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Absorbing Boundary Conditions for Linear Hyperbolic Systems

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Declaration

I declare that the master thesis entitled (Absorbing Boundary Conditions for Linear Hyperbolic Systems) is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Dedications

I dedicate my thesis to my husband, parents, friends and teachers who supported me on each step of the way.

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Finally, I must express my very profound gratitude to my husband, my husband family and to my parents, friends and sisters for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of writing this thesis.

Abstract

This thesis is a survey study of the numerical approximation of hyperbolic partial differential equations, that posed on an unbounded domain in one dimension and two dimension. For computational reasons the infinite domain is truncated and we develop boundary conditions at the resulting artificial boundaries. A finite different scheme is constructed for solving initial boundary value problem. Then We establish the GKS-stability of the corresponding finite different scheme.
Numerical examples are presented to compare the absorbing boundary conditions and far field boundary conditions, for short time and long time behaviour.

Introduction

In this thesis we discuses the far field boundary conditions to approximate solutions of partial differential equations. The main idea is to chose appropriate boundary conditions for the resulting artificial boundary. Such that these boundaries should prevent any reflection of outgoing wave and easy to implement. We shall consider hyperbolic systems in one and two dimensions.
Typical examples of first order hyperbolic equations are the Euler equations, shallow water equations and the hydrodynamic model of the semiconductor equations. We confine the domain by introducing artificial boundaries without making any changes to the considered differential equation, and these boundaries are called artificial, non-reflecting, transparent or absorbing boundary conditions (ABCs).
Many researchers have been active in this area recently, their work has been mainly concerned with ABCs that are better suited for a transient solution than for a steady solution, and most of these boundary conditions may lead to steady solutions of poor accuracy.
In this survey we are discussing ABCs which are designed, such that the solution of the bounded domain approximates well the solution on the original unbounded domain, and leads to stable and accurate solutions. We base our considerations on the paper of Engquist and Halpern . They constructed a new class of boundary conditions that combines the properties of ABCs for transient solutions and the properties of transparent boundary conditions for steady state problems. These boundary conditions, which are called far field boundary conditions (FBCs), can be used in both the transient regime and when the solution approaches the steady state. In this sense, they can be applied when the evanescent and traveling waves are present in the time-dependent calculation or when a time-dependent formulation is used for computations until the steady state.
For hyperbolic systems, these FBCs are defined up to matrix factor in front of the steady terms . How to choose this factor in a way to accelerate the convergence to the steady state, and to improve the accuracy of the transient solution is the problem. Since the problem has wavelike solutions, these FBCs must model the radiation of energy out of the computational domain. An incorrect specification of these boundary conditions can cause spurious reflected waves to be generated at the artificial boundary, if the time-dependent equations are only an intermediate step toward computing the steady state, then a flow of energy into computational do-

main can destroy the convergence to the steady state, these waves represent energy propagating back into computational domain . Since they are not part of the desired solution, they can substantially reduce the accuracy of the computed solution. Conversely, the correct specification of FBCs can accelerate the convergence. Thus, an answer of the above question consists in minimizing the spurious reflections.
Our work consists of three chapters, the first one discuss linear hyperbolic for one and two dimensional systems, as well as the absorbing boundary conditions for one dimensional case and for two dimensional case(Engquist and Majda). The second chapter deal with the numerical part, numerical absorbing boundary condition and the stability of the finite difference scheme. In the last one, we present two examples, the first one, discuss briefly a 2×2 model system and show the convergence of this system with first order FBCs to the correct steady state. In the second example, we compare the numerical approximations for different choices of the scaling matrices for a 3×3 system.

Contents

Chapter 1

Linear Hyperbolic Systems

1.1 Linear hyperbolic one dimensional systems

First we consider system of the form:

$$u_t + A(x,t)u_x + B(x,t)u = F(x,t), \quad x \in I\!R, \quad t \geq 0, \tag{1.1}$$

with initial function $u(x,0) = f(x)$, and $A, B \in C^\infty(I\!R \times [0,\infty), I\!R^{n\times n})$ such as $F \in C^\infty(I\!R \times [0,\infty), I\!R^n)$, $f \in C^\infty(I\!R, I\!R^n)$. Now, we want to focus on hyperbolic systems.

Definition 1.1. [26] *(Hyperbolic system)*
A system of form (1.1) *is called hyperbolic if for all* $x \in I\!R$ *and* $t \geq 0$, *we have* $A(x,t)$ *is diagonalisable matrix and has only real valid function eigenvalues* $\lambda_j(x,t),\ j = 1,...,n$. *If* $\lambda_j(x,t),\ j = 1,...,n$ *are in different pair* ($\lambda_j \neq \lambda_i$ *for* $i \neq j$) *then* (1.1) *is called strictly hyperbolic system.*

For a hyperbolic system (1.1) with $A = A(x,t)$ and using definition (1.1) then there is a regular matrix $P = P(x,t)$, such that

$$\Lambda := P^{-1}AP = diag(\lambda_1,...,\lambda_n), \quad \lambda_j = \lambda_j(x,t), \quad j = 1,...,n.$$

Define a new variable,

$$\upsilon(x,t) := P^{-1}(x,t)u(x,t),$$

v is called characteristic variables

$$v_t = (P^{-1}u)_t = P_t^{-1}u + P^{-1}u_t,$$

substituting in (1.1)

$$\begin{aligned} v_t &= P_t^{-1}u + P^{-1}(-Au_x - Bu + F) \\ &= P_t^{-1}u - P^{-1}Au_x - P^{-1}Bu + P^{-1}F \\ &= -P^{-1}APP^{-1}u_x - (P^{-1}B + P_t^{-1})u + P^{-1}F \\ &= -\Lambda v_x + \Lambda P_x^{-1}u - (P^{-1}B + P_t^{-1})u + P^{-1}F. \end{aligned}$$

Thus (1.1) can be transformed into the following characteristic form:

$$v_t + \Lambda(x,t)v_x + \tilde{B}(x,t)u = \tilde{F}(x,t), \tag{1.2}$$

where

$$\begin{aligned} \tilde{B} &= (P^{-1}B + P_t^{-1} - P^{-1}APP_x^{-1})P, \\ \tilde{F} &= P^{-1}F, \end{aligned}$$

and the initial values in (1.1) are transformed accordingly.
In the following we assume that (1.1) is strictly hyperbolic. Otherwise, we go to the characteristic variables $v(x,t) := P^{-1}(x,t)u(x,t)$ and treat the transformed system.

General solution for equation (1.1)

Consider first the decoupled case, which means that $B = 0$ and use the method of the characteristics. The linear system now decomposed into n independent scalar hyperbolic equations of form:

$$\frac{\partial u_j}{\partial t} + \lambda_j \frac{\partial u_j}{\partial x} = F_j, \quad j = 1, ..., n.$$

As further simplification we let $F_j = 0$. This yields

$$\frac{\partial u_j}{\partial t} = -\lambda_j \frac{\partial u_j}{\partial x}, \ j = 1, ..., n. \tag{1.3}$$

If we use the method of the characteristics for the description of the solution we obtain n scalar differential equation for u_j along the characteristic curve $(x(t), t)$, which is defined by

$$\frac{dx}{dt} = \lambda_j(x,t).$$

Because of

$$\frac{d}{dt}(u_j(x(t),t)) = \frac{du_j}{dx}\frac{dx}{dt} + \frac{du_j}{dt} = \lambda_j \frac{du_j}{dx} + \frac{du_j}{dt} =^{(1.3)} 0,$$

u_j is obviously constant along the characteristic curve and (1.3) is equivalent to:

$$\frac{d}{dt}(u_j(x(t),t)) = 0 \ \text{ for } \ \frac{dx}{dt} = \lambda_j(x,t), \ j = 1, ..., n. \tag{1.4}$$

for the solution of (1.4) we consider two cases below :
(i) $\lambda_j = constant$
Consider the n independent initial value problems (IVP) of the form:

$$u_t + \lambda_j u_x = 0, \ x \in I\!R, \ t > 0$$
$$u(x,0) = f(x), \ \ x \in I\!R, \text{ for } j = 1, ..., n.$$

The characteristics here are straight lines of the form $x_0 = x - \lambda_j t$ in the xt-plane, which are given by

$$x(t) = \lambda_j,$$
$$x(0) = x_0,$$

are defined. Accordingly, solutions of the system are given by:

$$u(x,t) = f(x - \lambda_j t). \tag{1.5}$$

Further properties of the solution:

- The solution at time t_0 is a copy of f , shifted by amount $\mid \lambda_j \mid t_0$ to the left if λ_j is negative and to the right if λ_j is positive .
- The solution at a point (x,t) depends only on $f(\xi)$, where $\xi = x - \lambda_j t$.

If one understands x as a spatial variable and t as the time, starting from $x_0 = \lambda_j t$ it is clear that λ_j must have the dimension that depends on time. Thus, λ_j is called the propagation velocity along the characteristic. This terminology motivates the following statement:
In equation (1.5), initial data spreads with speed λ_j, so the solution of (1.5) can be considered as a wave that propagates at velocity λ_j, but does not change its shape. If we consider a point $(\bar{x}, \bar{t})$ in the xt-plane, the function value of the solution at this point depends only on the function value of the initial data at point x_0, which is

clearly determined by $x_0 = \bar{x} - \lambda_j \bar{t}$. So we can change the initial data at every point except x_0 without influencing the solution in $(\bar{x}, \bar{t})$. We call the set $D(\bar{x}, \bar{t}) = x_0$ a dependency domain, in this case this set consists of only one point.
In the case of a hyperbolic system, this is generally no longer the case, but a fundamental property of hyperbolic equations is that this area always remains limited due to the finite speed of propagation.
The size of area D naturally grows with $\bar{t}$, but there is a natural limit of the form

$$D \subset \{|x - \bar{x}| \leq a_{max}\bar{t}\}.$$

Here a_{max} is a positive constant that depends on the specific system and its characteristic speeds.
We have thus answered the question, how much of the beginning has an influence on the solution at a certain point, Conversely, we can also look at this effect from the point of view of x_0 and ask ourselves on which points in the xt-plane the initial value in x_0 has an effect.
In this way, initial data at a point x_0 only have an influence on the solution within a cone $\{(x,t) : |x - x_0| \leq a_{max}t\}$ in the xt-plane, here we speak of the area of influence of x_0.
Let us now turn again to the solution u itself and its properties. For u to be the solution of (1.5) in the classical sense, there must be at least C^1- smoothness in place and time. The observation that the solution only depends on $f(x_0)$ along a characteristic shows that smoothness is not necessary to construct u from f in this way. It turns out, however, that every singularity in the initial data is also retained along the corresponding characteristic.
On the other hand, smooth areas in the initial data will lead to smooth areas at the corresponding points in the solution. This, too, is a fundamental property of linear hyperbolic equations, which is expressed by the following sentence:
Singularities only move along the characteristics.
Of course u can no longer be a solution in the classical sense. If one extends the concept of solution to weak solutions, which essentially corresponds to a conversion (and thus generalization) into an integral representation u is the solution of (1.5), provided that the set of singularities of u_0 remains at most countable.
(ii) $\lambda_j = \lambda_j(x,t)$
In this case, too, the method of characteristics will help us. For the lake of simplicity, we assume that λ_j is a smooth function, similar to the case:

- Let's call a curve $(x(t), t), t > 0$ Characteristic of the gable (1.5) if:

$$\frac{dx}{dt}(t) = \lambda_j(x(t), t),\ t \geq 0. \tag{1.6}$$

So, if the solution of (1.5) $(x(t), t)$ is a characteristic, using (1.6) it follows that:

$$\frac{d}{dt}u(x(t), t) = u_t + u_x\frac{dx}{dt} = u_t + u_x\lambda_j(x, t) = 0.$$

So let's just say that:

$$\frac{d}{dt}(u_j(x(t), t)) = 0 \text{ for } \frac{dx}{dt} = \lambda_j(x, t),$$

also valid here.
However, the characteristic curves are now not straight lines any more . Now, for $F \neq 0$. We are addressing the problem

$$u_t + \lambda_j(x, t)u_x = F(x, t),\ u(x, 0) = f(x),\ x \in I\!R. \tag{1.7}$$

The characteristics is defined as in (1.6)

$$\frac{d}{dt}u(x(t), t) = F(x(t), t),$$

(i.e)

$$u(x(t), t) = f(x_0) + \int_0^t F(x(\tau), \tau)d\tau,\ x(0) = x_0.$$

As was to be expected, that depends Solution along a characteristic on the one hand from the corresponding initial value in x_0 and on the other hand from the development of the inhomogeneity in the time span $[0, t]$ along the characteristic. If there are F jump positions, these are smoothed by the integral and the solution u is continuous. If f is continuous, u_0 jump, so they also wander along the characteristics. Discontinuities in inhomogeneity just don't produce new jumps in u, but in the derivations [**23**].

The general case

for equations of form:

$$u_t + \lambda_j(x, t)u_x = b(x, t)u + F(x, t). \tag{1.8}$$

If we define characteristics again by (1.6) if u is the solution of the general equation (1.8), then along the characteristics we have curve $(x(t), t)$ by

$$\frac{d}{dt}u(x(t),t) = b(x(t),t)u(x(t),t) + F(x(t),t),$$

and we obtain $u(x(t), t)$ by solving a linear ordinary differential equation.

Remark:(coupled case)

If B is not a diagonal shape, the system is coupled by equations in (1.1), but only in the non-differential terms. The expression Bu causes growth, evaporation, or oscillations in the solution, but it does not affect the actual propagation of the solution along the characteristics of system speeds, which depend only on higher order.

1.2 Locally absorbing boundary conditions for one-dimensional case

Let us define a boundary for the (local) area, we get the following problem:

$$\begin{aligned} u_t + A(x,t)u_x + B(x,t)u &= F(x,t), \\ u(x,0) &= f(x), \end{aligned} \tag{1.9}$$

here a solution is sought for $x \in [0, L]$, $L > 0$ and $t \geq 0$. For the lake of simplicity, we first consider the homogeneous scalar case again:

$$u_t + \lambda_j u_x = 0, \quad x \in [0, L], t > 0,$$

$$u(x, 0) = f(x), \quad x \in [0, L].$$

As we have already described in the previous section, the characteristics of the characteristics depend on the actual value of λ_j.

If $\lambda_j > 0$ the characteristics are from left to right and you now needs additional boundary data for $x = 0$, the so-called inflow edge. Where as the specification of border data leads to problems at $x = L$, since the solution there is already determined by the initial data or the edge data in $x = 0$. At this point, the characteristics leave the considered area and one therefore speaks of the outflow edge. With a similar consideration, we obtain that the specification of boundary data for $\lambda_j = 0$ is neither necessary nor allowed, and in the case $\lambda_j < 0$ the inflow and outflow edges

in the comparison with $\lambda_j > 0$.
If we consider a hyperbolic system (1.9), then the possibility of choosing boundary data depends on which sign the corresponding eigenvalues of Λ we have. For this, the following assumption is needed.

Theorem 1.2. *(constant partition at the edge)*

$$\lambda_j(0,t) \;\; and \;\; \lambda_j(L,t), \;\; j = 1, ..., n,$$

do not change their sign as functions of time. Thus, it will be either > 0, $= 0$ *or* < 0 *for all time* $t > 0$.

