Hebron University

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Mathematics Department

Numerical Solution of the Heat Equation Using Finite Element Method

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Dedication

I dedicate my thesis to my family and friends who supported me each step of the way.

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I am grateful to my supervisor Dr. Hasan Almanasreh, for his continuous support and encouragement, which enabled me to write and understand the subject.

I acknowledge the support of Hebron University for this work, and I appreciate the effort of all respected staff in the Department of Mathematics and thank them for it.

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I must express my deep gratitude and appreciation to my family, who have always encouraged and supported me both in my university studies or during the research and writing process of this thesis.

As well as for those who facilitated my mission in preparing the thesis.

<u>الإقرار</u>

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان :

الحل التقريبي لمعادلة الحرارة باستخدام طريقة العناصر المحدودة

Numerical Solution of the Heat Equation Using Finite Element Method

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص،
باستثناء ما تمت الإشارة إليه حيثما ورد، و أن هذه الرسالة ككل لم تقدم
من قبل لنيل أية درجه علمية أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية او بحثية أخرى.

<u>Declaration</u>

The work provided in this thesis, unless otherwise referenced, is the result of the researcher's work, and has not been submitted elsewhere for any other degree or qualification.

Abstract

This thesis treats the heat equation and its approximate solution using the finite element method. For this purpose, a general theory of the finite element method and its strategy are explained carefully. Then after, the heat equation and its derivation and applications are considered in details. The finite element method is then applied to approximate the solution of the heat equation in one and two dimensional spaces. In this essence, error analysis is studied for the heat equation in both categories, a posteriori and a priori error estimates.

Also, the thesis discusses the numerical solution for the heat equation throughout examples in one and two dimensions using the MATLAB software. Since the heat equation requires time discretization besides the spatial one, then iterative methods are needed for the time solution. In one dimension, we apply two methods for the time iteration, the backward Euler method and theta method. It is found that Theta method in the FEM is more stable than the Backward Euler method: increasing number of nodal points provides better convergence to the exact solution, i.e., less error. Been studying of θ, several values of θ are tested to reach the optimal value which makes the error as minimum as possible. After examining several choices of θ, we arrive at the conclusion that θ should live in the interval $[0.65, 0.8]$ to obtain better convergence.

الملخص

تتناول هذه الرسالة معادلة الحرارة وحلها التقريبي باستخدام طريقة العناصر المحدودة. لهذا الغرض، يتم شرح النظرية العامة لطريقة العناصر المحدودة واستراتيجيتها بعناية. ثم بعد ذلك، يتم النظر بالتفصيل في معادلة الحرارة واشتقاقها وتطبيقاتها. ثم يتم تطبيق طريقة العناصر المحدودة لتقريب حل معادلة الحرارة في مسافات أحاديةو ثنائية الأبعاد. في هذا الجوهر، يتم دراسة تحليل الخطأ لمعادلة الحرارة في كلتا الفئتين، تقديرات الخطأ اللاحقة والسابقة. تناقش الأطروحة أيضًا الحل العددي لمعادلة الحرارة من خلال أمثلة في بعد واحد أو بعدين باستخدام برنامج الماتلاب.

نظرًا لأن معادلة الحرارة تتطلب تحديدًا زمنيًا إلى جانب المعادلة المكانية، فإن هناك حاجة إلى طرق تكرارية لحل الوقت في أحد الأبعاد ، نطبق طريقتين لتكرار الوقت، طريقة أويلر العكسية وطريقة ثيتا، وجد أن طريقة ثيتا هي أكثر استقرارًا من طريقة أويلر العكسية: زيادة عدد العقد توفر النقاط تقارباً أفضل للحل الدقيق، أي خطأ أقل .

Contents

Chapter 1

Introduction

The finite element method (FEM) is a numerical technique for solving problems which are described by partial differential equations that arise in scientific and engineering applications. The FEM uses a variational form of the problem that involves an integral form of the differential equation over a given domain where this domain is divided into a number of subdomains called finite elements.

The heat equation is a second order partial differential equation that describes heat transfer by conduction and heat change to objects, as it automatically flows from places where it is higher towards the places where it is lower and is considered a special case of the diffusion equation, mathematically $v_t = \alpha v_{xx}$.

This thesis is concerned with the finite element method for the heat equation. In terms for the method used and examples time dependent or independent and discuss that in one and two dimensional in detail and clarify the mechanism of action in each case.

Also the a priori and a posteriori errors resulting from the use of the finite element method of the heat equation is studied. The advantage of this is to know the amount of error in the approximate solution in order to make it as minimal as possible and thus get the best approximation of the solution.

The essence of this work begins with the debate of the variational formulation and the discretization of the problem with homogenous and mixed boundary conditions. We construct a variational (weak) formulation by multiplying both sides of the differential equation by a test function $w(x)$ and integrating over the interval by parts. The aim of introducing the notation of weak formulation is to give access to the existence and uniqueness results for the solutions which is well suited for the numerical approximation of such problems.

In the discretization we construct suitable finite element dimensional spaces for the unknown and test functions. Moveover, we discuss the error estimate, which is the difference between the approximate solution v_h and the exact solution v, for both types of error a priori and a posteriori. The first type is error bounds given by known information on the solution of the variational problem and the finite element function space, where the second type is error bounds given by information on the numerical solution obtained on the finite element function space.

For a complete picture, we talk about the steady state of heat equation which does not depend on time, mathematically $v_t = 0$, this becomes elliptic equation. The simplest nontrivial examples of elliptic partial differential equation are the Laplace equation:

$$\Delta v = v_{xx} + v_{yy} = 0,$$

and the Poisson equation,

$$\Delta v = v_{xx} + v_{yy} = g(x, y).$$

We will discuss some examples using the MATLAB program through drawing the solution, taking several divisions of the same example, and showing the results of each of them.
This project consists of four chapters. Chapter One will be about the FEM in general. Chapter Two talks about heat equation. The error estimation for its both types, a posteriori and a priori, will be explained in Chapter Three. The computation of heat equation is discussed in Chapter Four.

Chapter 2

Finite Element Method (FEM)

In this chapter, we present the theory of the FEM. This method is an approximation technique for solving differential equations using piecewise polynomials.

2.1 Preliminaries

In this section we will review some of the concepts we need while explaining the method.

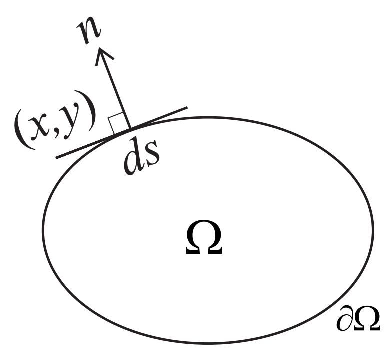


Figure 2.1: Ω and $\partial\Omega$

Theorem 2.1. *"Green's Formula" Let $v \in C^2(\Omega)$, then :*

$$\int_\Omega \triangle v w dx = \int_{d\Omega} (\nabla v.n) w ds - \int_\Omega \nabla v.\nabla w dx,$$

where

1. $v \in C^2(\Omega)$: *all partial derivatives of v of order 2 are continuous.*

2. $n(x,y)$ *is the outward unit normal at the boundary point* $(x,y) \in \partial\Omega$ *and* ds *is a curve element on* $\partial\Omega$.

Definition 2.1.

$$L^p\text{- } norm : \|g\|_{L_p} = \left(\int_\Omega |g(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

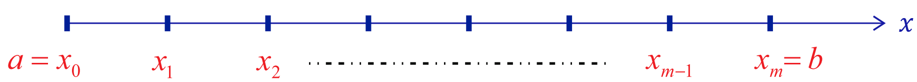
Definition 2.2. *Some of the finite element spaces :*

1. $L^P(\Omega) = \{w(x) : \|w\|_{L^P(\Omega)} < \infty\}$.

2. $H^1(\Omega) = \{w(x) : \|w\|_{L^P(\Omega)} + \|\nabla w\|_{L^P(\Omega)} < \infty\} = \{w(x) : \int_\Omega (w^2(x) + \mid \nabla w \mid^2) dx < \infty\}$.

3. $H^1_0(\Omega) = \{w(x) : w \in H^1(\Omega) \quad and \quad w = 0 \quad on \quad \partial\Omega\}$.

Definition 2.3. *Linear Basis Functions in* $1D$ *:*
Let W *be the space of continuous functions on* $[a,b]$, *and* W^L *be the subspace consists of linear functions on* $[a,b]$. *Define* W^L_h *to be the finite subspace of* W^L *on the partition* $a = x_0 < x_1 < ... < x_{m-1} < x_m = b$.



A basis for W^L_h *is* $\{\Psi_j\}_{j=0}^m$,

$$\Psi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h_j}, & x_{j-1} \le x < x_j, \\ \dfrac{x_{j+1} - x}{h_{j+1}}, & x_j \le x \le x_{j+1}, \\ 0, & O.W, \end{cases}$$

where :

- $h_j = x_j - x_{j-1}$.

