S)$, this proves (1).

[3]If S is any ring and A is a general ring (no unity) with $J(A) = A$, then the abelian group $S \oplus A$ becomes an ideal extension if we define multiplication by $(s, a)(t, b) = (st, sb + at + ab)$.

[4]The Dorroh extension is also known as the **unitization**.

2. Assume that S is $\mathcal{L}$-stable and let $r \in R$, we must show r is $\mathcal{L}$-stable in R. Write $r = s + a$, $s \in S$, $a \in A$. Since $A \subseteq (R)$, it suffices (by Lemma 3.2.6) to show that s is $\mathcal{L}$-stable in R. To that end, let $Rs + L = R$, $L \in \mathcal{L}(R)$, say $ps + l = 1$, $p \in R$, $l \in L$. Then $1 = \overline{1} = \overline{ps} + \overline{l}$, so $S = Ss + \overline{L}$. Moreover $\overline{L} \in \mathcal{L}(S)$ because θ is $\mathcal{L}$-fit, so $s - u \in \overline{L}$ for some $u \in U(S) \subseteq U(R)$. If $s - u = \overline{x}$ where $x \in L$, then $s - u - x \in ker(\theta) = A$, say $s - u - x = a \in A$. Finally $s - (u + a) = x \in L$, and we are done because $u + a$ is a unit of R.

□

Lemma 4.4.4. ([63]) Let $R = S \oplus A$ be an ideal extension. Define $\theta : R \mapsto S$ by $\theta(s+a) = s$ for all $s \in S$ and $a \in A$. Then for any $c \in S$, $\mathtt{l}_S(c) = \theta[\mathtt{l}_R(c)]$. (In particular, θ is $\mathcal{K}$-full).

Proof. For convenience, write $\theta(r) = \overline{r}$ for all $r \in R$. and recall that $\overline{s} = s$ for all $s \in S$. $\mathtt{l}_S(c) \subseteq [\mathtt{l}_R(c)]$. If $s \in \mathtt{l}_S(c)$ then $s = \theta(s) \in \theta[\mathtt{l}_R(c)]$.

Next, $l_S(c) \supseteq [l_R(c)]$. If $b \in l_R(c)$ then $bc = 0$ so $\theta(b)c = \overline{b}c = \overline{b}\overline{c} = \overline{bc} = \overline{0} = 0$, that is $\theta(b) \in l_S(c)$. □

Corollary 4.4.5. ([63]) Let $R = S \oplus A$ be an ideal extension. Then

1. R has SR1, IC or DF if and only if S has the same property.
2. If R is left UG, then S is left UG. The converse holds if for each $b \in R$, $\theta[\mathtt{l}_R(b)] = \mathtt{l}_S(s)$ for some $s \in S$.

Proof. Define $\theta : R \mapsto S$ by $\theta(s + a) = s$ for all $s \in S$ and $a \in A$. Observe that $\ker \theta = A \subseteq J(R)$.

- SR1. If $\mathcal{B}(R) = \{L \mid L \text{ is a left ideal of } R\}$, then θ is both $\mathcal{B}$-fit and $\mathcal{B}$-full and Theorem 4.4.3 adapts.
- IC. Use $\mathcal{E}(R) = \{Re \mid e^2 = e \in R\}$. Then θ is $\mathcal{E}$-full because $\theta(Re) = Se$ for all $e^2 = e \in S$, and θ is $\mathcal{E}$-fit because $\theta(Rf) = S(f)$ for all $f^2 = f \in R$. Hence we are done by Theorem 4.4.3.
- DF. Using $\mathcal{T}(R) = \{0\}$, again θ is both $\mathcal{T}$-fit and $\mathcal{T}$-full, so Theorem 4.4.3 applies.
- Left UG. Use $\mathcal{K}(R) = \{\mathtt{l}(a) : a \in R\}$: Then θ is $\mathcal{K}$-full because of the result of Lemma 4.4.4, R is left UG implies S is left UG by Theorem 4.4.3. By the same theorem, the converse holds if θ is $\mathcal{K}$-fit (for each $b \in R$, $\theta[\mathtt{l}_R(b)] = \mathtt{l}_S(s)$ for some $s \in S$).