Notation 1.3. u^+, u^0, u^- *consist of those components* $u_j = u_j(x,t)$ *with index* j, *for which* $\lambda_j > 0$, $\lambda_j = 0$ *or* $\lambda_j < 0$ *respectively.*

With this notation, can satisfy the boundary conditions in the form

$$u^+(0,t) = g_0(t), \quad u^-(L,t) = g_L(t), \quad t > 0. \tag{1.10}$$

For the components which depend only on the initial data, so for $\lambda_j = 0$, we get accordingly

$$u^0(x,t) = f^0(x), \;\; 0 \le x \le L, \;\; t > 0,$$

so we only put conditions on the incident variable or on the inrush edges.
Now let's look specifically at the characteristics that start at the vertices $(x,t) = (0,0)$ and $(x,t) = (L,0)$, then the solution will have an discontinuity along this, if the boundary and the initial data do not match, i.e., if $f^+(0) \neq g_0(0)$ or $f^-(L) \neq g_L(0)$. To avoid this, in the following let's assume that the initial data are compatible with the boundary data.

- **Compatibility of the input data**

The boundary data g_0 and g_L are C^∞-smooth and compatible with the initial distribution f,

$$\text{i.e.,} \;\; f^+(0) = g_0^-(0), \;\; f(L) = g_L(0),$$

and equal heat in arbitrarily averaged outliers. The boundary conditions given by equation (1.10) can be generalized as

$$\begin{aligned} u^+(0,t) &= S_0(t)u^-(0,t) + g_0(t), \\ u^-(L,t) &= S_L(t)u^+(L,t) + g_L(t), \;\; t > 0, \end{aligned} \tag{1.11}$$

where S_0 and S_L are matrices of appropriate dimensions.
It is then said that the inflowing characteristic variables are expressed by the term of the out-flowing. This clearly means that waves leaving the area have an effect on the waves entering the area. Since we are dealing with transparent boundary conditions in this work, we will give below a derivation for this particular kind of boundary conditions.
Introducing ABC means that the solution to our (locally) constrained problem is to conform to the whole-space solution, which has been restricted to the considered subdomain. That means waves, to hit the edges, not to notice anything about them. In particular, there should be no reflections at the edges that lead back to the domain. For our one-dimensional scalar problem, this is easy to realize by assuming

$$u_j(0,t) = 0 \text{ for } \lambda_j > 0,$$

as boundary conditions or

$$u_j(L,t) = 0 \text{ for } \lambda_j < 0,$$

select, i.e., we forbid the information of inward waves at the respective inflow edge.

Notation 1.4. [**20**] *We assume that all input data (initial conditions and inhomogeneities) are located in* $[0, L]$*, i.e., the corresponding carriers lie within* $[0, L]$*.*
In this case, all waves that emanate from the edge are really undesirable, since they would not occur in the whole-space solution. If the system already exists in its characteristic form, then it is particularly easy to formulate the ABC. If this is not the case, then the question is how to express the corresponding conditions - which are valid for the transformed system - by means of the output variables shallow-water equations.

Example: The shallow-water equations are given by:

$$\begin{aligned} \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} &= 0, \\ \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} &= 0, \end{aligned}$$

where U and u are the mean liquid velocity and its perturbation, H and h are the average liquid level and its perturbation, and g denoting the constant gravitational acceleration.

The characteristic shape is given by:

$$\begin{pmatrix} d \\ e \end{pmatrix}_t + \begin{pmatrix} U-c & 0 \\ 0 & U+c \end{pmatrix} \begin{pmatrix} d \\ e \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the characteristic variables are $d = u - \frac{\Phi}{c}$ and $e = u + \frac{\Phi}{c}$ in which $c = \sqrt{gH}$ and $\Phi = gH$ holds.
We now want to derive the ABC for the boundary $x = L$ with $0 < U < c$.
Since the including characteristics for $x = L$ and $t > 0$ are equals zero, then also $d(x,t) = 0$ must hold for all $(x,t) \in \{(\tilde{x},\tilde{t}) | (U-c)\tilde{t} + L < \tilde{x} < L\}$.
We again assume that the input data are localized in $[0, L]$ and thus obtain for all $t > 0$

$$u(x,t) = \frac{\Phi(x,t)}{c}, \tag{1.12}$$

in a small environment around $x = L$. In a similar manner the outgoing characteristics are equal to zero.

$$\frac{\partial}{\partial t} \underbrace{(u + \frac{\Phi}{c})}_{=e} + (U+c)\frac{\partial}{\partial x} \underbrace{(u + \frac{\Phi}{c})}_{=e} = 0.$$

Using (1.12) to eliminate u and $\frac{\Phi}{c}$, we get ABC at $x = L$ for u and Φ :

$$\begin{aligned} \frac{\partial u}{\partial t} + (U+c)\frac{\partial u}{\partial x} &= 0, \\ \frac{\partial \Phi}{\partial t} + (U+c)\frac{\partial \Phi}{\partial x} &= 0. \end{aligned}$$

Similarly, we get ABC at $x = 0$:

$$\begin{aligned} \frac{\partial u}{\partial t} + (U-c)\frac{\partial u}{\partial x} &= 0, \\ \frac{\partial \Phi}{\partial t} + (U-c)\frac{\partial \Phi}{\partial x} &= 0. \end{aligned}$$

Alternatively, one can also motivate this representation as follows:
The general solution for the velocity u of the linearised shallow-water system can be represented as:

$$u(x,t) = F_r[x - (U+c)t] + F_l[x - (U-c)t].$$

Here, the components F_r and F_l stand for waves that travel to the right or to the left. The above conditions are fulfilled at each edge only by the wave travelling outwards. This means that a wave that moves inwards can not occur, it does not fulfil the boundary condition. In this sense, the derived conditions do not create any reflections, so they are transparent.
For the locally one-dimensional case we obtain the following theorems:

Theorem 1.5. [5] *(one dimensional case)*
If the boundary is not characteristic and the input data g_0, g_L, F, f are compatible at $t = 0$. Then, the symmetric hyperbolic initial boundary value problem

$$\begin{aligned} u_t &= \Lambda u_x + Bu + F, \quad 0 \le x \le L, \\ u(x,0) &= f(x), \quad 0 \le x \le L, \\ u_-(x,0) &= S_0 u^+(0,t) + g_0(t), \\ u_+(L,t) &= S_L u^-(L,t) + g_L(t), \end{aligned}$$

has a unique solution. This solution is $C^\infty-$smooth and for any finite time interval $0 \le t \le T$, and

$$\|u(.,t)\|^2 + \int_0^t |u(0,\tau)|^2 + |u(L,\tau)|^2 d\tau \le C_T[\|f\|^2 + \int_0^t |g_0(\tau)|^2 + |g_L(\tau)|^2 + \|F(.,\tau)\|^2 d\tau,$$

for $0 < t < T$, where C_T is independent from the data g_0, g_L, F, f.

A general result for the two-dimensional pallet is as follows:

Theorem 1.6. [27] *Given the symmetric hyperbolic initial boundary value problem*

$$\begin{aligned} u_t &= \Lambda u_x + By + Cu + F, \\ u(x,y,0) &= f(x,y), \\ u^-(0,y,t) &= S_0(y,t)u^+(0,y,t) + g_0(y,t), \\ u^+(L,y,t) &= S_L(y,t)u^-(L,y,t) + g_L(y,t), \end{aligned}$$

for $0 \le x \le L$, $-\infty \le y \le \infty$, $t \ge 0$. Λ is non singular and the matrices S_0 and S_L are in the norm (for example $S_0 = S_L = 0$). If the data is compatible, so there is a unique smooth solution and the problem is well posed.

1.3 Absorbing boundary conditions for systems in two dimensions (Engquist and Majda)

In the following derivation we are guided by the work of Engquist and Majda [8]. We consider the two dimensional hyperbolic system

$$u_t + A(x,y,t)u_x + B(x,y,t)u_y + C(x,y,t)u = F(x,y,t), \qquad (1.13)$$
$$u(x,y,0) = f(x,y), \quad x,y \in I\!R,\ t \geq 0.$$

All appearing coefficients and the initial distribution f are $C^\infty-$ smooth. The characterization of the hyperbolicity is more difficult here. We first introduce two concepts:

Definition 1.7. [28] *:(Strictly Hyperbolic) The system* (1.13) *is called strictly hyperbolic if, for all* k_1, k_2 *with* $\mid k_1 \mid + \mid k_2 \mid \neq 0$, *the matrix* $k_1A + k_2B$ *has different real eigenvalues.*

Definition 1.8. [28] *:(Symmetrically Hyperbolic) The system* (1.13) *is called symmetric hyperbolic if the matrices* A, B *are symmetric for all* $(x,y,t) \in I\!R \times I\!R \times I\!R_0^+$.

It is no longer as simple as in the one dimensional case to construct and analyse the solution using characteristics.

Definition 1.9. [30] *(Characteristic in one point)*
A (hyper) surface $\Gamma \in I\!R^2 \times [0,\infty)$ *with the equation* $\phi(x,y,t) = 0$ *is for the system* (1.13) *Characteristic in the point* (x_0, y_0, t_0), *if the matrix*

$$\frac{\partial \phi}{\partial t} I + A\frac{\partial \phi}{\partial x} + B\frac{\partial \phi}{\partial y},$$

singular on this point.
We say that Γ *is characteristic if* Γ *characteristic is in every point. With these characteristics flat it is also possible in the multidimensional case to construct solutions* [4].

For the treatment of the two-dimensional case we consider the following strictly hyperbolic system of first order with variable coefficients:

$$u_t + \Lambda(x,y,t)u_x + B(x,y)u_y + C(x,y,t)u = 0, \quad x \geq 0, \ -\infty < y < \infty, \ t \geq 0, \qquad (1.14)$$

where Λ and B are symmetric $n \times n$ matrices. Furthermore, we assume that Λ for all $(x, y, t) \in I\!R_0^+ \times I\!R \times I\!R_0^+$ is nonsingular and therefore can be written without loss of generality can be written in the form

$$\Lambda = diag(\lambda_1, ..., \lambda_m, \lambda_{m+1}, ..., \lambda_n),$$

with $\lambda_j > 0$ for $1 \leq j \leq m$ and $\lambda_j < 0$ for $m + 1 \leq j \leq n$. Since the system is strictly hyperbolic then,

$$\lambda_i \neq \lambda_j \text{ for } i \neq j.$$

In our notation of section (1.2), we obtain

$$\Lambda^+ = diag(\lambda_1, ..., \lambda_m) \in I\!R^{m \times m},$$

$$\Lambda^- = diag(\lambda_{m+1}, ..., \lambda_n) \in I\!R^{(n-m)\times(n-m)}.$$

We will now derive boundary conditions for the boundary at $x = 0$ and write (1.14) in the form:

$$u_x = \Lambda^{-1} u_t - \Lambda^{-1} B u_y + \tilde{C} u, \tag{1.15}$$

with $\tilde{C} = -\Lambda^{-1} C$. Defining $M = M(\xi, \omega)$ by

$$M(\xi, \omega) := -\Lambda^{-1} i\xi - \Lambda^{-1} i\omega,$$

that's how we obtain

$$u_x = M(\xi, \omega) u + \tilde{C} u. \tag{1.16}$$

This corresponds to the representation ξ and ω are the dual variables to t and y.
As in the one dimensional case, we now want the positive and negative eigenvalue λ_j.
Decouple and specify transparent boundary conditions at the corresponding inflow edges.
For this we use the construction of Taylor [**30**], according to which there is a smooth matrix $V(\xi, \omega, x, y, t)$ that is invertible for all (ξ, ω) with $\mid \omega \mid / \mid \xi \mid + \mid \omega \mid < c_0$ and $c_0 > 0$ and equation (1.15) after transformation

$$\omega = V(\frac{\partial_t}{i}, \frac{\partial_x}{i}, x, y, t) u,$$

the form

$$\omega_x = \begin{pmatrix} \Omega_{11} & 0 \\ \Omega_{12} & \Omega_{22} \end{pmatrix} \omega, \tag{1.17}$$

accepts.
Here assumes

$$\Omega_{11} = \Omega_{11}(\frac{\partial_t}{i}, \frac{\partial_x}{i}, x, y, t)u,$$

is a $m \times m$- pseudo-differential operator of order 1, Ω_{11} and Ω_{22} are $(n-m) \times m$, $(n-m) \times (n-m)$ respectively -Pseudo differential operators of order 1.
Now Ω_{11} contains the positive eigenvalues, that belong to the inflow components components at $x = 0$. The ABCs must eliminate at $x = 0$, therefore, we obtain the condition to these components

$$\omega^+ \mid_{x=0} = (Vu)^+ \mid_{x=0} = 0. \tag{1.18}$$

Now we develop $V(\xi, \omega, x, y, t)$ asymptotically as follows :

$$V(\xi, \omega, x, y, t) \cong V_0(\xi, \omega, x, y, t) + \xi^{-1}V_{-1}(\xi, \omega, x, y, t) + \xi^{-2}V_{-2}(\xi, \omega, x, y, t) + ..., \tag{1.19}$$

where $\cong$ is the asymptotic of pseudo-differential operators ([**29**],[**30**]).
From the strict hyperbolicity of (1.14) we obtain that $M(1, 0)$ is a diagonal matrix with pairwise distinct entries. Therefore, there exists a constant c_0 such that $M(\xi, \omega)$ in a conic environment of $(1, 0)$, so for $\mid \omega \mid / \mid \xi \mid + \mid \omega \mid < c_0$, also have different eigenvalues in pairs. We choose V_0 such that

$$V_0 M V_0^{-1} = \begin{pmatrix} {}_0\Omega_{11} & 0 \\ {}_0\Omega_{12} & {}_0\Omega_{22} \end{pmatrix}, \tag{1.20}$$

with dim $({}_0\Omega_{ij}) = \dim({}_0\Omega_{ij})$.
Remark:
V_0 is not uniquely determined by (1.15). In the one-dimensional case ($B = 0$) or when B is diagonal, $M(\xi, \omega)$ is diagonal matrix and $V_0 = \alpha I, \alpha \in I\!R$, can be chosen. If one breaks off the asymptotic evolution of V after finitely many steps, one obtains from (1.18) a hierarchy of absorbing boundary conditions (ABC).
We would like to use a 2-term evolution of V_0 in order to obtain a formula of the form V_0 except for an error of the order $O(\xi^{-1})$ at symbol level.
We write

$$V_0 + \xi^{-1}V_{-1} + O(\xi^{-2}) = (1 + K_1)V_0 + O(\xi^{-2}),$$

with $K^1 V_0 = \xi^{-1}V_{-1}$, and define

$$\tilde{W} = (1 + K_1)V_0 u.$$

Assuming that K_1 has the form

$$K_1 = \begin{pmatrix} 0 & \tilde{K} \\ 0 & 0 \end{pmatrix},$$

where $(1+K_1)^{-1} = (1-K_1)$.
Because of $V^{-1} \cong V_0^{-1}(1-K_1) + O(\xi^{-2})$, (1.16) and $W = Vu$ we get

$$W_x = V_x u + V u_x = V_x u + V(Mu + \tilde{C}u) = V_x V^{-1} W + V(M + \tilde{C})V^{-1} W.$$

From this we get:

$$\begin{aligned} &V_{0x}V_0^{-1} + (1+K_1)V_0(M+\tilde{C})V_0^{-1}(1-K_1) \\ &\quad = V_{0x}V_0^{-1} + V_0 M V_0^{-1} + K_1)V_0 M V_0^{-1} \\ &\quad\quad -V_0 M V_0^{-1} K_1 + V_0 \tilde{C} V_0^{-1} + O(\xi^{-1}), \end{aligned}$$

where only the asymptotic evolution from V to the term of order ξ^{-1} and was used the composition formula for pseudo differential operators ([**29**],[**30**]).
To obtain the form (1.17), $\tilde{K}$ must be chosen such that the term

$$V_{0x}V_0^{-1} + K_1)V_0 M V_0^{-1} - V_0 M V_0^{-1} K_1 + V_0 \tilde{C} V_0^{-1},$$

becomes a lower block triangle matrix. Since (1.14) symmetrically hyperbolic, $V_o = I$ can be chosen and $\tilde{K}$ must satisfy the equation

$$\tilde{K}\Omega_{22} - \Omega_{11}\tilde{K} + \tilde{C_{12}} = 0.$$

In an environment around (ξ, ω) applies $\Omega_{11} = -(\Lambda^+)^{-1}$ and $\Omega_{22} = -(\Lambda^-)^{-1}$ was with $\tilde{C_{12}} = -(\Lambda^+)^{-1} C_{12}$ gives the following:

$$\tilde{K} = (k_{ij}) \text{ with } k_{ij} = \frac{\lambda_j}{\lambda_j - \lambda_i} c_{ij}, \;\; 1 \le i \le m, \;\; m+1 \le j \le N.$$

Remark 1.10. *$\tilde{K}$ is unique if $(\Lambda^+)^{-1}$ and $(\Lambda^-)^{-1}$ have disjoint spectra. Since (Λ^+) and (Λ^-) contain the positive and negative eigenvalues, this is obviously fulfilled here. If we now continue to follow the procedure of Engquist and Majda* [**8**]*, we first find that we can choose $V_0(1,0) = I$ and thus*

$$\xi^{-1} V_{-1}(1, 0, x, y, t) = K_1(x, y, t),$$

applies.