- $h_{j+1} = x_{j+1} - x_j$.

Note that :

$$\Psi_j(x_i) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

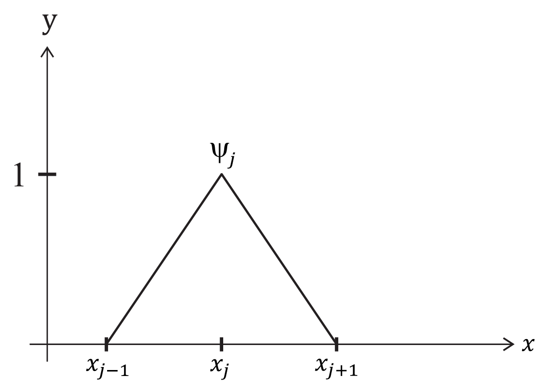


Figure 2.2: Ψ_j

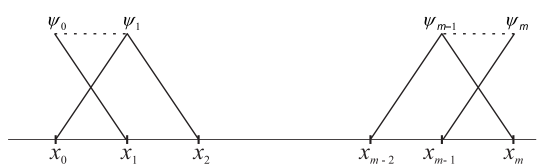


Figure 2.3: Example of the half basis

Remark 2.1. *There are two types of mesh :*

1. *Uniform mesh,* $h_j = h_{j+1} = h$, $\forall j$.

2. *Non-uniform mesh,* $h_j \neq h_{j+1} \neq h$, $\forall j$.

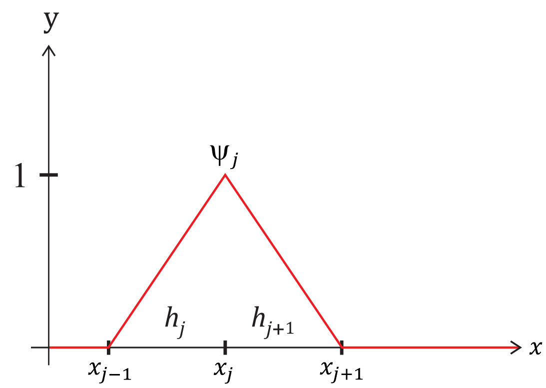


Figure 2.4: non-uniform mesh

Remark 2.2. *If $v \in W_h^L$, then $v(x) = \sum_{j=0}^{m} \xi_j \Psi_j$, where $\xi_j = v(x_j)$.*

$$\begin{aligned}\Longrightarrow v(x) &= \xi_0\Psi_0 + \xi_1\Psi_1 + ... + \xi_{m-1}\Psi_{m-1} + \xi_m\Psi_m \\ &= v(\xi_0)\Psi_0 + \xi_1\Psi_1 + ... + \xi_{m-1}\Psi_{m-1} + v(\xi_m)\Psi_m \\ &= \xi_1\Psi_1 + \xi_2\Psi_2 + ... + \xi_{m-1}\Psi_{m-1} \textit{(in case } v \textit{ is zero on the boundaries)} \\ &= \sum_{j=1}^{m-1} \xi_j\Psi_j.\end{aligned}$$

Definition 2.4. *Let $w \in C[a,b]$, then in the partition : $a = x_0 < x_1 < ... < x_m = b$ and $h = (x_{j+1} - x_j), j = 0, ..., m$, the continuous piecewise linear interpolant of w is defined by $\pi_h w(x) = \sum_{j=0}^{m} w(x_j)\Psi_j(x), x \in [a,b]$.*
Here the sub-index h refers to the mesh function $h(x)$, [7]. Note that $\pi_h w(x) = w(x_j), j = 0, ..., m$.

Theorem 2.2. *[7] Let $\pi_h w(x)$ be the piecewise linear interpolant of $w(x)$, then there is an interpolation constant c_i such that :*

- $\|\pi_h w - w\|_{L_p} \leq c_i \|h^2 w''\|_{L_p}, \quad 1 \leq p \leq \infty.$
- $\|(\pi_h w)' - w'\|_{L_p} \leq c_i \|hw''\|_{L_p}.$
- $\|\pi_h w - w\|_{L_p} \leq c_i \|hw'\|_{L_p}.$

Definition 2.5. *A norm associated with this scalar product is defined by*

$$\|v\| = \sqrt{\langle v, v\rangle} = \langle v, v\rangle^2 = (\int_0^T |v(x)|^2 dx)^{\frac{1}{2}},$$

and is called the L_2 norm of $v(x)$.

Remark 2.3. *In the differential equation, if the value of the unknown function is given on the boundary, then the test function in the FEM can be assumed on the boundary. This is because the exact value of the unknown function is given at the boundary and we do not want to approximate it.*

Definition 2.6. *Let v and w be two vectors in $\mathbb{R}^n$. The Cauchy-Schwartz inequality states that*

$$|\langle v, w\rangle| \leq \|v\|.|\|w\|.$$

Definition 2.7. *$f(x)$ and $g(x)$ are orthogonal on Ω if $\langle f, g\rangle = \int_\Omega f(x)g(x)dx = 0$.*

Definition 2.8. *Choose a subspace $W_h^L \subset W$ of dimension h and solve the projected problem: Find $w_h \in W_h^L$ such that for all $w_h \in W_h^L, a(v_h, w_h) = f(w_h)$, here, $a(\cdot, \cdot)$ is a bilinear form and f is a bounded linear functional on W. We call this* ***the Galerkin equation****.*

Galerkin orthogonality: *The key property of the Galerkin approach is that the error is orthogonal to the chosen subspaces. Since $W_h^L \subset W$ we can use w_h as a test vector in the original equation. Subtracting the two,we get the Galerkin orthogonality relation for the error, $\epsilon_n = v - v_h$ which is the error between the solution of the original problem, v and the solution of the Galerkin equation, v_h.*

$$a(\epsilon_n, w_h) = a(v, w_h) - a(v_h, w_h) = f(w_h) - f(w_h) = 0$$

2.2 Mechanism of The FEM

In this section, we will start by explaining the method. Let us consider the following problem : Find v such that

$$\begin{aligned} -v'' &= g, \quad x \in (0, 1), \\ v(0) &= v(1) = 0, \end{aligned} \tag{2.1}$$

where g is a given function. Sometimes this problem is easy to solve analytically for example, if $g = 2$ then $v = -x^2 + x$ by integrating g twice and using the boundary conditions $v(0) = v(1) = 0$. Nevertheless, for other problems, where the exact solution is not in hand, the numerical approximation is the only choice, so, here, we proceed using FEM. The technique of the FEM is based on the variation formulation and discretization as follows :

Variation Formulation (VE):

To drive the variational formulation of equation (2.1), we multiply it by a test function w, and integrate over $(0, 1)$,

$$\int_0^1 -v''wdx = \int_0^1 gwdx.$$

Integrating by parts on the left side of this equation provides,

$$\int_0^1 v'w'dx - v'(1)w(1) + v'(0)w(0) = \int_0^1 gwdx. \tag{2.2}$$

Since $v(1) = 0$ and (2.2) holds $\forall w \in H_0^1((0,1))$, hence

$$\int_0^1 v'w'dx = \int_0^1 gwdx.$$

We now state the following variational formulation of equation (2.1): find $v(x) \in H_0^1(\Omega)$ such that

$$\int_0^1 v'w'dx = \int_0^1 gwdx, \forall w \in H_0^1.$$

Discretization:

Assume the partition $a = x_0, x_1, ..., x_m = b$ where $x_i = a + ih, i = 0, ..., m, h = \dfrac{b-a}{m}$, m is the number of subintervals.

Let $v \in W_h^L \implies v(x) = \sum_{j=1}^{m-1} \xi_j \Psi_j$, where Ψ_j is the linear basis function defined by Definition 2.3., and $\xi_j = v(x_j)$.