□

4.5 Polynomial Rings

This section consists of the following one and only result.

Theorem 4.5.1. ([62]) Let $\mathcal{L}$ be any left idealtor. For the polynomial ring $R = S[x]$ over the ring S, we have the following:

1. If R is SR1, then so is S. The converse need not be true in general.

2. If R is left UG, then so is the ring S.

3. If R is IC, then so is the ring S. The converse need not to be true in general.

4. R is DF if and only if S is DF.

5. If S is left $\mathcal{L}$-stable, then R is not left $\mathcal{L}$-stable in general.

Proof. 1. Since R is a homomorphic image of $R[x]$. The converse fails because $\mathbb{R}[x]$ is SR1 while $\mathbb{R}$ is SR1.

2. By Theorem 2.2.20, every left UG ring is left AS ring and vice versa. And so Theorem 2.2.26 finishes the proof.

3. Theorem 2.3.16 adapts. The converse is denied by Example 2.3.11.

4. Theorem 2.4.31 gives more than required.

5. This follows because the SR1 and the IC conditions do not pass to polynomial rings by (1) and (3)

$\square$

4.6 Matrix Rings

Consider the Morita context ring $R = \begin{bmatrix} R_1 & V \\ W & R_2 \end{bmatrix}$ where R_1 and R_2 are rings with bimodules $V =_{R_1} V_{R_2}$ and $W =_{R_2} W_{R_1}$. If $VW = 0$ and $WV = 0$ then R is called the **context-null extension** of R_1 and R_2 by the bimodules V and W, and the multiplication takes the form

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix} \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} = \begin{bmatrix} aa' & av' + vb' \\ wa' + bw' & bb' \end{bmatrix}$$

Note that the diagonals multiply "directly" as in a direct product. With this in mind, write $S = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & V \\ W & 0 \end{bmatrix}$. Then the context-null extension R takes the form $R = S \oplus A$ and so is an ideal extension ($A \subseteq J(R)$ because $A^2 = 0$). Hence Theorem 4.4.3 can be applied. Rather than state the details here, we are going to generalize this to the $n \times n$ case.

Let $R_1, \ldots, R_n$ be rings and, whenever $i \neq j$, let V_{ij} be an R_i-R_j-bimodule. Assume that there exist multiplications $V_{ij}V_{ji} \subseteq R_i$ for each i, j, and $V_{ij}V_{jk} \subseteq V_{ik}$ when $i \neq k$, such that

$$R = \mathbb{M}_n[R_i, V_{ij}] = \begin{bmatrix} R_1 & V_{12} & \cdots & V_{1n} \\ V_{21} & R_2 & \cdots & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{n1} & V_{n2} & \cdots & R_n \end{bmatrix}$$

is an associative ring using matrix operations, called a generalized $n \times n$ matrix ring over the rings R_i. The prototype example is $R = \text{End}({}_RM)$ where $M = M_1 \oplus \cdots \oplus M_n$, $R_i = \text{End}({}_RM_i)$ for each i, and $V_{ij} = \text{Hom}_R(M_i, M_j)$ when $i \neq j$.

Definition 4.6.1. ([63]) A generalized matrix ring $R = \mathbb{M}_n[R_i, V_{ij}]$ over the rings $R_1, \dots, R_n$ is called a **context-null extension** of the rings R_i, denoted by $R = CN_n[R_i, V_{ij}]$, if $V_{pj}V_{jq} = 0$ whenever $j \neq p$ or $j \neq q$.