Now we develop V_0 in a Taylor series around the vertical angle of incidence θ (it is $\omega/\xi = sin\theta$) and obtain

$$V_0(\xi,\omega) = V_0(1,\omega/\xi) = (I + \frac{\omega}{\xi}\frac{\partial}{\partial\omega}V_0)(1,0) + O(\mid \frac{\omega}{\xi} \mid^2). \tag{1.21}$$

If you choose the approach $\frac{\partial}{\partial\omega}V_0 = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, you get a calculation similar to K for the condition

$$-X(\Lambda^-)^{-1} + (\Lambda^+)^{-1}X - (\Lambda^+)^{-1}B_{12} = 0,$$

and therefore with

$$X = (\chi_{jl})_{m+1\le l\le N}^{1\le j\le m} \text{ with } \chi_{jl} = \frac{\lambda_l}{\lambda_l - \lambda_j}b_{jl}.$$

From (1.19) and (1.21) we get a hierarchy of ABCs :
Zero order ABCs: Error $O(\mid \omega/\xi \mid +1/ \mid \xi \mid)$

$$u^+(0,y,t) = 0. \tag{1.22}$$

First or two order ABCs: Error $O(\mid \omega/\xi \mid +1/ \mid \xi \mid^2)$

$$u_t^+(0,y,t) + K(0,y,t)u^-(0,y,t) = 0. \tag{1.23}$$

First order ABCs: Error $O(\mid \omega/\xi \mid^2 +1/ \mid \xi \mid^2)$

$$u_t^+(0,y,t) + X(0,y)u_y^-(0,y,t) + K(0,y,t)u^-(0,y,t) = 0. \tag{1.24}$$

Remark 1.11. **[1]** *The derived boundary conditions do not have the form of (1.10) or (1.11). The correct status of the(ABCs) can also be demonstrated in this case.*

1.4 Model examples

The following section presents the model examples on which the numerical tests were performed. These are conservation laws. These are the Euler equations, the shallow water equations, and the hydrodynamic model of the semiconductor equations. How these are related to hyperbolic equations is described. Under certain special cases, the shallow water equations can be derived from the Euler equations,. Since these equations are non-linear partial differential equations, the linearised equations are also given immediately. The construction of the corresponding absorbing boundary conditions takes place at a suitable place in the text.

Euler equations

Euler's equations are a first-order partial differential equation system in the field of flow mechanics. They are a special case of the Navier-Stokes equations for frictionless fluids (zero viscosity fluids). They are composed of the conservation equations for mass continuity equation, momentum and energy [**9**].
One dimensional Euler equation

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \upsilon(E+p) \end{pmatrix}_x = 0, \tag{1.25}$$

and two dimensional Euler equation

$$\begin{pmatrix} \rho \\ \rho u \\ \rho \upsilon \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u \upsilon \\ u(E+p) \end{pmatrix}_x + \begin{pmatrix} \rho \upsilon \\ \rho u \upsilon \\ \rho \upsilon^2 + p \\ \upsilon(E+p) \end{pmatrix}_y = 0, \tag{1.26}$$

where ρ is the density, E the energy, p the pressure and (u, υ) the speed.
Accordingly, Euler equations is written in the form

$$u_t + (f(u))_x = 0,$$

or

$$u_t + (f(u))_x + (g(u))_y = 0,$$

where $u = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}$, $f(u) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \upsilon(E+p) \end{pmatrix}$.

- **Linearisation**

In order to be able to process this system, we need a state equation which combines the pressure p with the other variables [**18**]. For an ideal gas whose internal energy e is proportional to the temperature the equation of state holds where:

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho \mid u \mid^2,$$

where

$$\gamma = \frac{\alpha + 2}{\alpha},$$

and α is the number of degrees of freedom of a gas molecule. γ is called Isene bropenexponent. We start from a monatomic gas (three degrees of freedom) and thus, we get $\gamma = \frac{5}{3}$. Then, for the pressure

$$p = \frac{2}{3}E - \frac{1}{3}\rho \mid u \mid^2 .$$

Now we write (1.25) in the form

$$u_t + f^{'}(u)u_x = 0.$$

We get in a place dimension:

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} 0 & 1 & 0 \\ \frac{-2}{3}u^2 & \frac{4}{3}u & \frac{2}{3} \\ \frac{2}{3}u^3 - \frac{5}{3}\frac{Eu}{\rho} & \frac{5}{3}\frac{Eu}{\rho} - u^2 & \frac{5}{3}u \end{pmatrix} \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_x = 0.$$

After evaluating $f^{'}$ at one point $(\overline{\rho}, \overline{\rho u}, \overline{E})$ we obtain the linear system

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + A \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_x = 0,$$

with $A = f^{'}(\overline{\rho}, \overline{\rho u}, \overline{E})$.
In two dimensions

$$\begin{pmatrix} \rho \\ \rho u \\ \rho \upsilon \\ E \end{pmatrix}_t + A \begin{pmatrix} \rho \\ \rho u \\ \rho \upsilon \\ E \end{pmatrix}_x + B \begin{pmatrix} \rho \\ \rho u \\ \rho \upsilon \\ E \end{pmatrix}_y = 0.$$

With $A = f^{'}(\overline{\rho}, \overline{\rho u}, \overline{\rho \upsilon}, \overline{E}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-2}{3}u^2 + \frac{1}{3}\upsilon^2 & \frac{4}{3}u & \frac{-2}{3}\upsilon & \frac{2}{3} \\ -u\upsilon & \upsilon & u & 0 \\ \frac{-5}{3}E\frac{u}{\rho} + \frac{2}{3}u^3 + \frac{2}{3}u\upsilon^2 & \frac{5}{3}\frac{E}{\rho} - u^2 - \frac{1}{3}\upsilon^2 & \frac{-2}{3}u\upsilon & \frac{5}{3}u \end{pmatrix}$

and $B = g^{'}(\overline{\rho}, \overline{\rho u}, \overline{\rho \upsilon}, \overline{E}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -u\upsilon & \upsilon & u & 0 \\ \frac{-2}{3}\upsilon^2 + \frac{1}{3}u^2 & \frac{-2}{3}u & \frac{4}{3}\upsilon & \frac{2}{3} \\ \frac{-5}{3}E\frac{\upsilon}{\rho} + \frac{2}{3}u^2\upsilon + \frac{2}{3}\upsilon^3 & \frac{-2}{3}u\upsilon & \frac{5}{3}\frac{E}{\rho} - \upsilon^2 - \frac{1}{3}u^2 & \frac{5}{3}\upsilon \end{pmatrix}$.

Shallow-Water equations

Now consider the movement of water or any other incompressible fluid with free surface and small depth. In this case, one speaks of shallow water. More specifically, we have the condition that the height of the surface above the ground is small compared to a typical horizontal travel length (e.g., wavelength).

- **Derivation**

Let $h(x,t)$ denote the height of the water surface, which is small compared to a typical wavelength, and assume that the water bottom is horizontal and neglect friction. Furthermore, the density of the water is normalized to one. Let $u(x,t)$ denote the velocity, p_o the atmospheric pressure, and $p_o + p(x,t)$ the water pressure.

$$\frac{d}{dt}\int_{x_1}^{x_2} h(x,t)dx = q_2 - q_1,$$

where $q = uh$ is the mass flow and q_1, q_2 are the corresponding values of q at positions x_1 and x_2.
If we now carry out the limit value process $x_2 \longrightarrow x_1$, we obtain

$$h_t + q_x = 0,$$

or

$$h_t + (uh)_x = 0. \tag{1.27}$$

A second relation between u and h is obtained from the conservation of momentum in the x-direction. If we look again at a range $x_2 \leq x \leq x_1$, then the constant pressure p_o is on the whole edge in equilibrium. Thus, only the excess pressure p has contribution to the momentum balance. If $P(x,t)$ is the total pressure along a vertical section, i.e.,

$$P = \int_0^h p dy,$$

so the momentum balance is given by:

$$\frac{d}{dt}\int_{x_2}^{x_1} hu\, dx = hu^2 \mid_{x=x_2} - hu^2 \mid_{x=x_1} + P_2 - P_1, \tag{1.28}$$

with $P_i = P(x_i,t), i = 1,2$.
The term on the left is the total impulse rate of change in region $x_2 \leq x \leq x_1$. The

terms $hu^2\ |_{x=x_i}$ describe the impulse transport through the surfaces at $x = x_i (i = 1, 2)$.
The basic assumption in the shallow water theory is that the pressure is hydrostatic, i.e., it applies

$$\frac{\partial p}{\partial y} = -g, \tag{1.29}$$

and g is the gravitational constant. Integrating (1.29) and assuming that $p = p_o$ at the surface $y = h$, we obtain for the pressure

$$p = p_o + g(h - y).$$

Therefore, the total pressure is given by

$$p = \int_0^h g(h - y)dy = \frac{1}{2}gh^2.$$

Together with (1.28) we get :

$$\frac{d}{dt}\int_{x_2}^{x_1} hu \ \ dx + [hu^2 + \frac{1}{2}gh^2]_{x_2}^{x_1} = 0,$$

and in the limit $x_2 \longrightarrow x_1$ results

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = 0. \tag{1.30}$$

Remark (Related to the Euler equations):
Equations (1.27) and (1.30) correspond to the first two equations of the Euler equations. Since incompressible fluids are considered in the shallow-water theory, the density remains constant and instead it varies the height of the fluid surface. After the pressure has been assumed to be hydrostatic, the energy conservation equation is decoupled from that of mass and momentum conservation. Therefore, the system is reduced by one equation.
(1.27) and (1.30) give us the one dimensional shallow-water equations:

$$u_t + uu_x + \phi_x = 0, \tag{1.31a}$$

$$\phi_t + (\phi u)_x = 0. \tag{1.31b}$$

Here $\phi = gh$ is the earth potential, which corresponds to a factor of the height of the fluid surface. A similar derivation leads [**31**] to the following in two spatial

dimensions System (two dimensional shallow water equations):

$$u_t + uu_x + \upsilon u_y + \phi_x = 0, \tag{1.32a}$$

$$\upsilon_t + u\upsilon_x + \upsilon\upsilon_y + \phi_y = 0, \tag{1.32b}$$

$$\phi_t + (\phi u)_x + (\phi u)_y = 0. \tag{1.32c}$$

- **Linearisation**

The linearised form is obtained from these equations by linearising around a constant value $\overline{U} = (\overline{u}, \overline{v}, \overline{\phi})$. For a solution U of (1.31) we choose the perturbation $U = \overline{U} + U'$ with the small perturbation $U' = (u', \upsilon', \phi')^T$. This approach is used in (1.31) and terms of quadratic order in the perturbation they don't get enough attention. The linearised form then has the form:

$$\begin{pmatrix} u' \\ \phi' \end{pmatrix}_t + \begin{pmatrix} \overline{u} & 1 \\ \overline{\phi} & \overline{u} \end{pmatrix} \begin{pmatrix} u' \\ \phi' \end{pmatrix}_x = 0. \tag{1.33}$$

Since we usually need the diagonalized shape to formulate the ABCs, and in the general case it is very simple compared to the Euler equations, we still carry out the corresponding variable transformation. To simplify the notation we use $u' = u, \phi' = \phi, \overline{u} = U$ and $\overline{\phi} = \Phi$, then (1.33) has the clearer shape

$$\begin{pmatrix} u \\ \phi \end{pmatrix}_t + \begin{pmatrix} U & 1 \\ \Phi & U \end{pmatrix} \begin{pmatrix} u \\ \phi \end{pmatrix}_x = 0,$$

the matrices

$$T^{-1} = \begin{pmatrix} 1 & -\frac{1}{c} \\ 1 & \frac{1}{c} \end{pmatrix}, T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{c}{2} & -\frac{c}{2} \end{pmatrix},$$

with $c = \sqrt{\Phi}$, transform (1.31) into the characteristic form.

$$\begin{pmatrix} d \\ e \end{pmatrix}_t + \begin{pmatrix} U - c & 0 \\ 0 & U + c \end{pmatrix} \begin{pmatrix} d \\ e \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.34}$$

because of

$$\begin{pmatrix} d \\ e \end{pmatrix} = T^{-1} \begin{pmatrix} u \\ \phi \end{pmatrix},$$

are the characteristic variables given by $d = u - \frac{\phi}{c}$ and $e = u + \frac{\phi}{c}$.
In the two dimensional case (1.32) the linearised form is given

$$\begin{pmatrix} u \\ \upsilon \\ \phi \end{pmatrix}_t + \begin{pmatrix} U & 0 & 1 \\ 0 & U & 0 \\ \Phi & 0 & U \end{pmatrix} \begin{pmatrix} u \\ \upsilon \\ \phi \end{pmatrix}_x + \begin{pmatrix} V & 0 & 0 \\ 0 & V & 1 \\ 0 & \Phi & V \end{pmatrix} \begin{pmatrix} u \\ \upsilon \\ \phi \end{pmatrix}_y = 0. \tag{1.35}$$

System (1.35) can be symmetrized by multiplying it by $\Phi > 0$ the positive-definite matrix diag $(\sqrt{\Phi}, \sqrt{\Phi}, 1)$. Accordingly, (1.35) is a hyperbolic system that can be symmetrized simultaneously. With $S = \text{diag}(\sqrt{\Phi}, \sqrt{\Phi}, 1)$ we obtain:

$$S \begin{pmatrix} U & 0 & 1 \\ 0 & U & 0 \\ \Phi & 0 & U \end{pmatrix} S^{-1} = \begin{pmatrix} U & 0 & \sqrt{\Phi} \\ 0 & U & 0 \\ \sqrt{\Phi} & 0 & U \end{pmatrix},$$

and

$$S \begin{pmatrix} V & 0 & 0 \\ 0 & V & 1 \\ 0 & \Phi & V \end{pmatrix} S^{-1} = \begin{pmatrix} V & 0 & 0 \\ 0 & V & \sqrt{\Phi} \\ 0 & \sqrt{\Phi} & V \end{pmatrix},$$

so that the system (1.35) can be transformed to

$$\tilde{W}_t + \begin{pmatrix} U & 0 & c \\ 0 & U & 0 \\ c & 0 & U \end{pmatrix} \tilde{W}_x + \begin{pmatrix} V & 0 & 0 \\ 0 & V & c \\ 0 & c & V \end{pmatrix} \tilde{W}_y = 0,$$

with $\tilde{W} = (cu, cv, \phi)^T$. The quantity $c = \sqrt{\Phi}$ is the velocity of gravitational waves in shallow water, so we also get the physical restrictions $0 < U^2 + V^2 < c^2, c > 0$. In order to diagonalize the coefficient matrix of the r-derivative, we perform another variable transformation with the orthogonal matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}. \tag{1.36}$$

Thus, we obtain for $W := T^{-1}\tilde{W}$ the form

$$W_t + \underbrace{\begin{pmatrix} U+c & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & U-c \end{pmatrix}}_{=:\Lambda} W_x + \underbrace{\begin{pmatrix} V & \frac{C}{\sqrt{2}} & 0 \\ \frac{C}{\sqrt{2}} & V & -\frac{C}{\sqrt{2}} \\ 0 & \frac{C}{\sqrt{2}} & V \end{pmatrix}}_{=:B} W_y = 0. \tag{1.37}$$

The Hydrodynamic Model of the Semiconductor Equations

The hydrodynamic equations essentially consist of the Euler equations. Considered in this case is a charged gas of one-atom particles (three degrees of freedom, $\gamma = \frac{5}{3}$) in an electric field with additional relaxation time terms. For the system to become hyperbolic, we neglect the heat conduction terms. The system consists of the three conservation laws for particle density, momentum (density), and energy (density), which are as follows:

- **particle density (electron density):**

$$n_t + (nv)_x = C_n,$$

where n is the electron concentration, v is the mean velocity of the electrons and C_n is a measure of the recombination and regeneration processes, where C_n becomes zero because we consider a unipolar model.