Substitute $v(x) = \displaystyle\sum_{j=1}^{m-1} \xi_j \Psi_j$ in $\displaystyle\int_0^1 v'w'dx = \int_0^1 gwdx$ to get

$$\int_0^1 (\sum_{j=1}^{m-1} \xi_j \Psi_j)'w'dx = \int_0^1 gwdx.$$

Since ξ_j is constant, this means that the derivative will be in Ψ_j,

$$\int_0^1 \sum_{j=1}^{m-1} \xi_j \Psi_j^{'} w' dx = \int_0^1 g w dx$$

$$\sum_{j=1}^{m-1} \xi_j \int_0^1 \Psi_j^{'} w' dx = \int_0^1 g w dx$$

As $w \in W_h^L$ we may take $w = \Psi_i, i = 0, ..., m-1$, thus,

$$\sum_{j=1}^{m-1} \xi_j \int_0^1 \Psi_j^{'} \Psi_i^{'} dx = \int_0^1 g \Psi_i dx$$

$$\xi_1 \int_0^1 \Psi_1^{'} \Psi_i^{'} dx + \xi_2 \int_0^1 \Psi_2^{'} \Psi_i^{'} dx + ... + \xi_{m-1} \int_0^1 \Psi_{m-1}^{'} \Psi_i^{'} dx = \int_0^1 g \Psi_i dx.$$

$$\begin{array}{lllll}
\text{When} & i = 1 & \xi_1 \int_0^1 \Psi_1^{'} \Psi_1^{'} dx + \xi_2 \int_0^1 \Psi_2^{'} \Psi_1^{'} dx + ... + \xi_{m-1} \int_0^1 \Psi_{m-1}^{'} \Psi_1^{'} dx & = & \int_0^1 g \Psi_1 dx. \\
\text{When} & i = 2 & \xi_1 \int_0^1 \Psi_1^{'} \Psi_2^{'} dx + \xi_2 \int_0^1 \Psi_2^{'} \Psi_2^{'} dx + ... + \xi_{m-1} \int_0^1 \Psi_{m-1}^{'} \Psi_2^{'} dx & = & \int_0^1 g \Psi_2 dx. \\
 & & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots & = & \vdots \\
\text{When} & i = m-1 & \xi_1 \int_0^1 \Psi_1^{'} \Psi_{m-1}^{'} dx + \xi_2 \int_0^1 \Psi_2^{'} \Psi_{m-1}^{'} dx + ... + \xi_{m-1} \int_0^1 \Psi_{m-1}^{'} \Psi_1^{'} dx & = & \int_0^1 g \Psi_{m-1} dx.
\end{array}$$

In matrix form, this is equivalent to

$$A\xi = b, \tag{2.3}$$

where

$$A = \begin{pmatrix} \int_0^1 \Psi_1' \Psi_1' dx & \int_0^1 \Psi_2' \Psi_1' dx & \dots & \int_0^1 \Psi_{m-1}' \Psi_1' dx \\ \vdots & \vdots & \vdots & \vdots \\ \int_0^1 \Psi_1' \Psi_{m-1}' dx & \int_0^1 \Psi_2' \Psi_{m-1}' dx & \dots & \int_0^1 \Psi_{m-1}' \Psi_{m-1}' dx \end{pmatrix}.$$

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{m-1} \end{pmatrix}.$$

$$b = \begin{pmatrix} \int_0^1 g\Psi_1 dx \\ \vdots \\ \int_0^1 g\Psi_{m-1} dx \end{pmatrix}.$$

Equation (2.3) is a linear system, where ξ is the unknown that is desired to be determined, this means that solving this linear system implies the solution of (2.1) has been found. We refer to A as the stiffness matrix and b as the load vector,[2],[4],[17].

After this simple example on the FEM, we summarize it in the following basic steps.

Basic Finite Element Algorithm :

1. Create a grid with m elements on a certain interval and determine the corresponding continuous piecewise linear functions W_h.

2. Calculate the $(m-1) \times (m-1)$ matrix of A and the $(m-1) \times 1$ vector b, with entries $A_{ij} = \int_\Omega \Psi_j' \Psi_i' dx, b_i = \int_\Omega g\Psi_i dx$.

3. Resolve the linear system $A\xi = b$.

4. Set $v(x) = \sum_{j=1}^{m-1} \xi_j \Psi_j$.

2.3 FEM Matrices in 1D

In this section, we will provide some information about the matrices that appear when solving differential equations of order 2 in one dimension using linear basis functions,[15, 17]. At in the beginning, the following matrices are denoted :

- $A = [a_{ij}] = \int_\Omega \Psi_j' \Psi_i' dx$: stiffness matrix.
- $M = [m_{ij}] = \int_\Omega \Psi_j \Psi_i dx$: mass matrix.
- $C = [c_{ij}] = \int_\Omega \Psi_j' \Psi_i dx$: convection matrix.

Below, a computation on each of them is performed.

<u>The Stiffness Matrix A</u>:

To compute the entries a_{ij} of the stiffness matrix A using linear basis, we need to determine $\Psi_j'(x)$,

$$\Psi_j'(x) = \begin{cases} \frac{1}{h_j}, & x_{j-1} < x < x_j, \\ \frac{-1}{h_{j+1}}, & x_j < x < x_{j+1}, \\ 0, & \text{O.W.} \end{cases}$$

As for $i = j$:

$$\begin{aligned} a_{i,i} &= \int_\Omega \Psi_i' \Psi_i' dx \\ &= \int_{x_{i-1}}^{x_i} (\frac{1}{h_i})^2 dx + \int_{x_i}^{x_{i+1}} (\frac{-1}{h_{i+1}})^2 dx \\ &= \frac{x_i - x_{i-1}}{h_i^2} + \frac{x_{i+1} - x_i}{h_{i+1}^2} \\ &= \frac{h_i}{h_i^2} + \frac{h_{i+1}}{h_{i+1}^2} \\ &= \frac{1}{h_i} + \frac{1}{h_{i+1}}. \end{aligned}$$

When $j = i + 1$:

$$\begin{aligned}
a_{i,i+1} &= \int_\Omega \Psi'_{i+1}\Psi'_i dx \\
&= \int_{x_{i-1}}^{x_i} \Psi'_{i+1}\Psi'_i dx + \int_{x_i}^{x_{i+1}} \Psi'_{i+1}\Psi'_i dx + \int_{x_{i+1}}^{x_{i+2}} \Psi'_{i+1}\Psi'_i dx \\
&= 0 + \int_{x_i}^{x_{i+1}} \Psi'_{i+1}\Psi'_i dx + 0 \\
&= \int_{x_i}^{x_{i+1}} \frac{1}{h_{i+1}}\frac{-1}{h_{i+1}} dx \\
&= \frac{-h_{i+1}}{h_{i+1}^2} = \frac{-1}{h_{i+1}}.
\end{aligned}$$

The stiffness matrix is symmetric, so $a_{i+1,i} = a_{i,i+1} = \dfrac{-1}{h_{i+1}}$.

To summarize,

$$A = \begin{pmatrix}
\underbrace{\frac{1}{h_1}}_{\text{half basis}} & \frac{-1}{h_2} & 0 & \dots & 0 \\
\frac{-1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & \frac{-1}{h_3} & 0 & 0 \\
0 & \dots & \dots & \dots & 0 \\
\dots & 0 & \dots & \dots & \frac{-1}{h_m} \\
0 & \dots & 0 & \frac{-1}{h_m} & \underbrace{\frac{1}{h_m}}_{\text{half basis}}
\end{pmatrix}.$$

With a uniform mesh, i.e, $h_i = h$, for all i,

$$A = \frac{1}{h}\begin{pmatrix}
\underbrace{1}_{\text{half basis}} & -1 & 0 & 0 & \dots & 0 \\
-1 & 2 & -1 & 0 & \dots & 0 \\
0 & -1 & 2 & -1 & \dots & 0 \\
0 & 0 & -1 & \ddots & \dots & 0 \\
\vdots & & & & \ddots & \vdots \\
0 & \dots & \dots & 0 & -1 & \underbrace{1}_{\text{half basis}}
\end{pmatrix}.$$

The Mass Matrix M :

We use the same manipulation used in the previous matrix to find the form of M.

$$j = i \Longrightarrow m_{i,i} = \int_{\Omega} \Psi_i \Psi_i dx = \int_{x_{i-1}}^{x_i} \Psi_i \Psi_i dx + \int_{x_i}^{x_{i+1}} \Psi_i \Psi_i dx = \frac{h_i + h_{i+1}}{3}.$$

$$j = i+1 \Longrightarrow m_{i,i+1} = \int_{\Omega} \Psi_{i+1} \Psi_i dx = \int_{x_i}^{x_{i+1}} \Psi_{i+1} \Psi_i dx = \frac{h_{i+1}}{6}.$$

$$j = i-1 \Longrightarrow m_{i,i-1} = \int_{\Omega} \Psi_{i-1} \Psi_i dx = \int_{x_{i-1}}^{x_i} \Psi_{i-1} \Psi_i dx = \frac{h_i}{6}.$$

To summarize,

$$M = \begin{pmatrix} \frac{h_1}{3} & \frac{h_2}{6} & 0 & \dots & 0 \\ \frac{h_2}{6} & \frac{h_1 + h_2}{3} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{h_m}{6} \\ 0 & \dots & 0 & \frac{h_m}{6} & \frac{h_m}{3} \end{pmatrix}.$$

With a uniform mesh, i.e, $h_i = h$,

$$M = \frac{h}{6} \begin{pmatrix} \underbrace{2}_{\text{half basis}} & 1 & 0 & 0 & \dots & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 1 & \underbrace{2}_{\text{half basis}} \end{pmatrix}.$$

Note that the mass matrix is also symmetric.