Thus the case $n = 2$ is described above. For $n = 4$ the multiplication in $CN_4[R_i, V_{ij}]$ becomes

$$\begin{bmatrix} a & v_{12} & v_{13} & v_{14} \\ v_{21} & b & v_{23} & v_{24} \\ v_{31} & v_{32} & c & v_{34} \\ v_{41} & v_{42} & v_{43} & d \end{bmatrix} \begin{bmatrix} p & u_{12} & u_{13} & u_{14} \\ u_{21} & q & u_{23} & u_{24} \\ u_{31} & u_{32} & r & u_{34} \\ u_{41} & u_{42} & u_{43} & s \end{bmatrix} = \begin{bmatrix} ap & au_{12} + v_{12}q & au_{13} + v_{13}r & au_{14} + v_{14}s \\ v_{21}p + bu_{21} & bq & bu_{23} + v_{23}r & bu_{24} + v_{24}s \\ v_{31}p + cu_{31} & v_{32}q + cu_{32} & cr & cu_{34} + v_{34}s \\ v_{41}p + du_{41} & v_{42}q + du_{42} & v_{43}r + du_{43} & ds \end{bmatrix}$$

where the diagonals multiply "directly" as in the 2×2 case above. Furthermore, by deleting pairs of columns and the corresponding rows, each of the 2×2 rings $CN_2[R_i, V_{ij}]$ arises as a corner of $CN_4[R_i, V_{ij}]$.

In the general $n \times n$ case, write $R = CN_n[R_i, V_{ij}]$. If $R = \begin{bmatrix} R_1 & V_{12} & \cdots & V_{1n} \\ V_{21} & R_2 & \cdots & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{n1} & V_{n2} & \cdots & R_n \end{bmatrix}$, let $S = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix}$ and $A = \begin{bmatrix} 0 & V_{12} & \cdots & V_{1n} \\ V_{21} & 0 & \cdots & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{n1} & V_{n2} & \cdots & 0 \end{bmatrix}$. Then S is a subring of R, $A \lhd R$, and $A \subseteq J(R)$ because $A^2 = 0$. That is, $R = S \oplus A$ is an ideal extension. Hence we obtain.

Corollary 4.6.2. ([63]) The ring $CN_n[R_i, V_{ij}]$ has SR1, IC or DF if and only if each factor ring R_i has the same property.

Proof. Since $R = S \oplus A$ is an ideal extension and $S \cong R_1 \times \cdots \times R_n$ as rings, the result follows using Theorem 3.1.10, Corollary 4.2.4 and Corollary 4.4.5. $\square$

Theorem 4.6.3. ([63]) Let $R_1, \cdots, R_n$ be rings and let $R = CN_n[R_i, V_{ij}]$ be a generalized context-null extension. Then (with the notation above) we have:

$$R = S \oplus A \text{ is an ideal extension and } A \subseteq J(R) \text{ because } A^2 = 0.$$

Define $\theta : R \mapsto S$ by $\theta(s + a) = s$ where $s \in S$ and $a \in A$. If $\mathcal{L}$ is a left idealtor then

1. R is $\mathcal{L}$-stable. Each R_i is $\mathcal{L}$-stable provided
   (a) $X \in \mathcal{L}(S)$ implies $X = \theta(L)$ for some $L \in \mathcal{L})(R)$.
   (b) $L_i \in R_i$ for each i implies $\Pi_{i=1}^{n} L_i \in \mathcal{L}(S)$.
2. Each R_i is $\mathcal{L}$-stable implies R is $\mathcal{L}$-stable provided:
   (c) $L \in \mathcal{L}(S)$ implies $L = \Pi_{i=1}^{n} L_i$ for $L_i \in \mathcal{L}(R_i)$.
   (d) $L \in L(R)$ implies $\theta(L) \in \mathcal{L}(S)$.

Proof. We have $R \overset{\theta}{\mapsto} S \overset{\sigma}{\mapsto} \Pi_{i=1}^{n} L_i$ where $\sigma[diag(r_1, \ldots, r_n)] = (r_1, \ldots, r_n)$ where $ri \in Ri$ for each i. Since σ is an isomorphism we have (by Lemma 2.9) that θ is $\mathcal{L}$-fit/$\mathcal{L}$-full if and only if $\sigma \circ \theta$ is $\mathcal{L}$-fit/$\mathcal{L}$-full. Hence, for determining whether θ is $\mathcal{L}$-fit/$\mathcal{L}$-full we may assume that $S = \Pi_{i=1}^{n} R_i$, and apply Theorem 4.2.1.