- **Impulse:**

$$(nv)_t + (nv^2 + p)_x = fn + \frac{C_p}{m},$$

where $f = f(x,t)$ is an external force due to the electric field, m is the effective mass of an electron and C_p is a measure of the electron-electron and electron-lattice collisions. As with the Euler equations, we must also specify the pressure p here. Let p be given by $p = \kappa_B nT/m$.

- **Energy:**

$$[n(\frac{1}{2}v^2 + e_I)]_t + [nv(\frac{1}{2}v^2 + e_I) + pv]_x = fnv + \frac{C_w}{m},$$

where e_I is the internal energy and C_w is a collision term. We further assume that the energy bands are parabolic and that with the energy density $w = mne_I + mnv^2/2 = \kappa_B nT/(\gamma - 1) + mnv^2/2$ and the momentum density $p = mnv$ we obtain

$$\begin{aligned} \rho_t + p_x &= 0, \\ p_t + (pv + \kappa_B nT)_x &= f\rho + C_p, \\ w_t + [v(w + \kappa_B nT)]_x &= fp + C_w, \end{aligned}$$

where we have multiplied the equations by m and introduced the density $\rho = mn$. If we additionally use $\kappa_B nT = (\gamma - 1)(w - pv/2)$, the last two equations yield

$$\begin{aligned} p_t + [(\gamma - 1)w + \frac{1}{2}(\gamma - 3)pv]_x &= f\rho + C_p, \\ w_t + [v(\gamma - \frac{1}{2}(\gamma - 3)pv)]_x &= fp + C_w. \end{aligned}$$

In our application example we use the collision terms C_p, C_w with relaxation times τ_p, τ_w, [**2**] and [**3**] to be proposed:

$$\begin{aligned} C_p &= -\frac{p}{\tau_p}, \\ C_w &= -\frac{w}{\tau_w} + \frac{1}{\gamma - 1}\frac{\rho \kappa_B T_0}{m \tau_w}, \\ \tau_p &= \tau_p(T) = \frac{m \lambda_0 T_0}{eT}, \\ \tau_w &= \tau_w(T) = \frac{\tau_p}{2} + \frac{3 \lambda_0 \kappa_B T T_0}{2 e \upsilon_s^2 (T + T_0)}. \end{aligned}$$

We also use the following physical parameters for silicon at lattice temperature $To = 300K$ [**3**]:

$$\begin{aligned} \text{effective mass of an electron} \quad m &= 0.26.m_e, \\ \text{saturation velocity} \quad \upsilon_s &= 1.035 - 10^5 m s^{-1}, \\ \text{mobility of the electrons} \quad \lambda_0 &= 0.145 m^2 s^{-1} V^{-1}. \end{aligned}$$

For simplification we use constant relaxation times $\tau_p = \tau_p(T_0), \tau_w = \tau_w(T_0)$, with $\upsilon = p/\rho$ the equations can now be written in conservation form.

$$u_t + (f(u))_x + h(u) = 0,$$

where

$$\begin{aligned} u &= \begin{pmatrix} \rho \\ p \\ w \end{pmatrix}, \\ f(u) &= \begin{pmatrix} p \\ (\gamma - 1)w - \frac{1}{2}(\gamma - 3)\frac{p^2}{\rho} \\ \gamma \frac{pw}{\rho} - \frac{1}{2}(\gamma - 1)\frac{p^3}{\rho^2} \end{pmatrix}, h(u) = \begin{pmatrix} 0 \\ -f\rho + \frac{p}{\tau_p} \\ -fp + \frac{w}{\tau_w} - \frac{1}{\gamma - 1}\frac{\rho \kappa_B T_0}{m \tau_w} \end{pmatrix}. \end{aligned}$$

The initial state is determined by:

$$n(x,0) = n_D(x), \;\; \upsilon(x,0) = 0, \;\; T(x,0) = T_0,$$

where n_D is the doping profile. The initial conditions are:

$$\rho(x,0) = m n_D(x), \;\; p(x,0) = 0, \;\; w(x,0) = \frac{1}{\gamma - 1}\kappa_B T_0 n_D(x).$$

1.4.1 Diagonalization of the matrix system

In order to be able to formulate the boundary conditions and numerical methods more easily later, we consider the equations in the form

$$u_t + Au_x + Cu = 0.$$

The coefficients are given by

$$A = \frac{\partial f}{\partial u} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)v^2 & -(\gamma-3)v & \gamma-1 \\ -\gamma\frac{wv}{\rho} + (\gamma-1)v^3 & \gamma\frac{w}{\rho} - \frac{3}{2}(\gamma-1)v^2 & \gamma v \end{pmatrix}, v = \frac{p}{\rho},$$

and

$$C = \begin{pmatrix} 0 & 0 & 0 \\ -f & -\tau_p^{-1} & 0 \\ \frac{-\kappa_B T_0}{(\gamma-1)m\tau_w} & -f & \tau_w^{-1} \end{pmatrix}.$$

In addition, with the speed of sound we get $c, c^2 = \gamma\kappa_B T/m$ from the equation for the energy density w:

$$\gamma\frac{w}{\rho} = \frac{1}{\gamma-1} + \frac{1}{2}\gamma v^2.$$

The system matrix can thus be written as

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)v^2 & -(\gamma-3)v & \gamma-1 \\ -v[\frac{c^2}{\gamma-1} + (1-\frac{1}{2}\gamma)v^2] & \frac{c^2}{\gamma-1} + (\gamma-\frac{3}{2})v^2 & \gamma v \end{pmatrix}.$$

In addition, A has the eigenvalues

$$\lambda_2 = v, \quad \lambda_1, \lambda_3 = v \pm c,$$

with the associated eigenvectors

$$v_2 = \begin{pmatrix} 1 \\ v \\ \frac{v^2}{2} \end{pmatrix}, v_{1,3} = \begin{pmatrix} 1 \\ v \pm c \\ \frac{v^2}{2} \pm vc + \frac{c^2}{\gamma-1} \end{pmatrix}. \tag{1.38}$$

1.5 Derivation of far field boundary conditions

In this section we will talk about the derivation of hierarchy of far field boundary conditions (FBCs)[**6**], at $x = 0$ and $x = 1$, for a strictly hyperbolic system of the form

$$u_t + \Lambda u_x + Cu = F(x), \qquad x \in I\!R, \quad t > 0, \tag{1.39}$$

with the initial function

$$u(x,0) = f(x), \qquad x \in I\!R, \tag{1.40}$$

where C and Λ are $n \times n$ constant matrices, u is a vector with n components, and we assume that $f(x)$ and $F(x)$ are $C^{\infty}[0,1]$-smooth functions. Furthermore, due to the strict hyperbolicity the eigenvalues of Λ are distinct ($\lambda_j \neq \lambda_i$ for $i \neq j$) and not equal zero.
Therefore, Λ can be written in the form

$$\Lambda = diag(\lambda_1, ..., \lambda_m, \lambda_{m+1}, ..., \lambda_n) = \begin{pmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{pmatrix}, \tag{1.41}$$

with $\Lambda^+ = \text{diag}(\lambda_1, ..., \lambda_m), \quad \lambda_j > 0, \quad \Lambda^- = \text{diag}(\lambda_{m+1}, ..., \lambda_n), \quad \lambda_j < 0,$
this assumption does not hold for systems with characteristic boundary, such as Maxwell's equations [**32**], but it will possible with t. We will further assume that

$$\acute{C} = \frac{C^T + C}{2} \geq \delta I, \quad \delta > 0, \tag{1.42}$$

this condition is necessary to ensure the convergence of the whole space problem to the steady state as $t \to \infty$, for more information we can see [**32**]. Furthermore, we assume that $\Lambda^{-1}C$ has distinct eigenvalues. We need m boundary conditions at $x = 0$, and $(n - m)$ at $x = L$.

Notation 1.12. *Any $n \times n$-matrix X is partitioned as*

$$X = \begin{pmatrix} X^{++} & X^{+-} \\ X^{-+} & X^{--} \end{pmatrix},$$

where X^{++}, X^{+-}, X^{-+}, and X^{--} are $m \times m, m \times (n-m), (n-m) \times m$, and $(n-m) \times (n-m)$-matrices, respectively.
Also $X^+ := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} X$ *and* $X^- := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} X$.

Solutions of equation (1.39) consist of n different patterns propagating at different speeds. A critical step in developing boundary conditions for (1.39) is to determine the propagation direction for each mode, and to distinguish between incoming and outgoing modes at the boundary.
If we take a Laplace transform in t, with the dual variable s

$$\tilde{u}(x,s) = \int_0^{\infty} e^{-st} u(x,t) dt, \quad s \in \mathbb{C}, \quad \Re s > 0,$$

the system becomes

$$\tilde{u}_x + sE(s)\tilde{u} = \Lambda^{-1}\tilde{f}, \tag{1.43}$$

where $E(s) := \Lambda^{-1} + \frac{1}{s}\Lambda^{-1}C$. We want to separate $\tilde{u}$ into 'rightgoing' and 'leftgoing' modes. Each of these modes corresponds to the eigenvalue of E (s).

with positive real part,in the first case and modes that are corresponds to eigenvalue of $E(s)$ with negative real parts in the second case.

Lemma 1.13. [**32**] *Let G, H be $n \times n$-matrices with H Hermitian and regular, suppose $HG + G^*H$ is a positive semi-definite and $i_0(G) = 0$. Then $i(G) = i(H)$.*

Lemma 1.14. [**32**] *For $\Re s > 0$, $E(s)$ has exactly m eigenvalues with positive real part and $(n - m)$ with negative real part; i.e., $i(\Lambda) = i(E)$.*

From Lemma (1.14) and [**13**], there is $\eta_0 > 0$ and a nonsingular transformation $T(s)$ such that for $\Re s > \eta_0$,

$$XEX^{-1} = D = \begin{pmatrix} D^+ & 0 \\ 0 & D^- \end{pmatrix}, \qquad D^+ > 0, \quad D^- < 0, \tag{1.44}$$

where $D(s)$ is the matrix of eigenvalues of $E(s)$, arranged so that $D^+(s)$ is an $m \times m$ positive definite matrix, corresponding to rightgoing solutions, and positive eigenvalues of $E(s)$, $D^-(s)$ is an $(n-m)\times(n-m)$ negative definite matrix, corresponding to leftgoing solutions, and negative eigenvalues of $E(s)$. Here, we drop the explicit s-dependence. From now on all matrices are a function of s unless otherwise noted. The system (1.43) can be written in different way With the characteristic variables $\tilde{v} := X\tilde{u}$ as

$$\tilde{v}_x + sD\tilde{v} = X\Lambda^{-1}\tilde{F},$$

and then partitioned into

$$\frac{d}{dx}\begin{pmatrix} \tilde{v}^+ \\ \tilde{v}^- \end{pmatrix} - s\begin{pmatrix} D^+ & 0 \\ 0 & D^- \end{pmatrix}\begin{pmatrix} \tilde{v}^+ \\ \tilde{v}^- \end{pmatrix} = X\Lambda^{-1}\tilde{F},$$

where $\tilde{v}^+$ and $\tilde{v}^-$ represent "rightgoing" and "leftgoing" modes respectively.
Now, we restrict the domain of x in (1.39) to $(0, 1)$. The exact nonreflecting boundary conditions follow immediately. Since there are no incoming modes at a nonreflecting boundary, at right boundary, there should be no leftgoing modes, so an exact perfectly ABC is

$$\tilde{v}^- = [X\tilde{u}]^- = 0, \qquad x = 1. \tag{1.44a}$$

At the left boundary $x = 0$ there should be no rightgoing modes, so an exact perfectly ABC is

$$\tilde{v}^+ = [X\tilde{u}]^+ = 0, \qquad x = 0. \tag{1.45a}$$

We face two difficulties in implementing the above mentioned boundary conditions. The first difficulty, because the boundary condition is expressed in the Laplace transform (x, s)-space, and the matrix $X(s)$ contains irrational functions of s (for example, square roots), when we turn back into physical (x, t)-space, the boundary conditions will be non-local in time.
From a computational perspective, we prefer a local boundary condition, which can be obtained by approximating the non-rational elements of X with the by rational functions of s.
The second difficulty arises when approximations are introduced: then the resulting IBVP may beill-posed.
For $s \to \infty$, we have $E(s) \to \Lambda^{-1}$ and hence $X(s) \to I$. Following standard practice in [**7**] we shall hence make a high frequency expansion of X for $\Re s > \eta_0$:

$$X(s) = I + \frac{1}{s}X_1 + \frac{1}{s^2}X_2 + O\left(\frac{1}{|s|^3}\right). \tag{1.46}$$

The zero order ABCs are then

$$\begin{aligned} \tilde{u}^+ &= 0, \qquad x = 0, \\ \tilde{u}^- &= 0, \qquad x = 1. \end{aligned}$$

First and second order ABCs are respectively

$$\begin{aligned} \left[(I + \frac{1}{s}X_1)\tilde{u}\right]^+ &= 0, \qquad x = 0, \\ \left[(I + \frac{1}{s}X_1)\tilde{u}\right]^- &= 0, \qquad x = 1, \end{aligned}$$

$$\begin{aligned} \left[(I + \frac{1}{s}X_1 + \frac{1}{s^2}X_2)\tilde{u}\right]^+ &= 0, \qquad x = 0, \\ \left[(I + \frac{1}{s}X_1 + \frac{1}{s^2}X_2)\tilde{u}\right]^- &= 0, \qquad x = 1. \end{aligned}$$

We transformed first and second order ABCs to the physical space by the substitution $s \to \frac{\partial}{\partial t}$.