<u>The Convection Matrix</u> C:

Similarly, following the technique of calculations performed before, the convection matrix with uniform and non-uniform mesh is given by

$$C = \frac{1}{2}\begin{pmatrix} \underbrace{-1}_{\text{half basis}} & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 \\ 0 & 0 & -1 & \ddots & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & -1 & \underbrace{1}_{\text{half basis}} \end{pmatrix}.$$

The load vector $b = \int_\Omega g\Psi_i dx$, can be calculated directly, or by numerical integration techniques in case that g is not simple, thus, knowing these elements integration will save time and effort instead of calculating each value a lone.

Consider the following equation

$$\begin{aligned} -v'' + v' + v = g, \quad x \in (0,1), \\ v(1) = v(0) = 0. \end{aligned} \tag{2.4}$$

The FEM for the problem starts by multiplying (2.4) by a test function w such that $w(0) = w(1) = 0$ and integrate over $(0,1)$,

$$\int_0^1 -v''wdx + \int_0^1 v'wdx + \int_0^1 vwdx = \int_0^1 gwdx.$$

Integrate by parts of $\int_0^1 -v''wdx$,

$$-v'w|_0^1 + \int_0^1 v'w'dx + \int_0^1 v'wdx + \int_0^1 vwdx = \int_0^1 gwdx. \tag{2.5}$$

VF: Find $v \in H_0^1((0,1))$ such that following holds $\forall w \in H_0^1((0,1))$,

$$\int_0^1 v'w'dx + \int_0^1 v'wdx + \int_0^1 vwdx = \int_0^1 gwdx. \tag{2.6}$$

Let $v = \sum_{j=1}^{m-1} \xi_j \Psi_j$ and take $w = \Psi_i$ in (2.6), where $i = 1, ..., m-1$

$$
\begin{aligned}
\int_0^1 \sum_{j=1}^{m-1} \xi_j \Psi_j' \Psi_i' dx + \int_0^1 \sum_{j=1}^{m-1} \xi_j \Psi_j' \Psi_i dx + \int_0^1 \sum_{j=1}^{m-1} \xi_j \Psi_j \Psi_i dx &= \int_0^1 g \Psi_i dx, \quad i = 1, ..., m-1. \\
\sum_{j=1}^{m-1} \xi_j (\underbrace{\int_0^1 \Psi_j' \Psi_i' dx}_{A} + \underbrace{\int_0^1 \Psi_j' \Psi_i dx}_{C} + \underbrace{\int_0^1 \Psi_j \Psi_i dx}_{M}) &= \underbrace{\int_0^1 g \Psi_i dx}_{b}, \quad i = 1, ..., m-1.
\end{aligned}
$$

In the system of matrices the above equation can be written as : $(A + C + M)\xi = b$. Numerical examples will be presented to further clarify the idea and explain how to use it in details.

2.4 Numerical Examples

In this section, the FEM is explained through numerical examples.

Example 2.1. *Let v be the solution to*

$$
\begin{aligned}
-v'' = 0, &\quad x \in (0,1), \\
v'(0) - 3(v(0) - 1) = 0,& \\
v(1) = 0.&
\end{aligned} \tag{2.7}
$$

Let $I = (0,1)$ be divided into a uniform mesh, calculate the FEM solution of v for $m = 3$.

The exactly solution is $v(x) = \frac{-3}{4}x + \frac{3}{4}$.

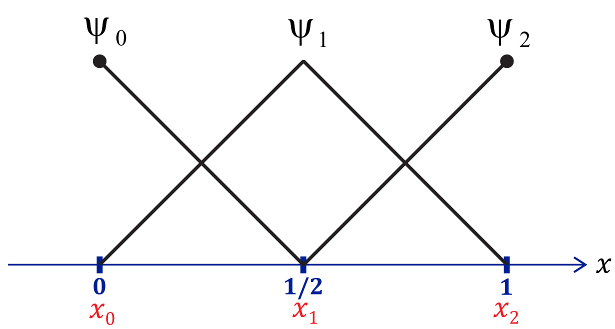


Figure 2.5: Ψ_0, Ψ_1 and Ψ_2

Solution: To solve the example, firstly multiply equation (2.7) by a test function and integrate over $(0,1)$,

$$-\int_0^1 v''w dx = \int_0^1 0(w)dx.$$

Integrate by parts of the left side of equation,

$$-v'w|_0^1 + \int_0^1 v'w'dx = 0.$$

Equivalently,

$$-v'(1)w(1) + v'(0)w(0) + \int_0^1 v'w'dx = 0.$$

$$v'(0)w(0) + \int_0^1 v'w'dx = 0.$$

Substitute the value of $v'(0) = 3(v(0) - 1)$ (boundary condition)

$$3(v(0) - 1)w(0) + \int_0^1 v'w'dx = 0.$$

$$3v(0)w(0) + \int_0^1 v'w'dx = 3w(0). \tag{2.8}$$

VF: Find $v \in H^1((0,1))$ such that $v(1) = 0$ and (2.8) holds for $\forall w \in H^1((0,1))$ with $w(1) = 0$. To proceed to the discretization step, v is written as

$$\begin{aligned}
v &= \sum_{j=0}^{3} \xi_j \Psi_j \\
&= \sum_{j=0}^{2} \xi_j \Psi_j + \xi_3 \Psi_3 \quad \text{since} \quad \xi_3 = v(x_3) = v(1) = 0 \\
&= \sum_{j=0}^{2} \xi_j \Psi_j.
\end{aligned}$$

Substitute the value of v in (2.8) and take $w = \Psi_i, i = 0,1,2$

$$3\xi_0 \Psi_i(x_0) + \sum_{j=0}^{2} \xi_j \int_0^1 \Psi_j' \Psi_i' dx = 3\Psi_i(x_0).$$

Equivalently,

$$\begin{pmatrix} 3\xi_0 \\ 0 \\ 0 \end{pmatrix} + (\frac{1}{\frac{1}{3}}) \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned} 3\xi_0 + 3\xi_0 - 3\xi_1 &= 3. \\ 3\xi_0 + 6\xi_1 - 3\xi_2 &= 0. \\ 3\xi_1 + 6\xi_2 &= 0. \end{aligned}$$

When we solve this system we have, $\xi_0 = \dfrac{15}{32}, \xi_1 = \dfrac{-3}{16}, \xi_2 = \dfrac{3}{32}$.
Therefore,

$$\begin{aligned} v(x) &= \xi_0\Psi_0 + \xi_1\Psi_1 + \xi_2\Psi_2 \\ &= \frac{15}{32}\Psi_0 + \frac{-3}{16}\Psi_1 + \frac{3}{32}\Psi_2. \end{aligned}$$

An example that contains additions to the first example will be discussed below.

Example 2.2. *Consider*

$$\begin{gathered} -((x+2)v')' + v = 3x, \\ v'(0) = v(1) = 0, \end{gathered} \tag{2.9}$$

construct the finite element solution with two subintervals.

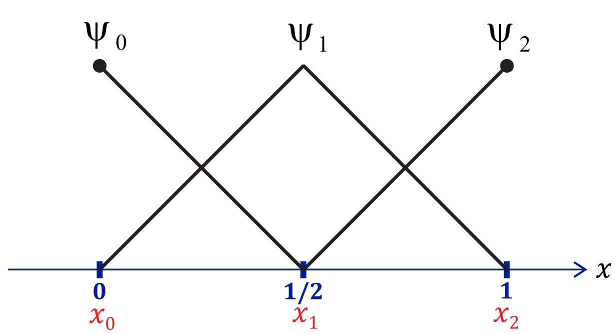


Figure 2.6: Ψ_0, Ψ_1 and Ψ_2

Solution: Multiply (2.9) by a test function w such that $w(1) = 0$ and integrate over $(0, 1)$

$$-\int_0^1 ((x+2)v')'wdx + \int_0^1 vwdx = \int_0^1 3xwdx.$$

Integrate by parts of $-\int_0^1 ((x+2)v')'wdx$ to get

$$-(x+2)v'w|_0^1 + \int_0^1 (x+2)v'w'dx + \int_0^1 vwdx = \int_0^1 3xwdx. \tag{2.10}$$

Since $w(1) = v'(0) = 0$, then (2.10) becomes

$$\int_0^1 (x+2)v'w'dx + \int_0^1 vwdx = \int_0^1 3xwdx. \tag{2.11}$$

VF: Find $v \in H^1((0,1))$ such that $v(1) = 0$ and (2.11) holds for $\forall w \in H^1((0,1))$ with $w(1) = 0$.