1. Assume R is $\mathcal{L}$-stable. Then S is $\mathcal{L}$-stable by Theorem 4.4.3 using (a). Now, with (b), each R_i is $\mathcal{L}$-stable by Theorem 4.2.1.

2. Assume each R_i is $\mathcal{L}$-stable. Then $S = \Pi_{i=1}^{n} R_i$ is $\mathcal{L}$-stable by (c) and Theorem 4.2.1. Hence, because of (d), $S \oplus A$ is $\mathcal{L}$-stable by Theorem 4.4.3.

□

If $V_{ij} = 0$ whenever $i > j$ then the generalized matrix ring $\mathbb{M}_n[R_i, V_{ij}]$ becomes upper triangular, and is called an $n \times n$ **generalized upper triangular matrix** ring over the rings R_i, and denoted by $\mathbb{T}_n[R_i, V_{ij}]$. The case $n = 2$ is the usual split-null extension $\begin{bmatrix} R_1 & V_{12} \\ 0 & R_2 \end{bmatrix}$.

The following theorem is the analogue of Theorem 4.6.3 for general context-null extensions. The routine proof is omitted.

Theorem 4.6.4. ([63]) Let $R_1, \ldots, R_n$ be rings and let $R = \mathbb{T}_n[R_i, V_{ij}]$ be a generalized upper triangular matrix ring over the R_i. Let $S \subseteq R$ be the subring of diagonal matrices, and let $A \lhd R$ denote the ideal of matrices with zero diagonal. Then all the conclusions of Theorem 4.6.3 are valid.

Related Open Questions

In this last chapter, we leave some open questions:

Question 4.6.5. ([63]) Is the left UG condition left-right symmetric? If not, then when exactly?

Question 4.6.6. ([63]) When the monoid $S_{\mathcal{L}}(R)$ becomes ring?

Question 4.6.7. Can the condition "exchange" be weakened so that

$$\text{SR1} \iff \text{Left UG} \iff \text{IC?}$$

Question 4.6.8. Can the condition "π-regular" be weakened to "Zorn" so that

$$\text{SR1} \iff \text{Left UG} \iff \text{IC?}$$

Question 4.6.9. Can the ring-theoretic condition "right self-injective" be weakened so that

$$\text{SR1} \iff \text{Left UG} \iff \text{IC} \iff \text{DF?}$$

Question 4.6.10. Are the ring classes $\{\text{stably IC}\}, \{\text{stably DF}\}$ and $\{\text{perspective}\}$ affordable? (Note that they all lie strictly between $\{\text{SR1}\}$ and $\{\text{DF}\}$).

Question 4.6.11. Does there exist a closed left idealtor affording the left UG rings? The IC rings?

Question 4.6.12. Does there exists an affordable class of rings with unique corresponding idealtor? If so, must it be closed?

Question 4.6.13. Is there a module-theoretic characterization for left UG rings? Is there a weaker condition than "exchange" so that any left UG ring is right UG and conversely?

Question 4.6.14. We know that the known four affordable classes $\{\text{SR1}\} \subseteq \{\text{left UG}\} \subseteq \{\text{IC}\} \subseteq \{\text{DF}\}$ form a chain. So we ask: Do all affordable classes form a chain?

Question 4.6.15. Can the module-theoretic condition in Theorem 2.4.50 be further weakened?

Question 4.6.16. Is it true that $\{\text{left UG}\} \subseteq \{\text{right UG}\}$? If so, then is $\{\text{left } \mathcal{L}\text{-stable}\} \subseteq \{\text{right } \mathcal{L}\text{-stable}\}$? If not, then when exactly? That is, when the notion of $\mathcal{L}$-stability becomes left-right symmetric?

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