Hence, the first and second order ABCs are respectively [8]

$$\left[(\frac{\partial}{\partial t}+X_1)u\right]^+=0, \qquad x=0, \tag{1.47a}$$

$$\left[(\frac{\partial}{\partial t}+X_1)u\right]^-=0, \qquad x=1, \tag{1.47b}$$

$$\left[(\frac{\partial^2}{\partial t^2}+X_1\frac{\partial}{\partial t}+X_2)u\right]^+=0, \qquad x=0, \tag{1.48a}$$

$$\left[(\frac{\partial^2}{\partial t^2}+X_1\frac{\partial}{\partial t}+X_2)u\right]^-=0, \qquad x=1. \tag{1.48b}$$

For large $\Re s>\eta_0$, the term $\frac{1}{s}\Lambda^{-1}C$ in $E(s)$ is a perturbation of Λ^{-1}. In this case, D in (1.44) can be considered as a diagonal matrix. With a high frequency expansion D is written as

$$D(s)=\Lambda^{-1}+\frac{1}{s}D_1+\frac{1}{s^2}D_2+O\left(\frac{1}{|s|^3}\right),$$

where $D_j(s),\ j=1,2,...$ are diagonal.
Write (1.44) as $XE=DX$, then the $O\left(|s|^{-1}\right)$-equation reads

$$X_1\Lambda^{-1}+\Lambda^{-1}C=\Lambda^{-1}X_1+D_1.$$

Solving for X_1 and D_1 gives

$$D_1=diag\left(\frac{c_{11}}{\lambda_1},...,\frac{c_{nn}}{\lambda_n}\right),$$

and

$$(X_1)_{ij}=\begin{cases}0, & i=j,\\ \dfrac{\lambda_j c_{ij}}{\lambda_j-\lambda_i}, & i\neq j,\end{cases}$$

where c_{ij} is the $(i,j)^{th}$ entry of C. The second order expansion of (1.44) yields

$$X_2\Lambda^{-1}+X_1\Lambda^{-1}C=\Lambda^{-1}X_2+D_1X_1+D_2.$$

From solving X_2 and D_2, we have

$$D_2=diag\left(\sum_{j\neq 1}\frac{c_{1j}c_{j1}}{\lambda_j-\lambda_1},...,\sum_{j\neq n}\frac{c_{nj}c_{jn}}{\lambda_j-\lambda_n}\right),$$

and

$$(X_2)_{ij} = \begin{cases} 0, & i = j, \\ \dfrac{1}{\lambda_i - \lambda_j}\left[c_{ii}c_{ij}\dfrac{\lambda_j^2}{\lambda_j - \lambda_i} - \sum_{l\neq i} c_{il}c_{lj}\dfrac{\lambda_i\lambda_j}{\lambda_l - \lambda_i}\right], & i \neq j. \end{cases}$$

Now, the stationary problem corresponding to (1.39) :

$$u_x + \Lambda^{-1}Cu = \Lambda^{-1}F(x), \qquad x \in I\!R, \tag{1.49}$$

$$u(x) \to 0, \qquad x \to \pm\infty. \tag{1.50}$$

Where $\Lambda^{-1}C$ is assumed to have distinct eigenvalues $\neq 0$. If we take lemma (1.14) but for the case $s = 0$, then we have the following lemma

Lemma 1.15. $i(\Lambda) = i(\Lambda^{-1}C)$.

Proof. Apply Lemma 1.13 with $H := \Lambda$ and $G := \Lambda^{-1}C$

$$\Lambda(\Lambda^{-1}C) + (\Lambda^{-1}C)^T\Lambda = 2C_1 > 0.$$

Assume that $\Lambda^{-1}C$ has the purely imaginary eigenvalue $i\omega$. Then

$$i\omega\phi = \Lambda^{-1}C\phi \Leftrightarrow i\omega\Lambda\phi = C\phi.$$

But, on the other hand

$$\begin{aligned} 0 &= \langle i\omega\Lambda\phi, \phi\rangle + \langle \phi, i\omega\Lambda\phi\rangle \\ &= \langle C\phi, \phi\rangle + \langle \phi, C\phi\rangle = \left\langle \phi, (C + C^T)\phi\right\rangle \geq 2\delta\left|\phi\right|^2 > 0, \end{aligned}$$

and hence $i_0(\Lambda^{-1}C) = 0$. □

We will diagonalize the system (1.49), using Lemma (1.15) and that $\Lambda^{-1}C$ has distinct eigenvalues.

$$\omega_x + R\omega = S\Lambda^{-1}F(x), \tag{1.51}$$

where ω is given by $\omega := Su$,

$$S\Lambda^{-1}CS^{-1} = R = \begin{pmatrix} R^+ & 0 \\ 0 & R^- \end{pmatrix}, \tag{1.52}$$

$R^+ = \text{diag}(r_1, ..., r_m), \;\; \Re r_j > 0, \qquad R^- = \text{diag}(r_{m+1}, ..., r_n), \;\; \Re r_j < 0.$
The following boundary conditions for the steady problem on the bounded domain $(0, 1)$ are satisfied by the steady solution on the unbounded domain

$$(Su)^+ = 0, \qquad x = 0, \tag{1.53a}$$

$$(Su)^- = 0, \qquad x = 1. \tag{1.53b}$$

This is true since the general solution of (1.51) outside the support of F is

$$\omega(x) = \begin{pmatrix} \omega^+(0)e^{-R^+x} \\ \\ \omega^-(0)e^{-R^-x} \end{pmatrix}, \qquad x \leq 0, \qquad\qquad \omega(x) = \begin{pmatrix} \omega^+(1)e^{R^+(1-x)} \\ \\ \omega^-(1)e^{R^-(1-x)} \end{pmatrix}, \qquad x \geq 1,$$

where $\omega = (\omega^+, \omega^-)^T$ is partitioned in the same way as u. It is necessary that

$$\omega^+(0) = \omega^-(1) = 0.$$

For the decay condition (1.50) to be valid (1.53) is unique up to a multiplication by regular matrices V^+ and V^-, respectively

$$(VSu)^+ = 0, \qquad x = 0, \tag{1.54a}$$

$$(VSu)^- = 0, \qquad x = 1. \tag{1.54b}$$

In [**7**] the authors defined a family of first order FBCs from a combination of the first order ABCs (1.47) and the transparent steady boundary conditions (1.54):

$$\left[(\frac{\partial}{\partial t} + VS)u\right]^+ = 0, \qquad x = 0, \tag{1.55a}$$

$$\left[(\frac{\partial}{\partial t} + VS)u\right]^- = 0, \qquad x = 1, \tag{1.55b}$$

which is defined up to a matrix factor, $V = \begin{pmatrix} V^+ & 0 \\ 0 & V^- \end{pmatrix}$, in front of S. Higher order boundary conditions can formally be derived analogously [**11**],[**12**], and [**16**]

$$\left[(\frac{\partial^2}{\partial t^2} + X_1\frac{\partial}{\partial t} + VS)u\right]^+ = 0, \qquad x = 0, \tag{1.56a}$$

$$\left[(\frac{\partial^2}{\partial t^2} + X_1\frac{\partial}{\partial t} + VS)u\right]^- = 0, \qquad x = 1. \tag{1.56b}$$

The solution of the IVP (1.39) on $(0,1)$ with the boundary conditions (1.55) or (1.56), for arbitrary regular V, converges for long time to the steady state, see [**7**]. But since spurious reflections pollute the computed solution, a good choice of V^+ and V^- that annihilate the spurious reflections up to higher order can accelerate this convergence for long time computations and gives higher accuracy for short time computations. Hence, we choose V^+ and V^- in away to annihilate spurious reflections.

To clarify that, we transform the first order left boundary condition (1.55a) into Laplace space, and use the Notation (1.12)

$$\left[(I+\frac{1}{s}VS)\tilde{u}\right]^+ = \left(I^+ + \frac{1}{s}V^+S^{++} \quad , \quad \frac{1}{s}V^+S^{+-}\right)\tilde{u} = 0. \tag{1.57}$$

In terms of the characteristic variables, $\tilde{u} = X^{-1}\tilde{v}$, where

$$X^{-1}(s) = I - \frac{1}{s}X_1 - \frac{1}{s^2}(X_2 - X_1^2) + O\left(\frac{1}{|s|^3}\right),$$

(1.57) then becomes

$$\left(I^+ + \frac{1}{s}V^+S^{++} \quad , \quad \frac{1}{s}V^+S^{+-}\right)\begin{pmatrix} I^+ - \frac{1}{s}X_1^{++} & -\frac{1}{s}X_1^{+-} \\ -\frac{1}{s}X_1^{-+} & I^- - \frac{1}{s}X_1^{--} \end{pmatrix}\tilde{v} + O\left(\frac{1}{|s|^2}\right)$$

$$= \left[I^+ + \frac{1}{s}(V^+S^{++} - X_1^{++})\right]\tilde{v}^+ + \frac{1}{s}\left[V^+S^{+-} - X_1^{+-}\right]\tilde{v}^- + O\left(\frac{1}{|s|^2}\right) = 0.$$

Neglecting $O(|s|^{-2})$-terms, we may solve for the incoming (rightgoing) modes in terms of outgoing ones as long as $\left[I^+ + \frac{1}{s}(V^+S^{++} - X_1^{++})\right]$ is nonsingular (this holds true at least for $|s|$ large)

$$\tilde{v}^+(0) = -\left[sI^+ + (V^+S^{++} - X_1^{++})\right]^{-1}\left[V^+S^{+-} - X_1^{+-}\right]\tilde{v}^-(0) =: R_c^+\tilde{v}^-(0),$$

where R_c^+ is the matrix of reflection coefficients.

Similarly the right boundary condition (1.55b) may be written in term of the characteristic variables as

$$\left(\frac{1}{s}V^-S^{-+} \quad , \quad I^- + \frac{1}{s}V^-S^{--}\right)\begin{pmatrix} I^+ - \frac{1}{s}X_1^{++} & -\frac{1}{s}X_1^{+-} \\ -\frac{1}{s}X_1^{-+} & I^- - \frac{1}{s}X_1^{--} \end{pmatrix}\tilde{v}$$

$$= \frac{1}{s}\left[V^-S^{-+} - X_1^{-+}\right]\tilde{v}^+ + \left[I^- + \frac{1}{s}(V^-S^{--} - X_1^{--})\right]\tilde{v}^- + O\left(\frac{1}{|s|^2}\right) = 0.$$

Neglecting $O(|s|^{-2})$-terms and solving for the incoming (leftgoing) modes in terms of outgoing ones as long as $\left[I^- + \frac{1}{s}(V^- S^{--} - X_1^{--})\right]$ is nonsingular, gives

$$\tilde{v}^-(1) = -\left[sI^- + (V^- S^{--} - X_1^{--})\right]^{-1}\left[V^- S^{-+} - X_1^{-+}\right]\tilde{v}^+(1) =: R_c^- \tilde{v}^+(1),$$

where R_c^- is the matrix of reflection coefficients at the right boundary.
For the pair of boundary conditions to be absorbing up to order $O(|s|^{-2})$, the matrices R_c^+ and R_c^- must be identically zero, that is $V^+ S^{+-} - X_1^{+-}$ and $V^- S^{-+} - X_1^{-+}$ must be zero. So the optimal choices of V^+ and V^- are then given as solutions of

$$V^+ S^{+-} = X_1^{+-}, \tag{1.58a}$$

$$V^- S^{-+} = X_1^{-+}. \tag{1.58b}$$

If $(S^{+-})^{-1}$ exists, then $V^+ = X_1^{+-}(S^{+-})^{-1}$ and the first order FBC at $x = 0$ reads

$$u_t^+ + X_1^{+-}(S^{+-})^{-1} S^{++} u^+ + X_1^{+-} u^- = 0, \tag{1.59a}$$

which is different from the first order ABC (1.47a) only by the middle term.
Similarly if $(S^{-+})^{-1}$ exists, then $V^- = X_1^{-+}(S^{-+})^{-1}$ and the first order FBC at $x = 1$ is

$$u_t^- + X_1^{-+} u^+ + X_1^{-+}(S^{-+})^{-1} S^{--} u^- = 0. \tag{1.59b}$$

We shall denote these FBCs as

$$\left[(\frac{\partial}{\partial t} + \hat{X}_1)u\right]^+ = 0, \qquad x = 0, \tag{1.60a}$$

$$\left[(\frac{\partial}{\partial t} + \hat{X}_1)u\right]^- = 0, \qquad x = 1, \tag{1.60b}$$

where

$$\hat{X}_1 = \begin{pmatrix} X_1^{+-}(S^{+-})^{-1} S^{++} & X_1^{+-} \\ X_1^{-+} & X_1^{-+}(S^{-+})^{-1} S^{--} \end{pmatrix}.$$

Chapter 2

Numerical scheme

2.1 Finite difference scheme

In the following section we consider finite difference methods for the solution of one dimensional linear hyperbolic systems of the form:

$$u_t + Au_x + Cu = F(x,t), \quad 0 \le x \le L, 0 \le t, \tag{2.1a}$$

$$u(x,0) = f(x), \qquad 0 \le x \le L, \tag{2.1b}$$

where A and C are constant $n \times n-$ matrices. The case of location and time-dependent matrices can be viewed analogously, and the boundary conditions at $x = 0, L$ which we obtain from previous chapter section (2) :

$$u_t^+ + (Su)^+ = 0, \qquad x = 0, \tag{2.1c}$$

$$u_t^- + (Su)^- = 0, \qquad x = 1. \tag{2.1d}$$

We now discretize the $(x,t)-$ stripe $[0,L] \times I\!R_0^+$ with the mesh points $\Delta t = k$ and $\Delta x = h$ and look for approximations of the solution of (2.1a)

$$\begin{aligned} x_i &= ih, \ i = 0, 1, ..., I, \\ t_l &= lk, \ l = 0, 1, ... \end{aligned}$$

Firstly assume that $r = k/h$, is constant and we use it to satisfy CFL-condition to show the stability, and denote the notation $u_i^l \in I\!R^n$ to approximate the exact solution $u(x_i, t_l)$. According to the partition of u, we set

$$u_i^l = \begin{pmatrix} (u^+)_i^l \\ (u^-)_i^l \end{pmatrix}, (u^+)_i^l \in I\!R^m, (u^-)_i^l \in I\!R^{n-m}.$$

It is therefore sufficient to consider the problem on the positive semi-axis $0 \leq x$:

$$\begin{aligned} u_t + \Lambda u_x + Cu = F(x,t), 0 \leq x, 0 \leq t, \\ u(x,0) = f(x), 0 \leq x, \end{aligned} \tag{2.2}$$

as an additional simplification, we assume that the system already exists in a characteristic form, i.e.,

$$\Lambda = \begin{pmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{pmatrix},$$

with $\Lambda^+ = diag(\lambda_1, ..., \lambda_m)$, $\Lambda^- = diag(\lambda_{m+1}, ..., \lambda_n)$,
and $\lambda_1 > \lambda_2 > ... > 0 > \lambda_{m+1} > ... > \lambda_n$.