Discretization: Let v be as follows:

$$\begin{aligned} v &= \sum_{j=0}^{2} \xi_j \Psi_j \\ &= \sum_{j=0}^{1} \xi_j \Psi_j + \xi_2 \Psi_2, \end{aligned}$$

since $\xi_2 = v(x_2) = v(1) = 0$, then

$$v = \sum_{j=0}^{1} \xi_j \Psi_j.$$

Substitute v in (2.11) and take $w = \Psi_i, i = 0, 1$,

$$\begin{aligned} \int_0^1 (x+2) \sum_{j=0}^{1} \xi_j \Psi_j^{'} \Psi_i^{'} dx + \int_0^1 \sum_{j=0}^{1} \xi_j \Psi_j \Psi_i dx &= \int_0^1 3x \Psi_i dx, i = 0, 1. \\ \sum_{j=0}^{1} \xi_j \int_0^1 (x+2) \Psi_j^{'} \Psi_i^{'} dx + \sum_{j=0}^{1} \xi_j \int_0^1 \Psi_j \Psi_i dx &= \int_0^1 3x \Psi_i dx, i = 0, 1. \end{aligned}$$

When $i = 0$:

$$\xi_0 \int_0^1 (x+2) \Psi_0^{'} \Psi_0^{'} dx + \xi_0 \int_0^1 (x+2) \Psi_1^{'} \Psi_0^{'} dx + \xi_0 \int_0^1 \Psi_0 \Psi_0 dx + \xi_1 \int_0^1 \Psi_1 \Psi_0 dx = \int_0^1 3x \Psi_0 dx.$$

When $i = 1$:

$$\xi_0 \int_0^1 (x+2)\Psi_0^{'}\Psi_1^{'} dx + \xi_0 \int_0^1 (x+2)\Psi_1^{'}\Psi_1^{'} dx + \xi_0 \int_0^1 \Psi_0\Psi_1 dx + \xi_1 \int_0^1 \Psi_1\Psi_1 dx = \int_0^1 3x\Psi_1 dx.$$

All integrals in the equations above will be calculated and then ξ is calculated. The integrals containing the weight functions $(x+2)$ and x are calculated as a regular account because in this case it is not beneficial to use the general rule of the matrices mentioned previously since there is other functions within the integrals. Thus,

$$\begin{aligned}\int_0^1 (x+2)\Psi_0^{'}\Psi_0^{'} dx &= \int_0^{\frac{1}{2}} (x+2)(-2)(-2)dx + \int_{\frac{1}{2}}^1 (x+2)(0)dx.\\ &= \int_0^{\frac{1}{2}} (x+2)(4)dx.\\ &= 4(\frac{x^2}{2} + 2x|_0^{\frac{1}{2}}).\\ &= 4(\frac{\frac{1}{4}}{2} + 1).\\ &= \frac{9}{2}.\end{aligned}$$

All other integrals that contain weight functions will be calculated in the same way. After calculating all integrals we obtain the following two equations in ξ_0 and ξ_1

$$\frac{9}{2}\xi_0 - \frac{9}{2}\xi_1 + \frac{1}{6}\xi_0 + \frac{1}{12}\xi_1 = \frac{1}{8}.$$

$$\frac{-9}{2}\xi_0 + 10\xi_1 + \frac{1}{12}\xi_0 + \frac{1}{3}\xi_1 = \frac{3}{4}.$$

Solving the above two equations, yields

$$\begin{aligned}\xi_0 &= \frac{663}{4135}.\\ \xi_1 &= \frac{1167}{8270}.\end{aligned}$$

$$\begin{aligned}\Longrightarrow v(x) =& \xi_0\Psi_0 + \xi_1\Psi_1 \\ =& \frac{663}{4135}\Psi_0 + \frac{1167}{8270}\Psi_1.\end{aligned}$$

2.5 Example Of Two Dimensional Problem

Consider the following problem

$$\begin{aligned} -\nabla.(a\nabla v) + cv = g, \qquad & x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \\ -n.(a\nabla v) = \alpha(v - f_D) + f_N, \qquad & x = (x_1, x_2) \in \partial\Omega. \end{aligned} \tag{2.12}$$

The functions $a = a(x_1, x_2), c = c(x_1, x_2), g = g(x_1, x_2), f_N = f_N(x_1, x_2)$ and $f_D = f_D(x_1, x_2)$ are data of the problems.
We would like to approximate $v(x_1, x_2)$ using FEM, multiplying the differential equation (2.12) by a test function $w(x_1, x_2)$ and integrate over Ω,

$$-\iint_\Omega \nabla.(a\nabla v)w dx_1 dx_2 + \iint_\Omega cvw dx_1 dx_2 = \iint_\Omega gw dx_1 dx_2.$$

Integrate by parts using Green's formula

$$-\int_{\partial\Omega} n.(a\nabla v)w ds + \iint_\Omega a\nabla v.\nabla w dx_1 dx_2 + \iint_\Omega cvw dx_1 dx_2 = \iint_\Omega gw dx_1 dx_2.$$

Use the boundary condition $-n.(a\nabla v) = \alpha(v - f_D) + f_N, x \in \partial\Omega$, to obtain

$$\int_{\partial\Omega} (\alpha(v - f_D) + f_N)w ds + \iint_\Omega a\nabla v.\nabla w dx_1 dx_2 + \iint_\Omega cvw dx_1 dx_2 = \iint_\Omega gw dx_1 dx_2.$$

Any term in which there is no v will be moved to the right side of the equation, this yields

$$\int_{\partial\Omega} \alpha vw ds + \iint_\Omega a\nabla v.\nabla w dx_1 dx_2 + \iint_\Omega cvw dx_1 dx_2 = \iint_\Omega gw dx_1 dx_2 + \int_{\partial\Omega} (\alpha f_D - f_N)w ds. \tag{2.13}$$

VF: Find $v \in H^1(\Omega)$ such that (2.13) holds $\forall w \in H^1(\Omega)$.
Discretization: Introducing the vector space W_h^L of continuous piecewise linear functions on a triangulation of Ω.

We construct a set of basis which called the tent functions $\{\Psi_j\}_{j=1}^{nnodes} \subset W_h^L$, where

$$\Psi_j(N_i) = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

for $i, j = 1, ..., nnodes$, N_i are the nodes in the generated triangulation, refers to the number of nodes in the given triangulation.
Now, $\forall v(x_1, x_2) \in W_h^L$, $v(x_1, x_2)$ can be written as a unique linear combination of Ψ_j's, i.e., $v(x_1, x_2) = \sum_{j=1}^{nnodes} \xi_j \Psi_j(x_1, x_2)$, see[4, 14, 16, 20, 21].
For the construction of the discrete system of linear equations, we substitute $v(x_1, x_2) = \sum_{j=1}^{nnodes} \xi_j \Psi_j(x_1, x_2)$ and take $w = \Psi_i, i = 1, ..., nnodes$ in (2.13)

$$\sum_{j=1}^{nnodes} \xi(\underbrace{\int_{\partial\Omega} \alpha \Psi_j \Psi_i ds}_{P} + \underbrace{\iint_{\Omega} a \nabla \Psi_j . \nabla \Psi_i dx_1 dx_2}_{A} + \underbrace{\iint_{\Omega} c \Psi_j \Psi_i dx_1 dx_2}_{M_c})$$
$$= \underbrace{\iint_{\Omega} g \Psi_i dx_1 dx_2}_{b} + \underbrace{\int_{\partial\Omega} (\alpha f_D - f_N) \Psi_i ds}_{rv}, \quad i = 1, ..., nnodes.$$

Introduce the notation:

$$p_{ij} = \int_{\partial\Omega} \alpha \Psi_j \Psi_i ds.$$

$$a_{ij} = \iint_{\Omega} a \nabla \Psi_j . \nabla \Psi_i dx_1 dx_2.$$

$$m_{cij} = \iint_{\Omega} c \Psi_j \Psi_i dx_1 dx_2.$$

$$rv_i = \int_{\partial\Omega} (\alpha f_D - f_N) \Psi_i ds.$$

$$b_i = \iint_{\Omega} g \Psi_i dx_1 dx_2.$$

Using these notations, the above equation can be written as : $(P + A + M_c)\xi = rv + b$.

After the FEM has been demonstrated in one and two dimension and does not depend on time, below we will start discussing the time dependent problems which are our concern.

2.6 Time Dependent

In this section, time dependent FEM will be discussed in one and two dimensions.

2.6.1 One Dimensional Case

Consider the following time dependent model problem:

$$\begin{aligned} v_t - (av')' + bv' + cv = g, \quad \alpha < x < \beta, \quad 0 < t < T, \\ v(\alpha, t) = v(\beta, t) = 0, \\ v(x, 0) = v_0(x). \end{aligned} \tag{2.14}$$

The functions $a = a(x,t), b = b(x,t), c = c(x,t)$ and $g = g(x,t)$ are data of the problems. To apply the FEM to the problem, we first treat the spatial variable, i.e., freezing the variable t and perform the usual steps of the FEM as it is done before to the problem. Then after we treat the time dependent derivative using methods of numerical integration. Multiply the differential equation (2.14) by a test function $w(x) \in H_0^1((\alpha, \beta))$ and integrate over (α, β)

$$\int_\alpha^\beta v_t w dx - \int_\alpha^\beta (av')' w dx + \int_\alpha^\beta bv' w dx + \int_\alpha^\beta cvw dx = \int_\alpha^\beta gw dx, \quad 0 < t < T.$$

Integrate by parts for the derivative with respect to x,

$$\int_\alpha^\beta v_t w dx - av'w|_\alpha^\beta + \int_\alpha^\beta av'w' dx + \int_\alpha^\beta bv' w dx + \int_\alpha^\beta cvw dx = \int_\alpha^\beta gw dx, \quad 0 < t < T.$$

Substitute $w(\alpha) = w(\beta) = 0$ to get

$$\int_\alpha^\beta v_t w dx + \int_\alpha^\beta av'w' dx + \int_\alpha^\beta bv' w dx + \int_\alpha^\beta cvw dx = \int_\alpha^\beta gw dx, \quad 0 < t < T. \tag{2.15}$$

VF: Find $v(x,t)$ such that, for every fixed t: $v(x,t) \in H_0^1((\alpha, \beta))$ and (2.15) holds $\forall w(x) \in H_0^1((\alpha, \beta))$. To discretize the problem according to the space variable x, assume a partition, $\alpha = x_0 < x_1 < ... < x_m < x_{m+1} = \beta$.

Let $v(x,t) = \sum_{j=0}^{m+1} \xi_j(t) \Psi_j(x)$ and since $\xi_0(t) = v(x_0, t) = v(\alpha, t) = 0, \xi_{m+1}(t) =$

$v(x_{m+1},t) = v(\beta,t) = 0$ then

$$v(x,t) = \sum_{j=1}^{m} \xi_j(t)\Psi_j(x). \tag{2.16}$$

Substitute (2.16) into (2.15) and take $w = \Psi_i(x), i = 1,...,m$, to get

$$\begin{gathered}\textstyle\sum_{j=1}^{m} \dot{\xi}_j(t)(\displaystyle\int_\alpha^\beta \Psi_j(x)\Psi_i(x)dx) + \sum_{j=1}^{m} \xi_j(t)(\int_\alpha^\beta a(x,t)\Psi_j'(x)\Psi_i'(x)dx)+ \\ +\textstyle\sum_{j=1}^{m} \xi_j(t)(\displaystyle\int_\alpha^\beta b(x,t)\Psi_j'(x)\Psi_i(x)dx)+ \\ +\textstyle\sum_{j=1}^{m} \xi_j(t)(\displaystyle\int_\alpha^\beta c(x,t)\Psi_j(x)\Psi_i(x)dx) = \int_\alpha^\beta g(x,t)\Psi_i(x)dx, \quad 0 < t < T,\end{gathered} \tag{2.17}$$

where $\dot{\xi} = \dfrac{\partial \xi}{\partial t}$.
When placing the derivative consider whether it is a derivative with respect to x or derivative with respect to t, this means that: $\xi_j(t)$ and $\Psi_j(t)$ are functions of t and x respectively.
Introduce the notation:

$$m_{ij} = \int_\alpha^\beta \Psi_j(x)\Psi_i(x)dx.$$

$$a_{ij}(t) = \int_\alpha^\beta a(x,t)\Psi_j'(x)\Psi_i'(x)dx.$$

$$b_{ij}(t) = \int_\alpha^\beta b(x,t)\Psi_j'(x)\Psi_i(x)dx.$$

$$c_{ij}(t) = \int_\alpha^\beta c(x,t)\Psi_j(x)\Psi_i(x)dx.$$

$$d_i(t) = \int_\alpha^\beta g(x,t)\Psi_i(x)dx.$$

Using the notations above, equation (2.17) can be written as

$$\begin{gathered} m_{11}\dot{\xi_1}(t) + ... + m_{1m}\dot{\xi_m}(t) + (a_{11} + b_{11} + c_{11})\xi_1(t) + ... + (a_{1m} + b_{1m} + c_{1m})\xi_m(t) = d_1(t). \\ \vdots \qquad \vdots \\ m_{m1}\dot{\xi_1}(t) + ... + m_{mm}\dot{\xi_m}(t) + (a_{m1} + b_{m1} + c_{m1})\xi_1(t) + ... + (a_{mm} + b_{mm} + c_{mm})\xi_m(t) = d_m(t). \end{gathered}$$

In the system of matrices the above equation is equivalent to

$$M\dot{\xi}(t) + (A(t) + B(t) + C(t))\xi(t) = D(t), \quad 0 < t < T. \tag{2.18}$$

Time Discretization:
Let $0 = t_0 < t_1 < ... < t_R = T$ where $k_m = t_m - t_{m-1}, m = 1, ..., R$, and let ξ^m denote the value $\xi(t_m), m = 1, ..R, \Longrightarrow \xi^0 = \xi(0)$. Use the initial condition to get

$$\xi^0 = \begin{bmatrix} \xi_1(0) \\ \vdots \\ \xi_m(0) \end{bmatrix} = \begin{bmatrix} v_0(x_1) \\ \vdots \\ v_0(x_m) \end{bmatrix}.$$

Now, integrate (2.18) over the time interval $[t_{m-1}, t_m]$,

$$\int_{t_{m-1}}^{t_m} M\dot{\xi}(t)dt + \int_{t_{m-1}}^{t_m} (A(t) + B(t) + C(t))\xi(t)dt = \int_{t_{m-1}}^{t_m} D(t)dt.$$

Because M is a constant matrix, we can get it out of the integration,thus

$$M\int_{t_{m-1}}^{t_m} \dot{\xi}(t)dt + \int_{t_{m-1}}^{t_m} (A(t) + B(t) + C(t))\xi(t)dt = \int_{t_{m-1}}^{t_m} D(t)dt.$$

$$M(\xi(t_m) - \xi(t_{m-1})) + \int_{t_{m-1}}^{t_m} (A(t) + B(t) + C(t))\xi(t)dt = \int_{t_{m-1}}^{t_m} D(t)dt. \tag{2.19}$$

We can not find the integrals in equation (2.19) directly, so, numerical integration techniques are used. Hence, we may approximate the integrals in (2.19) using right end-point quadrature which gives the backward Euler method.

$$\begin{aligned}\int_{t_{m-1}}^{t_m} (A(t) + B(t) + C(t))\xi(t)dt \approx& (A(t_m) + B(t_m) + C(t_m))\xi(t_m)\int_{t_{m-1}}^{t_m} dt \\ =& (A(t_m) + B(t_m) + C(t_m))\xi(t_m)k_m.\end{aligned}$$

Similarly, $\int_{t_{m-1}}^{t_m} D(t)dt \approx D(t_m)k_m$.
Thus, we arrive at

$$M(\xi^m - \xi^{m-1}) + (A(t_m) + B(t_m) + C(t_m))\xi^m k_m = D(t_m)k_m,$$

where ξ^m is $\xi(t_m)$. The equation is simplified as

$$M\xi^m - M\xi^{m-1} + (A(t_m) + B(t_m) + C(t_m))\xi^m k_m = D(t_m)k_m$$

$$(M + k_m(A(t_m) + B(t_m) + C(t_m)))\xi^m - M\xi^{m-1} = D(t_m)k_m.$$

Solving for ξ^m yields

$$\xi^m = \frac{D_m k_m + M\xi^{m-1}}{M + k_m(A_m + B_m + C_m)}. \tag{2.20}$$

In (2.20) we have introduced the notation $A_m = A(t_m), B_m = B(t_m), C_m = C(t_m)$ and $D_m = D(t_m)$, see [7].

After knowing the mechanism of the FEM in one dimension, below we treat the two dimensional case .

2.6.2 Two Dimensional Case

Consider the following time dependent model problem

$$\begin{aligned} v_t - \nabla.(a\nabla v) + b.\nabla v + cv = g, &\quad x = (x_1, x_2) \in \Omega, \quad 0 < t < T, \\ v(x,t) = 0, &\quad x = (x_1, x_2) \in \partial\Omega, \quad 0 < t < T, \\ v(x,0) = v_0(x), &\quad x = (x_1, x_2) \in \Omega, \end{aligned} \tag{2.21}$$

where $v(x,t) = v(x_1, x_2, t)$ is the unknown function that we wish to compute, the functions $a = a(x,t), b = b(x,t), c = c(x,t)$, and $g = g(x,t)$ are data to the problem.