2.2 Numerical absorbing boundary conditions

In this section we introduce a finite difference scheme to solve the IBVP (2.1a) [**17**] Lax-Wendorf (LW-scheme)based on the Taylor expansion [**14**]

$$u(x, t+k) = u(x,t) + ku_t(x,t) + \frac{k^2}{2}u_{tt}(x,t) + O(k^3), \tag{2.3}$$

where u_{tt} can be determined as follows

$$\begin{aligned} u_{tt} &= (-\Lambda u_x - Cu + F)_t \\ &= -\Lambda u_{tx} - Cu_t + F_t \\ &= \Lambda(\Lambda u_x + Cu - F)_x + C(\Lambda u_x + Cu - F) + F_t \\ &= \Lambda^2 u_{xx} + (\Lambda C + C\Lambda)u_x + C^2 u + F_t - \Lambda F_x - CF. \end{aligned}$$

Substituting into equation (2.3) we have

$$\begin{aligned} u(x,t+k) = u(x,t) - k(-\Lambda u_x(x,t) - Cu(x,t) + F(x,t)) \\ + \frac{1}{2}k^2(\Lambda^2 u_{xx}(x,t) + (\Lambda C + C\Lambda)u_x(x,t) + C^2 u(x,t) \\ + F_t(x,t) - \Lambda F_x(x,t) - CF(x,t)) + O(k^3), \end{aligned} \tag{2.4}$$

in order to approximate the spatial derivatives of u, the LW-scheme uses central difference quotients. And the derivatives of F will be appropriately discertized. After

the terms have been sorted, we get:

$$\begin{aligned} u_i^{l+1} &= (\frac{1}{2}r\Lambda + \frac{1}{2}(r\Lambda)^2 - \frac{1}{4}rk(\Lambda C + C\Lambda))u_{i-1}^l \\ &+ (I_{n\times n} - r\Lambda - (r\Lambda)^2 + \frac{1}{2}(kC)^2)u_i^l \\ &+ (-\frac{1}{2}r\Lambda + \frac{1}{2}(r\Lambda)^2\frac{1}{4}rk(\Lambda C + C\Lambda))u_{i+1}^l \\ &+ \frac{1}{2}k(I_{n\times n} - kC)F_i^l + \frac{1}{2}kF_i^{l+1} - \frac{1}{4}rk\Lambda(F_{i+1}^l - F_{i-1}^l), \;\; l = 0, 1, ..., \; i = 1, ..., I-1, \end{aligned} \tag{2.5}$$

to solve (2.5)uniquely, we provide initial value

$$u_i^0 = f(x_i), i = 0, 1, ..., I, \tag{2.6}$$

and at each time level $t_l = lk, \; l = 1, 2, ...,$ boundary values $u_0^{l+1}, \;\; u_I^{l+1}$, and these values split into two groups:
the first one, which we refer to as inflow boundary condition is

$$(u^+)_0^{l+1}, \; (u^-)_I^{l+1}.$$

The second group is

$$(u^-)_0^{l+1}, \; (u^+)_I^{l+1},$$

which we refer to as outflow boundary conditions.
The inflow boundary conditions are determined in the previous chapter by the discretization of the boundary conditions (2.1c), (2.1d), while the outflow boundary conditions are obtained by introducing numerical boundary conditions. In our work we will consider two types of numerical boundary conditions, the first one is upwinding in which u at boundaries satisfy the homogeneous version of the system (2.1a), and the second type is first order extrapolation.

Remark 2.1. [10] *The general horizontal extrapolation of order q for the outflow data* u^- *at* $x = 0$ *is*

$$(E_+ - I)^{q+1}(u^-)_0^{l+1} = 0, \;\; q = 0, 1, ...,$$

and that of u^+ *at* $x = l$ *is is*

$$(I - E_+^{-1})^{q+1}(u^+)_I^{l+1} = 0, \;\; q = 0, 1, ...,$$

where $E_+ u_i := u_{i+1}$.
Using boundary condition (2.1c), *we write*

$$D_+^t(u^+)_0^l + ((Su)^+)_0^l = 0,$$

which gives

$$(u^+)_0^{l+1} = (u^+)_0^l - k((Su)^+)_0^l, \tag{2.7}$$

since F is compactly supported in $(0, l)$, the outflow part of (2.1a) *at $x = 0$ satisfies*

$$u_t^- + \Lambda^- u_x^- + (Cu)^- = 0,$$

which discretized as

$$D_+^t(u^-)_0^l + \Lambda^- D_+^x(u^-)_0^l + ((Cu)^-)_0^l = 0.$$

Hence

$$(u^-)_0^{l+1} = (I + r\Lambda^-)(u^-)_0^l - k((Cu)^-)_0^l - r\Lambda^-(u^-)_1^l. \tag{2.8}$$

An alternative numerical boundary condition is the first order extrapolation

$$(u^-)_0^{l+1} = 2(u^-)_1^{l+1} - (u^-)_2^{l+1}. \tag{2.9}$$

The discretization of the right boundary condition is treated in a similar way.

2.3 Stability of the finite difference scheme

In solving linear hyperbolic partial differential equations numerically by means of finite difference approximations, a principal difficulty both theoretically and in practice is the question of Stability. For 'Cauchy problem' on the unbounded domain , a fairly complete Stability theory based on the Fourier analysis has been worked out during the last few decades by von Neumann, Lax, Kreiss, and others [**22, 24, 25**]. However, Fourier analysis cannot be applied straight forward way, and progress has been slower and technically more complex. Important contributions in this area were made by *S.* Osher [**21**] and by H.-O. Kreiss [**19**], and are based on various kinds of normal mode analysis that extend the Fourier methods. A comprehensive theory of this type was presented in an influential paper by Gustafsson, Kreiss, and Sundstrom (briefly: GKS)[**14**]. The complicated algebraic conditions of the GKS-theory were simplified in following work of Goldberg and Tadmor[**15**]. In this section we apply the GKS-theory to show the stability of the difference approximation (2.5) – (2.7) – (2.8) and the corresponding right bonndary discretization. We intend to provide both sufficient and necessary conditions for the stability of this discrete IBVP. It appears that the IBVP does not have the standard form presented in the GKS-theory and

thus, this stability theory is not directly applicable [**32**]. The discrete IBVP under consideration is given with two boundaries. According to the theorem below, which is valid for any of the GKS stability definitions, it is sufficient to consider the problem on the positive plane $x \geq 0$, i.e., on the index range $n \geq 0$.

Theorem 2.2. [**14**] *Consider the difference approximation for* $t \geq 0$ *and* $0 \leq x \leq 1$ *and assume that the corresponding left and right quarter-plane problems (which we get by removing one boundary to infinity) are stable, then the original problem is also stable.*

The idea is that the case of basic difference scheme (2.5) and each of boundary conditions separated into the two quarter plane problems that are relatively nice to handle. Therefore, we will consider only the stability of the right quarter plane problem, while the left quarter one is analogue.
We will write (2.5) as

$$\begin{aligned} u_i^{l+1} &= Qu_i^l + kb_i^l, \\ u_i^0 &= f(x_i),\ i = 0, 1, 2, 3, ..., \end{aligned} \tag{2.10a}$$

where

$$Q = \sum_{\sigma=-1}^{1} \Lambda_\sigma E_+^\sigma, \quad E_+ u_i = u_{i+1},$$

$$\begin{aligned} \Lambda_0 &= I - kC - (r\Lambda)^2 + \frac{1}{2}(kC)^2, \\ \Lambda_{\pm 1} &= \pm\frac{1}{2}r\Lambda + \frac{1}{2}(r\Lambda)^2 \pm \frac{1}{4}rk(\Lambda C + C\Lambda), \\ b_i^l &= \frac{1}{2}(F_i^{l+1} + F_i^l) - \frac{1}{4}r\Lambda(F_{i+1}^l - F_{i-1}^l) - \frac{1}{2}kCF_i^l. \end{aligned}$$

The boundary values are written as

$$u_0^{l+1} = B_{0,0}u_0^l + B_{1,0}u_1^l + B_{1,1}u_1^{l+1} + B_{2,1}u_2^{l+1}, \tag{2.11}$$

where the above matrices are determined by the numerical boundary conditions under consideration. For the upwinding case for equations (2.7) and (2.8), we have

$$B_{0,0} = \begin{pmatrix} I^+ & 0 \\ 0 & I + r\Lambda^- \end{pmatrix} - k \begin{pmatrix} S^{++} & S^{+-} \\ C^{-+} & C^{--} \end{pmatrix},$$

$$B_{1,0} = \begin{pmatrix} 0 & 0 \\ 0 & -r\Lambda^- \end{pmatrix}, \quad B_{1,1} = B_{2,1} = 0. \tag{2.12a}$$

However, if extrapolation (2.7) – (2.9) is used, then

$$B_{0,0} = \begin{pmatrix} I^+ & 0 \\ 0 & 0 \end{pmatrix} - k \begin{pmatrix} S^{++} & S^{+-} \\ 0 & 0 \end{pmatrix},$$

$$B_{1,1} = \begin{pmatrix} 0 & 0 \\ 0 & -2I^- \end{pmatrix}, \quad B_{2,1} = \begin{pmatrix} 0 & 0 \\ 0 & -I^- \end{pmatrix}, \quad B_{1,0} = 0. \tag{2.12b}$$

GKS [**14**] discussed some possible definitions stability of finite difference schemes of which we choose the space to be considered with an inner product and a standard that allows us to make use of the available results.
Let $l^2(x)$ denote the space of all grid functions $u = (u_i)_{i=0}^{\infty}$, $u_i = u(x_i)$, $x_i = ih$, $i = 0, 1, ...$ with $h\sum_{i=0}^{\infty} |u_i|^2 < \infty$, and define the scalar product and norm by

$$(u, v)_h = \sum_{i=0}^{\infty} h u_i^* v_i, \|u\|_h^2 = (u, u)_h.$$

We define $l^2(t)$ and $l^2(x, t)$ in the corresponding way and denote by

$$(u, v)_k = k\sum_{l=0}^{\infty} u^*(t_l)v(t_l), \quad \|u\|_k^2 = (u, u)_k, \quad t_l = lk,$$

$$(u, v)_{h,k} = hk\sum_{i=0}^{\infty}\sum_{l=0}^{\infty} u_i^*(t_l)v_i(t_l), \quad \|u\|_{h,k}^2 = (u, u)_{h,k},$$

the corresponding norms and scalar products.

Remark 2.3. [**14**] *Assume that the initial function is zero. The difference scheme (2.10)-(*2.12a*)or(*2.12b*) is stable, if there exist constants $c_0 > 0, \alpha_0 \geq 0$ such that, for all $t = t_l = lk$, all $\alpha > \alpha_0$, and all h, an estimate*

$$(\frac{\alpha - \alpha_0}{\alpha k + 1})\|e^{-\alpha t}u_0\|_k^2 + (\frac{\alpha - \alpha_0}{\alpha k + 1})^2\|e^{-\alpha t}u\|_{h,k}^2 \leq c_0\|e^{-\alpha(t+k)}b\|_{h,k}^2$$

holds.

While here the vector b_i^l of the basic scheme (2.10a) is a general combination of F and its derivatives, in [**14**] we have $b_i^l = F_i^l$. However, Goldberg et al. [**15**] showed that this generalization does not affect the results of [**14**] and they raised

the question of stability in the sense of Definition (2.3).
The definition of stability for the difference scheme for the left quarter plane problem is the same, except that the norm is taken over the grid on $(-\infty, 1]$ and $u_0 = f$ is replaced by u_I.
In the following, we shall reduce the above stability question to that of the principal part of the scalar outflow approximations, i.e., the part obtained by eliminating the terms of order k, k^2, and all inhomogeneity vectors. This result is based on Theorem (4.3) of [**14**], which provides a necessary and sufficient determinantal stability criterion given entirely in terms of the principal part of the approximations. The mere existence of such a criterion implies that for the stability purposes we may consider a basic scheme of (2.10)-(2.12a)or(2.12b) of the form

$$u_i^{l+1} = \tilde{Q} u_i^l, \quad \tilde{Q} = \sum_{\sigma=-1}^{1} \tilde{\Lambda_\sigma} E_+^\sigma, \quad E_+ u_i = u_{i+1}, \tag{2.13}$$

here

$$\begin{aligned} \tilde{\Lambda_0} &= I_{n\times n} - (r\Lambda)^2, \\ \tilde{\Lambda_{\pm 1}} &= \pm\frac{1}{2} r\Lambda + \frac{1}{2}(r\Lambda)^2, \end{aligned}$$

and the boundary conditions

$$(u^+)_0^{l+1} = (u^+)_0^l, \tag{2.14}$$

$$(u^-)_0^{l+1} = (I_{n\times n} + r\Lambda^-)(u^-)_0^l - r\Lambda^-(u^-)_1^l, \tag{2.15}$$

or

$$(u^-)_0^{l+1} = 2(u^-)_1^{l+1} - (u^-)_2^{l+1}. \tag{2.16}$$

The scheme (2.13) is now consistent with

$$u_t + \Lambda u_x = 0.$$

We split the basic scheme and the boundary values into inflow and outflow parts respectively

$$\begin{aligned} \left(u^-\right)_i^{l+1} &= \left(u^-\right)_i^l - \frac{r\Lambda^-}{2}\left(\left(u^-\right)_{i+1}^l - \left(u^-\right)_{i-1}^l\right) \\ &+ \frac{\left(r\Lambda^-\right)^2}{2}\left(\left(u^-\right)_{i+1}^l - 2\left(u^-\right)_i^l + \left(u^-\right)_{i-1}^l\right), \end{aligned} \tag{2.17}$$

$$\left(u^-\right)_0^{l+1} = \left(I + r\Lambda^-\right)\left(u^-\right)_0^l - r\Lambda^-\left(u^-\right)_1^l, \tag{2.18a}$$

$$\left(u^-\right)_0^{l+1} = 2\left(u^-\right)_1^{l+1} - \left(u^-\right)_2^{l+1}, \tag{2.18b}$$

and

$$\begin{aligned}\left(u^+\right)_i^{l+1} &= \left(u^+\right)_i^l - \frac{r\Lambda^+}{2}\left(\left(u^+\right)_{i+1}^l - \left(u^+\right)_{i-1}^l\right) \\ &+ \frac{\left(r\Lambda^+\right)^2}{2}\left(\left(u^+\right)_{i+1}^l - 2\left(u^+\right)_i^l + \left(u^+\right)_{i-1}^l\right),\end{aligned} \tag{2.19}$$

$$\left(u^+\right)_0^{l+1} = \left(u^+\right)_0^l. \tag{2.20}$$

Obviously, (2.10)-(2.12a)or(2.12b) is stable if and only if both parts are stable. Before we proceed, we include the following assumptions that are necessary for the result contained in this section.

Assumption 2.4. *1. The associated initial value scheme is stable.*

2. The difference scheme is either dissipative or nondissipative.

A necessary condition for the stability of the initial value scheme is to satisfy the CFL(Courant-Friedrichs-Levy)-condition. CFL-condition simply asserts that the analytical domain of dependence is contained in the numerical domain of dependence. For the LW-scheme, this gives

$$\max_{\nu=1,\ldots,n} |\lambda_\nu r| \leq 1, \tag{2.21}$$

Remark 2.5. **[25]** *The difference scheme (2.13) is dissipative of order* $2s$ *if there exists* $c > 0$ *such that the eigenvalues* $\mu_\nu(\xi)$ *of the amplification matrix of* $\tilde{Q}$ *satisfies the following estimate*

$$|\mu_\nu(\theta)|^2 \leq 1 - c\,|\theta|^{2s}, \qquad |\theta| \leq \pi.$$

This condition is equivalent to (**[24]**)

$$|\mu_\nu(\theta)|^2 \leq 1 - \acute{c}\sin^{2s}(\theta/2), \quad \acute{c} > 0.$$

The amplification matrix of $\tilde{Q}$ reads

$$I - ir\Lambda\sin\theta - (r\Lambda)^2(1-\cos\theta),$$

with eigenvalues

$$\mu_\nu(\theta) = 1 - ir\lambda_\nu \sin\theta - 2r^2\lambda_\nu^2 \sin^2(\theta/2), \qquad \nu = 1, ..., n.$$

This gives

$$\begin{aligned} |\mu_\nu(\theta)|^2 &= [1 - 2(r\lambda_\nu)^2 \sin^2(\theta/2)]^2 + (r\lambda_\nu)^2 \sin^2\theta \\ &= 1 - 2(r\lambda_\nu)^2[4\sin^2(\theta/2) + 4(r\lambda_\nu)^2 \sin^4(\theta/2) + \sin^2\theta] \\ &= 1 - 4(r\lambda_\nu)^2[1 - (r\lambda_\nu)^2] \sin^4(\theta/2), \qquad \nu = 1, ..., n. \end{aligned} \tag{2.22}$$

Thus, the difference scheme (2.13) is dissipative of order 4 if r is chosen to satisfy

$$0 < |\lambda_\nu r| \leq 1, \qquad \nu = 1, ..., n.$$

Since Λ is regular, Assumption (2.4) is fulfilled if the CFL-condition (2.21) is satisfied.