Multiply the differential equation (2.21) by a test function $w(x_1, x_2)$ such that $w = 0$ on $\partial\Omega$ and integrate over Ω,

$$\iint_\Omega v_t w dx_1 dx_2 - \iint_\Omega \nabla.(a\nabla v) w dx_1 dx_2 + \iint_\Omega b.\nabla v w dx_1 dx_2 + \\ + \iint_\Omega cvw dx_1 dx_2 = \iint_\Omega g w dx_1 dx_2, \quad 0 < t < T.$$

Integrates by parts using Green's formula to get

$$\iint_\Omega v_t w dx_1 dx_2 - \int_{\partial\Omega} (n.(a(x,t)\nabla v) w ds + \iint_\Omega (a\nabla v).\nabla w dx_1 dx_2 + \iint_\Omega b(x,t).\nabla v w dx_1 dx_2 + \\ + \iint_\Omega cvw dx_1 dx_2 = \iint_\Omega g w dx_1 dx_2, \quad 0 < t < T.$$

As v is given on $\partial\Omega$, then $w = 0$ on $\partial\Omega$, thus

$$\begin{aligned} \iint_\Omega v_t w dx_1 dx_2 + \iint_\Omega (a(x,t)\nabla v).\nabla w dx_1 dx_2 + \iint_\Omega b.\nabla v w dx_1 dx_2 + \\ + \iint_\Omega c v w dx_1 dx_2 = \iint_\Omega g w dx_1 dx_2, \quad 0 < t < T. \end{aligned} \tag{2.22}$$

VF: Find $v(x_1, x_2, t)$ such that for every fixed t : $v(x_1, x_2, t) \in H_0^1(\Omega)$ and (2.22) holds $\forall w \in H_0^1(\Omega)$

Discretization: Introducing the vector space W_h^L of continuous piecewise linear functions on a triangulation of Ω, see [4, 14, 16, 20, 21].

We construct a set of basis which called the tent functions $\{\Psi_j\}_{j=1}^n \subset W_h^L$, where

$$\Psi_j(N_i) = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

for $i, j = 1, ..., n$, N_i are the nodes in the generated triangulation, and n refers to the number of the interior nodes in the given triangulation.

Now, $\forall v(x_1, x_2) \in W_h^L$, $v(x_1, x_2)$ can be written as a unique linear combination of Ψ_j's, i.e., $v(x_1, x_2, t) = \sum_{j=1}^{n} \xi_j(t)\Psi_j(x_1, x_2)$.

For the construction of the discrete system of linear equations, we substitute $v(x_1, x_2, t) = \sum_{j=1}^{n} \xi_j(t)\Psi_j(x_1, x_2)$ and take $w = \Psi_i, i = 1, ..., n$ in (2.22)

$$\begin{aligned} \sum_{j=1}^{n} \dot{\xi}_j(t)(\iint_\Omega \Psi_j(x_1, x_2)\Psi_i(x_1, x_2) dx_1 dx_2) + \\ + \textstyle\sum_{j=1}^n \xi_j(\iint_\Omega a(x,t)\nabla\Psi_j(x_1, x_2).\nabla\Psi_i(x_1, x_2) dx_1 dx_2) + \\ + \textstyle\sum_{j=1}^n \xi_j(t)(\iint_\Omega (b(x,t).\nabla\Psi_j(x_1, x_2))\Psi_i(x_1, x_2) dx_1 dx_2) + \\ + \textstyle\sum_{j=1}^n \xi_j(t)(\iint_\Omega c(x,t)\Psi_j(x_1, x_2)\Psi_i(x_1, x_2) dx_1 dx_2) \\ = \iint_\Omega g(x,t)\Psi_i(x_1, x_2) dx_1 dx_2, \quad i = 1, ..., n, \quad 0 < t < T. \end{aligned} \tag{2.23}$$

Introduce the notation

$$M = (m_{ij})_{ij=1}^n \text{where} \quad m_{ij} = \iint_\Omega \Psi_j(x_1, x_2)\Psi_i(x_1, x_2) dx_1 dx_2.$$

$$A = (a_{ij})_{ij=1}^{n} \text{where} \quad a_{ij}(t) = \iint_{\Omega} a(x,t)\nabla\Psi_j(x_1,x_2).\nabla\Psi_i(x_1,x_2)dx_1dx_2.$$

$$B = (b_{ij})_{ij=1}^{n} \text{where} \quad b_{ij}(t) = \iint_{\Omega} (b(x,t).\nabla\Psi_j(x_1,x_2))\Psi_i(x_1,x_2)dx_1dx_2.$$

$$C = (c_{ij})_{ij=1}^{n} \text{where} \quad c_{ij}(t) = \iint_{\Omega} c(x,t)\Psi_j(x_1,x_2)\Psi_i(x_1,x_2)dx_1dx_2.$$

$$D = (d_i)_{i=1}^{n} \text{where} \quad d_i(t) = \iint_{\Omega} g(x,t)\Psi_i(x_1,x_2)dx_1dx_2.$$

In the system of matrices, equation (2.23) can be written as

$$M\dot{\xi}(t) + (A(t) + B(t) + C(t))\xi(t) = D(t), \quad 0 < t < T. \tag{2.24}$$

Time Discretization:

To discretize (2.24) in time, let $0 = t_0 < t_1 < ... < t_R = T$ where $k_m = t_m - t_{m-1}$ and let ξ^m denote $\xi(t_m), m = 1, ..., R$.

Use the initial condition to get

$$\xi^0 = \begin{bmatrix} \xi_1(0) \\ \vdots \\ \xi_n(0) \end{bmatrix} = \begin{bmatrix} v_0(n_1) \\ \vdots \\ v_0(n_n) \end{bmatrix}.$$

As was done previously with the same details, integrate (2.24) over the time interval $[t_{m-1}, t_m]$ to get

$$\int_{t_{m-1}}^{t_m} M\dot{\xi}(t)dt + \int_{t_{m-1}}^{t_m} (A(t) + B(t) + C(t))\xi(t)dt = \int_{t_{m-1}}^{t_m} D(t)dt.$$

$$M(\xi^m - \xi^{m-1}) + \int_{t_{m-1}}^{t_m} (A(t) + B(t) + C(t))\xi(t)dt = \int_{t_{m-1}}^{t_m} D(t)dt. \tag{2.25}$$

Approximating the integrals in (2.25) using right endpoint quadrature gives the backward Euler method

$$M(\xi^m - \xi^{m-1}) + (A(t_m) + B(t_m) + C(t_m))\xi^m k_m = D(t_m)k_m,$$

where ξ^m is $\xi(t_m)$. Simplify the above equation provides

$$\xi^m = \frac{D_m k_m + M\xi^{m-1}}{M + k_m(A_m + B_m + C_m)}, \tag{2.26}$$

where $A_m = A(t_m), B_m = B(t_m), C_m = C(t_m)$ and $D_m = D(t_m)$.

Backward Euler Method for the Heat Equation Algorithm :

1. Over a certain interval, create a grid with m elements and determine the space corresponding to continuous piecewise linear functions W_h.
2. Meet a time mesh $0 = t_0 < t_1 < ... < t_R = T$ on the interval $0 < t < T$ with R time steps $k_m = t_m - t_{m-1}$.
3. Set $\xi_0 = \xi(0)$.
4. For $m = 1, 2, .., R$.
   (a) Calculate the $(m-1) \times (m-1)$ mass, stiffness and convection matrices M, A, B and C, and the $(m-1) \times 1$ load vector $D_m = D(t_m)$ with entries $M_{ij} = \int_0^1 \Psi_j \Psi_i dx$, $A_{ij} = \int_0^1 \nabla\Psi_j \nabla\Psi_i dx$, $B_{ij} = \int_0^1 b.\nabla\Psi_j \Psi_i dx$,$C_{ij} = \int_0^1 \Psi_j \Psi_i dx$ and $(D_m)_i = \int_0^1 g(t_m)\Psi_j dx$.
   (b) solve the linear system $(M + k_m(A(t_m) + B(t_m) + C(t_m)))\xi^m - M\xi^{m-1} = D(t_m)k_m$.
5. end For.

Chapter 3

Heat Equation

In this chapter, we talk about the heat equation in general before going further in the FEM solution of it, thus it must be known and explained previously so that there is no malfunction in understanding.
In physics and mathematics, the heat equation is a second degree partial differential equation that describes heat transfer by conduction and heat change to objects, as it automatically flows from places where it is higher towards the places where it is lower and is considered as a special case of the diffusion equation. This equation appears in several areas from the manufacture of engines to biology, the areas of which will be mentioned in detail later.
Below, the equation will be clarified in terms of its derivation, solving examples and mention some applications.