We split the outflow approximation (2.17)-(2.18a)(or(2.18b) into $n - m$ scalar components, each of the form

$$\begin{aligned} v_i^{l+1} &= v_i^l - \frac{\kappa}{2}(v_{i+1}^l - v_{i-1}^l) + \frac{\kappa^2}{2}(v_{i+1}^l - 2v_i^l + v_{i-1}^l) \\ &= \frac{1}{2}(\kappa^2 + \kappa)v_{i-1}^l + (1 - \kappa^2)v_i^l + \frac{1}{2}(\kappa^2 - \kappa)v_{i+1}^l, \end{aligned} \tag{2.23}$$

where $\kappa := r\lambda_\nu$, for fixed $\lambda_\nu \in \Lambda^-$, and

$$v_0^{l+1} = v_0^l - \kappa(v_1^l - v_0^l), \tag{2.24a}$$

or

$$v_0^{l+1} = 2v_1^{l+1} - 2v_2^{l+1}. \tag{2.24b}$$

The scheme (2.10)-(2.12a)or(2.12b) is stable if and only if (2.17)-(2.18a)(or(2.18b)) and (2.19)-(2.20) are stable, and the latter are stable if and only if their scalar components are. Lemma (2.3) of [**15**] shows that the scalar components of the inflow approximation (2.19)-(2.20) are stable (for $0 < \kappa \leq 1$). So we conclude the main result of this section.

Lemma 2.6. *The Approximation (2.10)-(*2.12a*)or(*2.12b *is stable if and only if the scalar outflow components (2.23)-(*2.24a*)(or(*2.24b*)) are stable.*

To discuss the stability of (2.23)-(2.24a)(or(2.24b)) we use the discrete Laplace transform, which is one of the few approaches available for analyzing the stability of difference schemes for initial boundary value problems. This approach is used to transform out the temporal differences (time derivatives) and consider the scheme in transform space as a difference scheme in i.

Chapter 3

Test Examples and Numerical Results

In the following numerical tests we will compare the ABCs and the FBCs. In addition a numerical approximation using FBCs have been tested and compered with the exact solution over the domain.

3.1 Example one

Consider the linear hyperbolic system

$$u_t + \Lambda u_x + Cu = F(x,t), \quad x \in I\!R, \tag{3.1a}$$

$$x_1(x,0) = x_1^0(x), \qquad x_2(x,0) = x_2^0(x), \tag{3.1b}$$

where x_1^0, x_2^0, f and g have compact support in $(0,1)$,

$$u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \ C = \begin{pmatrix} 2 & 2 \\ \frac{3}{2} & 2 \end{pmatrix} \text{ and } F = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}.$$

Now, this gives

$$\Lambda^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \ \tilde{C} = \Lambda^{-1}C = \begin{pmatrix} 1 & 1 \\ -\frac{3}{4} & -1 \end{pmatrix}, \text{ and } \Lambda^{-1}F = \begin{pmatrix} \frac{f(x)}{2} \\ -\frac{g(x)}{2} \end{pmatrix}.$$

The corresponding steady system on $I\!R$, with the decay condition

$$x_1, x_2 \to 0, \quad x \to \pm\infty$$

is given by

$$u_x + \Lambda^{-1}Cu = \Lambda^{-1}F,$$

then, we get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_x + \begin{pmatrix} 1 & 1 \\ -\frac{3}{4} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{f(x)}{2} \\ -\frac{g(x)}{2} \end{pmatrix}, \quad x \in I\!R. \tag{3.2}$$

The zero order ABCs for the restriction to the interval $0 \leq x \leq 1$ are

$$x_1 = 0, \qquad x = 0, \tag{3.3a}$$

$$x_2 = 0, \qquad x = 1. \tag{3.3b}$$

To drive the first order ABCs, following the deriviation in section (1.5), we evaluate the eigenvalues of $\tilde{C}$

$$\begin{vmatrix} 1-\lambda & 1 \\ -\frac{3}{4} & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) + \frac{3}{4} = 0 \Rightarrow \lambda = \pm\frac{1}{2},$$

and

$$X_1 = \begin{pmatrix} 0 & \frac{\lambda_2 c_{12}}{\lambda_2 - \lambda_1} \\ \frac{\lambda_1 c_{21}}{\lambda_1 - \lambda_2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{3}{4} & 0 \end{pmatrix}.$$

Now,

$$[(\frac{\partial}{\partial t} + X_1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}]^+ = 0, \quad x = 0, \tag{3.4a}$$

$$[(\frac{\partial}{\partial t} + X_1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}]^- = 0, \quad x = 1. \tag{3.4b}$$

Thus, we have the first order ABCs

$$(x_1)_t + x_2 = 0, \qquad x = 0, \tag{3.5a}$$

$$(x_2)_t + \frac{3x_1}{4} = 0, \qquad x = 1. \tag{3.5b}$$

Now, we turn to derive FBCs. Let

$$S = \begin{pmatrix} \alpha & \frac{2\alpha}{3} \\ \beta & 2\beta \end{pmatrix},$$

α, β are parameters to be determined later, such that

$$S\tilde{C}S^{-1} = \mathrm{diag}(\lambda_1, \lambda_2) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix}.$$

The zero order FBCs for the steady problem on the $(0, 1)$ are

$$(Su)^+ = 0, \quad x = 0,$$

$$(Su)^- = 0, \quad x = 1.$$

Then, we have

$$x_1 + \frac{2}{3}x_2 = 0, \quad x = 0, \tag{3.6a}$$

$$x_1 + 2x_2 = 0, \quad x = 1, \tag{3.6b}$$

and the first order FBCs are

$$[(\frac{\partial}{\partial t} + S)u]^+ = 0, \quad x = 0,$$

$$[(\frac{\partial}{\partial t} + S)u]^- = 0, \quad x = 1.$$

Hence, we have

$$(x_1)_t + \alpha(x_1 + \frac{2}{3}x_2) = 0, \quad x = 0, \tag{3.7a}$$

$$(x_2)_t + \beta(x_1 + 2x_2) = 0, \quad x = 1. \tag{3.7b}$$

Consider the system (3.1) with $f(x) = x_1^0(x) = x_2^0(x) = 0$, and

$$g(x) = \begin{cases} cos^2(\pi(x - 0.5)/0.8, & x \in (0.1, 0.9), \\ 0, & \text{otherwise.} \end{cases}$$

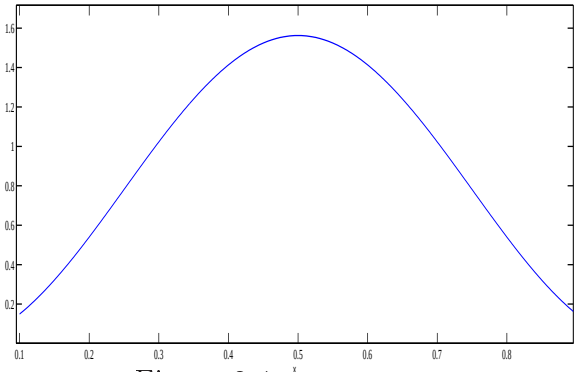

Figure 3.1: graph of $g(x)$.

With both boundary conditions (3.5) and (3.7). In this example we will test the convergence as $t \to \infty$ of the solution of the resulting Initial boundary value problem to the solution of the steady unbounded problem. The step sizes $h = 0.0005$ and $k = 0.00045$ are chosen so that $r = k/h = 0.9$ satisfies the CFL-condition [**14**]. Figure (3.2) shows the steady state solution of (3.2) in $(0, 1)$. The convergence will

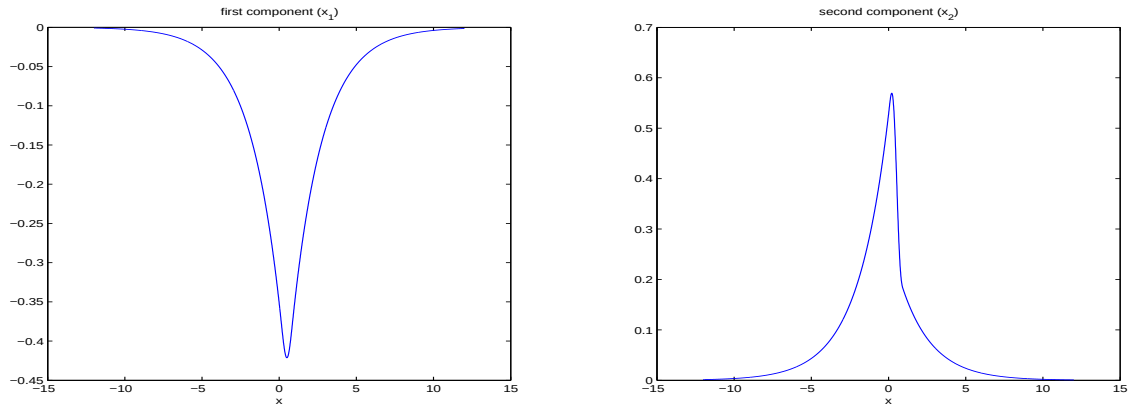

Figure 3.2: steady state solution

be presented in the following figures. In figures (3.4) the solution of the system (3.2) restricted to the domain $(0, 1)$ with FBCs, converges as $t \to \infty$ to the corresponding steady solution. However, this is not the case for the ABCs as figure (3.3) shows.

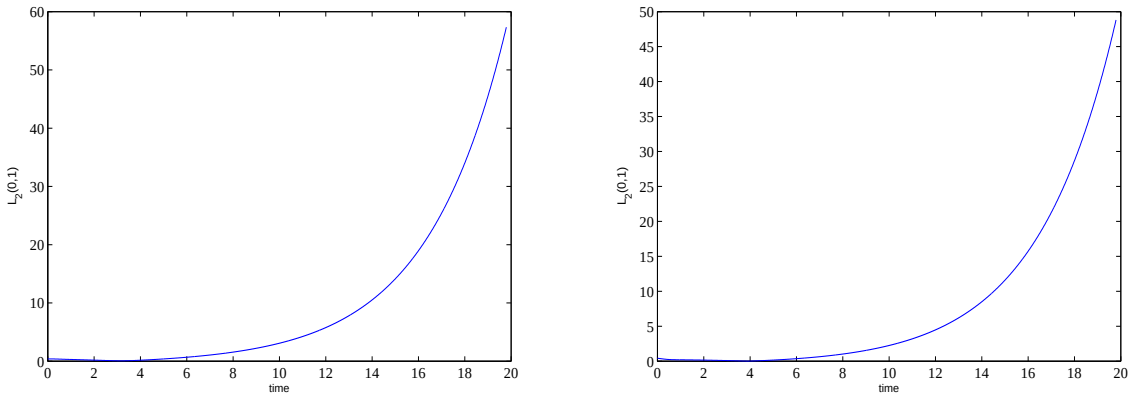

Figure 3.3: $L^2(0, 1)$ error between the steady state of x_1, x_2 and solution with boundary condition (3.5)

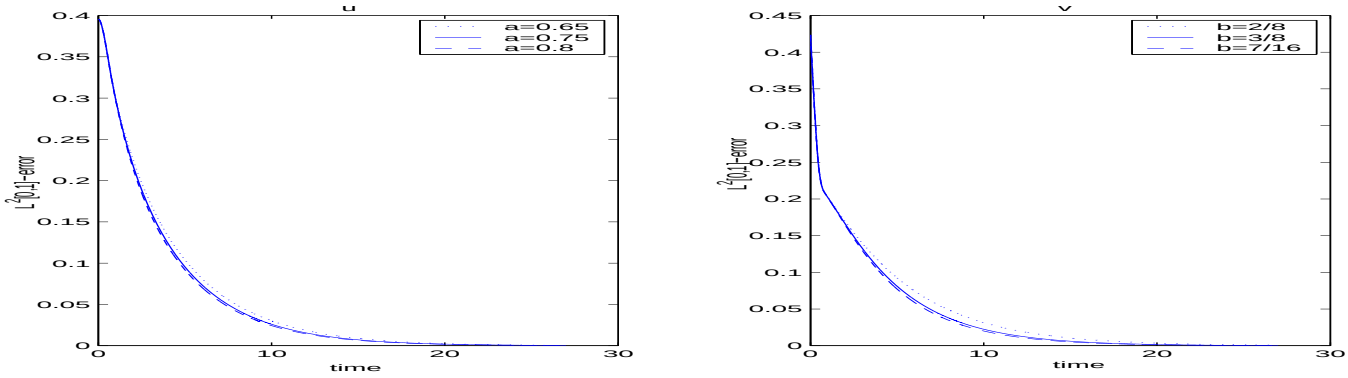

Figure 3.4: $L^2(0,1)$ error between the steady state of x_1, x_2 and solution with boundary condition (3.7)

This result gives the evidence that the far field boundary conditions are useful for long time. Moreover, for short time comparison, we consider the system (3.2) with a highly oscillating (wavering) initial condition

$$g(x) = \begin{cases} cos^2(2\pi(x-0.6)sin(2\pi px), & x \in (0.35, 0.85), \\ 0, & otherwise, \end{cases}$$

and $f(x) = g(x) = 0$. The cases of $p = 10, 20$ are shown in figure (3.5). Two types of artificial boundary conditions on $(0,1)$, ABCs and FBCs, will be used and the resulting solutions will be compared with the exact solution. The $L^2(0,1)$ error function, defined as $L^2 = \sum_{i=1}^{n}(y_{true} - y_{predicted})^2$, will be used to compare the different solutions.

α and β are arbitrary parameters, we can determined an optimal choice of α and β as [**6**] and this optimal choices are

$$\alpha = \frac{\lambda_2 c_{12}}{\lambda_2 - `\lambda_1} \frac{1}{s_{12}} = \frac{3}{2}, \qquad \beta = \frac{\lambda_1 c_{21}}{\lambda_1 - \lambda_2} \frac{1}{s_{21}} = \frac{3}{4},$$

and the FBCs become

$$(x_1)_t + \frac{3}{2}(x_1 + \frac{2}{3}x_2) = 0, \quad x = 0, \tag{3.8a}$$

$$(x_2)_t + \frac{3}{4}(x_1 + 2x_2) = 0, \quad x = 1. \tag{3.8b}$$

We showed in this test that this choices of α and β among other arbitrary options improve the approximation for short time, and accelerate the convergence to steady state solution for the long time. We test the approximation of the exact solution

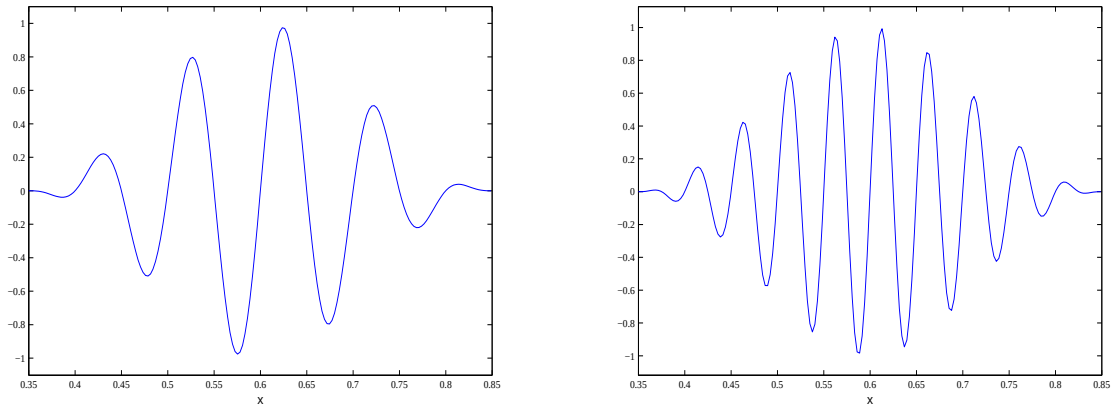

Figure 3.5: initial values for P=10 in the right and for P=20 in the left.

and solution with boundary condition (3.7) with different choices of α and β. The error function $L^2(0,1)$ is used for the approximation of inflow data x_1 at $x = 0$ and x_2 at $x = 1$, $h = 0.0005$ is chosen small enough in order to estimate the errors due to the boundary conditions and not the discretization errors.

3.2 Example two

Consider the linear hyperbolic system

$$u_t + \Lambda u_x + Cu = F(x,t), \quad x \in I\!R, \tag{3.9}$$

with

$$u = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & -1.6 \end{pmatrix}, \quad C = \begin{pmatrix} 0.1 & 0 & 0.5 \\ 0 & 0.2 & 1 \\ -0.5 & -1 & 0.5 \end{pmatrix}, \text{ and } F(x) = \begin{pmatrix} F_1(x) \\ F_1(x) \\ F_1(x) \end{pmatrix},$$

where

$$F_1(x) = \begin{cases} 10exp(-100(5x-1)^2), & x \in (0.1, 0.3) \\ 0, & \text{otherwise,} \end{cases}$$

and initial function

$$y^0(x) = \begin{cases} cos(\pi(x-0.5)/0.8 & , x \in (0.1, 0.9), \\ 0, & \text{otherwise.} \end{cases}$$

The ABCs and FBCs are derived as follows

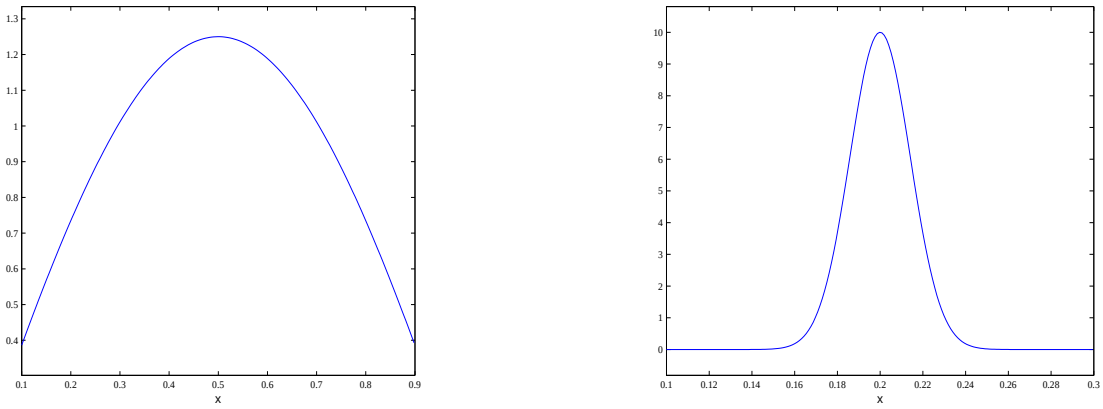

Figure 3.6: initial function and forcing function

$$\Lambda^{-1} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & -0.625 \end{pmatrix},$$

and

$$\tilde{C} = \Lambda^{-1}C = \begin{pmatrix} 0.05 & 0 & 0.25 \\ 0 & 0.5 & 2.5 \\ 0.3 & 0.625 & -0.3125 \end{pmatrix}.$$

The eigenvalues of $\tilde{C}$ are

$$\begin{vmatrix} 0.05-\lambda & 0 & .25 \\ 0 & 0.5-\lambda & 2.5 \\ 0.3 & 0.625 & -0.3125-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = -1.2599,\ \lambda_2 = 0.0695,\ \lambda_3 = 1.428,$$

and

$$X_1 = \begin{pmatrix} 0 & \frac{\lambda_2 C_{12}}{\lambda_2-\lambda_1} & \frac{\lambda_3 C_{13}}{\lambda_3-\lambda_1} \\ \frac{\lambda_1 C_{21}}{\lambda_1-\lambda_2} & 0 & \frac{\lambda_3 C_{23}}{\lambda_3-\lambda_2} \\ \frac{\lambda_1 C_{31}}{\lambda_1-\lambda_3} & \frac{\lambda_2 C_{32}}{\lambda_2-\lambda_3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0.266 \\ 0 & 0 & 1.051 \\ -0.234 & 0.051 & 0 \end{pmatrix},$$

then by equation (3.4) we have the zero order ABCS for the restriction to the interval $0 \le x \le 1$ are

$$y_1 = 0, y_2 = 0, \quad x = 0, \tag{3.10a}$$

$$y_3 = 0, \quad x = 1, \tag{3.10b}$$

where y_1, y_2 are inflow data at $x = 0$ while y_3 is the outflow data at $x = 1$, and the first order ABCs are

$$y_{1,t} + 0.266y_{2,t} = 0, \quad x = 0, \tag{3.11a}$$

$$y_{2,t} + 1.05y_{3,t} = 0, \quad x = 0, \tag{3.11b}$$

$$y_{3,t} - 0.234y_{1,t} + 0.05y_{2,t} = 0, \quad x = 1. \tag{3.11c}$$

The steady state of y_1, y_2, y_3 are shown in the following figures (3.7),(3.8)and (3.9) respectively. The convergence as $t \to \infty$ of the solution of the resulting IBVP to

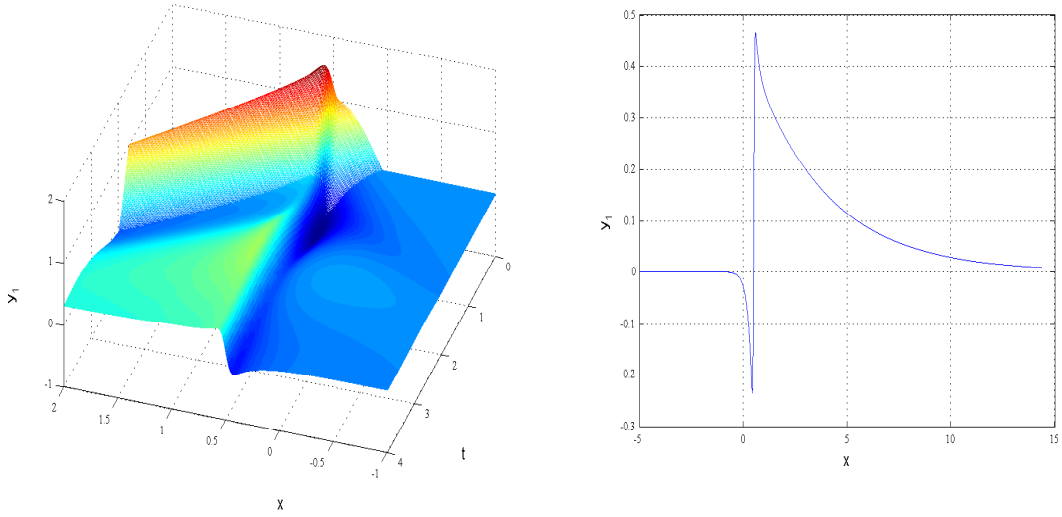

Figure 3.7: exact solution of $y_1(x)$ and the steady state solution

the solution of the steady unbounded problem has been tested with $h = 0.0005, k = 0.0004$ and $r = k/h = 0.8$.
The matrix S which diagonalize $\Lambda^{-1}C$,

$$S\Lambda^{-1}CS^{-1} = diag(\lambda_1, \lambda_2, \lambda_3),$$

is give by

$$S = \begin{pmatrix} -0.1093 & 0.9151 & -0.0630 \\ -0.8130 & -0.3974 & -0.9357 \\ 0.5719 & 0.0686 & -0.3470 \end{pmatrix}.$$

The first order FBCs read

$$[(\frac{\partial}{\partial t} + VS)u]^+ = 0, \quad x = 0,$$

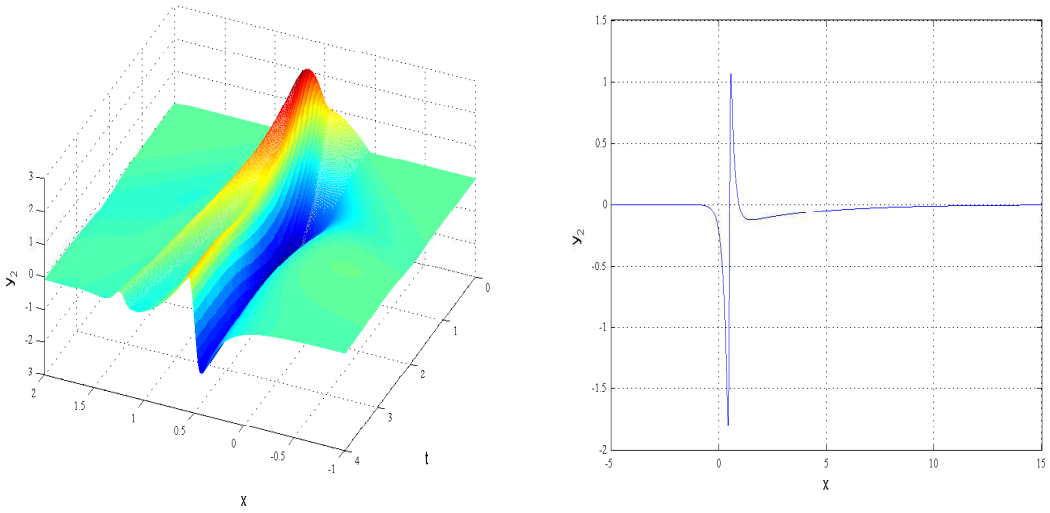

Figure 3.8: exact solution of $y_2(x)$ and the steady state solution

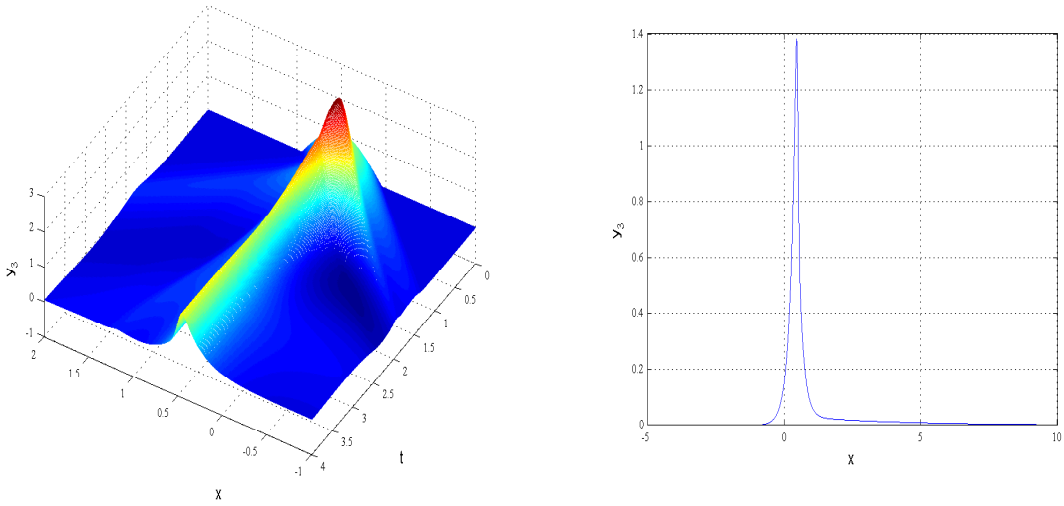

Figure 3.9: exact solution of $y_3(x)$ and the steady state solution

$$[(\frac{\partial}{\partial t} + VS)u]^- = 0, \quad x = 1.$$

Where $V = \begin{pmatrix} V^+ & 0 \\ 0 & V^- \end{pmatrix}$. Then, we get

$$u_t^+ + V^+(S^{++}u^+ + S^{+-}u^-) = 0, \qquad x = 0, \tag{3.12a}$$

$$u_t^- + V^-(S^{-+}u^+ + S^{--}u^-) = 0, \qquad x = 1. \tag{3.12b}$$

The scaling matrices

$$V^+ = \begin{pmatrix} \alpha & \beta \\ \gamma & \mu \end{pmatrix}, \quad V^- = \delta,$$

are chosen for the first order FBCs as follows

- a general solution of $V^+S^{+-} = X_1^{+-}$ for FBC at $x = 0$.
- $V^-S^{-+} = X_1^{-+}$, for FBC at $x = 1$.

If $(S^{+-})^{-1}$ exist, then $V^+ = X_1^{+-}(S^{+-})^{-1}$ and similarly $V^- = X_1^{-+}(S^{-+})^{-1}$.
The partition of S is $S^{++} = \begin{pmatrix} -0.1093 \\ -0.8130 \end{pmatrix}, S^{+-} = \begin{pmatrix} 0.9151 & -0.0630 \\ -0.3974 & -0.9357 \end{pmatrix}$,
$S^{-+} = \begin{pmatrix} 0.5719 \end{pmatrix}, S^{--} = \begin{pmatrix} 0.0686 & -0.3470 \end{pmatrix}$, and partition of X_1 is

$$X_1^{++} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, X_1^{+-} = \begin{pmatrix} 0 & 0.266 \\ 0 & 1.051 \end{pmatrix}, X_1^{-+} = \begin{pmatrix} -0.234 \end{pmatrix}, X_1^{--} = \begin{pmatrix} 0.051 & 0 \end{pmatrix}.$$

Then we have

$$V^{*+} = \begin{pmatrix} -0.1199 & -0.2762 \\ -0.4739 & -1.0913 \end{pmatrix}, \qquad V^{*-} = -0.4092.$$

The CFL-condition, $\max_{j=1,2,3} |r\lambda_j| < 1$, is satisfied, and we chose the stepsizes $h = 0.0005$, $k = 0.0004$ small enough in order to see the errors due to different boundary conditions and not the discretization errors. Tables 3.1-3.3 list the maximal absolute errors at the inflow data (y_1, y_2 at $x = 0$ and y_3 at $x = 1$). As well as the $L^2(0,1)-$error between exact solution and the solution with the boundary condition (3.7a) for different values of $\alpha, \beta, \gamma, \mu$, and δ are presented in tables (3.1),(3.2) and (3.3).

α,	β	abs. error at $x = 0$	$L^2(0,1)-$error
-0.2,	0	0.0703	0.0437
-0.8,	0.2	0.0718	0.0405
α^*,	β^*	0.0218	0.018

Table 3.1: Maximum errors due to the first order FBCs of y_1 for different choices of α and β,

γ,	μ	abs. error at $x = 0$	$L^2(0,1)-$error
-1,	0	0.2207	0.0654
-2.5,	0.5	0.1252	0.0420
γ^*,	μ^*	0.1243	0.0388

Table 3.2: Maximum errors of y_2 due to the first order FBCs for different choices of γ and μ,

δ	abs. error at $x = 1$	$L^2(0,1)-$error
1	0.1078	0.0421
2.5	0.0940	0.0470
δ^*	0.0875	0.0345

Table 3.3: Maximum errors due to the first order FBCs of y_3 for different choices of δ,

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