3.1 Derivation

We consider a rectangular parallel element of a conducting solid with dimensions $\Delta x, \Delta y$, and Δz, as in figure(3.1),[13]. The weight,Δw, of the element is

$$\Delta w = \rho \Delta x \Delta y \Delta z = g \Delta m,$$

where ρ is the density factor, Δm is the mass of the component, and g is the gravitational constant. The above equation can be written as :

$$\Delta m = \frac{\rho \Delta x \Delta y \Delta z}{g}.$$

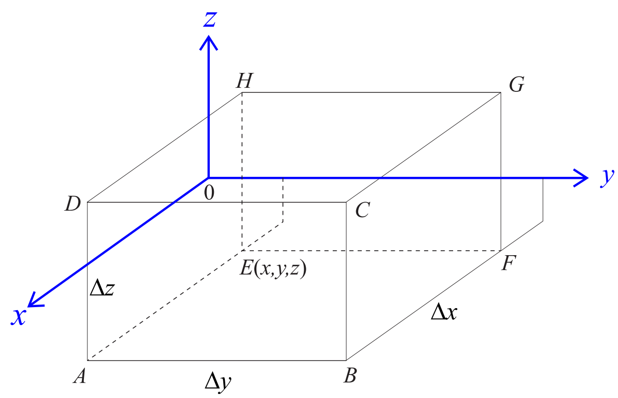


Figure 3.1: An element of a substance,[13]

If the temperature change is Δv for a period of time Δt, the amount of heat ΔQ stored in the element depending on the amount of heat lost or acquired from a substance due to a temperature change proportional with the mass, thus

$$\Delta Q = c\Delta m \Delta v. \tag{3.1}$$

Substitute the value of Δm in (3.1) to get

$$\Delta Q = \frac{c\rho}{g}\Delta x \Delta y \Delta z \Delta v, \tag{3.2}$$

since c, ρ and g are constants, then they are combined together.
Dividing equation (3.2) by Δt to have

$$\frac{\Delta Q}{\Delta t} = \frac{c\rho}{g}\Delta x \Delta y \Delta z \frac{\Delta v}{\Delta t}. \tag{3.3}$$

With reference to the temperature, it flows through an area at a rate proportional to the region and to the temperature of the natural gradient of the region. If heat flows into the element, the rate of flow is

$$-k\Delta A v_x, \tag{3.4}$$

where k is the thermal conductivity of the material, ΔA is the area of a face and v_x is the gradient. If the flow is outside the element sign figure(3.1) is reversed. The element ΔA is $\Delta y \Delta z$ for the faces $ABCD$ and $EFGH$, ΔA is $\Delta x \Delta z$ for $BCGF$ and $ADHE$; and for $ABFE$ and $CDHG$ the area ΔA is $\Delta x \Delta y$. The heat that causes the temperature change Δv comes from inside the material or from the transfer of heat through the sides of the

element.
We know $q(x, y, z, t)$ is the internal heat change per unit volume, the net rate of change of heat entering and leaving the element through faces $EFGH$ and $ABCD$ is $k\Delta y\Delta z[v_x|_{x+\Delta x} - v_x|_x]$ The quantity $v_x|_{x+\Delta x}$ refers to the partial derivative with respect to x evaluated at $x + \Delta x$ in the mid-point of the face (y_m, z_m).
Equating the rate in (3.3) with the contribution (3.4) for the total ΔA from the six faces of the element to get

$$\frac{c\rho}{g}\Delta x\Delta y\Delta z\frac{\Delta v}{\Delta t} = k[\Delta y\Delta z(v_x|_{x+\Delta x} - v_x|_x) + \Delta x\Delta z(v_x|_{y+\Delta y} - v_y|_y) + \Delta x\Delta y(v_z|_{z+\Delta z} - v_z|_z)] + $$
$$+\Delta x\Delta y\Delta z q(x, y, z, t).$$

Dividing the above equation by $\frac{c\rho}{g}\Delta x\Delta y\Delta z$ and allowing $\Delta x, \Delta y, \Delta z$ and Δt to go to zero to have

$$v_t = \alpha(v_{xx} + v_{yy} + v_{zz}) + \frac{g}{c\rho}q(x, y, z, t), \tag{3.5}$$

where $\alpha = \frac{kg}{c\rho}$. This is the heat equation in three dimensions.
If the heat inside the material is not lost or generated, then $q(x, y, z, t)$ is zero, and this will be our assumption.
If $v_t = 0$ and all changes relative to time have stopped, then we have steady state condition $v_{xx} + v_{yy} + v_{zz} = 0$.

3.2 Steady State

3.2.1 Steady State problems

The steady-state heat equation is by definition assume that the rate of change of heat with respect to time is zero, i.e., $v_t = 0$,[9].
Thus, equation (3.5) becomes

$$0 = \alpha(v_{xx} + v_{yy} + v_{zz}) + \frac{g}{c\rho}q(x, y, z, t).$$

$$\Longrightarrow \alpha(v_{xx} + v_{yy} + v_{zz}) = -\frac{g}{c\rho}q(x, y, z, t).$$

If α =constant, at this point we obtain
Poisson's equation:

$$\Delta v = -\frac{\frac{g}{c\rho}}{\alpha} q(x,y,z,t) \quad \text{where } \Delta v = v_{xx} + v_{yy} + v_{zz}.$$
$$= -\frac{\frac{g}{c\rho}}{\frac{kg}{c\rho}} q(x,y,z,t).$$
$$= \frac{-q(x,y,z,t)}{k}.$$

Moreover, if $q = 0$, then we get
Laplace's equation:

$$\Delta v = 0.$$

3.2.2 Steady State Solution

Consider the following example

$$v_t = \alpha v_{xx}.$$

The steady state problem is

$$\Longrightarrow \alpha v_{xx} = 0.$$

Since α is not zero, divide the equation by α,

$$v_{xx} = 0.$$

Integrate both sides with respect to x to obtain

$$v_x = A.$$

Integrate again to get

$$v(x,t) = Ax + B.$$

Example 3.1. *Consider an iron rod with boundary conditions* $v(0,t) = -4^0 C$, *and* $v(10,t) = 16^0 C$.

Solution : The steady state solution is

$$v(x,t) = Ax + B, \quad v(0,t) = -4^0C, \quad v(10,t) = 16^0C.$$

Use the boundary conditions to calculate the value of A and B ,

$$-4 = v(0,t) = A(0) + B \Longrightarrow B = -4.$$

Since $v(10,t) = 16 = A(10) - 4 \Longrightarrow A = 2$. Thus, $v(x,t) = 2x - 4$ is the steady state solution.

3.3 Statement Of The Equation

For a function $n(x,y,z,t)$ of three spatial variables (x,y,z) and the time variable t, the heat equation is $v_t = \alpha(v_{xx} + v_{yy} + v_{zz}) + q(x,y,z,t)$, where α is a real coefficient called the average diffusivity constant.
Utilizing Newton's notation for derivatives, and the notation of vector calculus, the heat equation can be written as $\dot{v} = \alpha\Delta v + q(x,y,z,t)$, where Δ is the Laplace operator and $\dot{v}$ is the time derivative.
In mathematical studies of the equation, one often specifies $\alpha = 1$, with this simplification, the heat equation is the prototypical parabolic partial differential equation,[10].

3.4 Physical Meaning

Initial condition and boundary condition:[5, 12]
The following data are always associated with the heat equation.

1. Initial Condition (IC): In this case, the initial temperature distribution in the rod is $v(x,0)$.

2. Boundary Condition (BC): In this case, the temperature of the rod is affected by what occurs at the ends, $x = 0, l$, as for one dimensional case.

Example 3.2. *Keep in mind a rod of length l with insulated sides is given an initial temperature distribution of $g(x)$ degree C, for $0 < x < l$. Find $v(x,t)$ at later time $t > 0$ if the ends of the rod are kept at 0^oC.*

The heat equation and the corresponding IC and BCs are the follows:
PDE:

$$v_t = kv_{xx}, \quad 0 < x < l, \quad t > 0,$$

IC:

$$v(x,0) = g(x), \quad 0 < x < l,$$

BC:

$$v(0,t) = v(l,t) = 0, \quad \forall t \geq 0.$$

Example 3.3. *Consider an initial boundary value problem (IBVP) for the heat equation*

$$\begin{aligned} v_t - v_{xx} =& g(x), \quad 0 < x < 1, \quad t > 0, \\ v(x,0) =& v_0(x), \quad 0 < x < 1, \\ v(0,t) =& v_x(1,t) = 0, \quad t \geq 0. \end{aligned}$$

The physical meaning to the (IBVP) where $g(x) = 10$.

- $v(x,t) =$ a certain value $\Longrightarrow$ temperature at x at a certain value t.
- $v(x,0) = v_0(x) \Longrightarrow v_0(x)$ the initial temperature at time $t = 0$.
- $v(0,t) = 0 \Longrightarrow$ the end $x = 0$ is kept at 0 temperature.
- $v_x(1,t) = 0 \Longrightarrow$ the end $x = 1$ is isolated.
- $g(x) = 10 \Longrightarrow$ heat source.

3.5 Solving Example Analytically

Consider the heat equation problem

$$\begin{aligned} v_t =& a^2 v_{xx}, \quad 0 < x < L, \quad t > 0, \\ v(0,t) =& v(L,t) = 0, \quad t > 0, \\ v(x,0) =& g(x), \quad 0 < x < L. \end{aligned}$$

Physically: heat distribution a long insulated lateral surface of rod of length L.

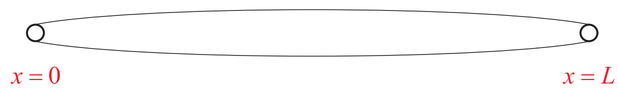


Figure 3.2: Metal rod

To solve the problem analytically, we will use separation of variables technique,[8].

(1) **Separation of variables**

Let $v(x,t) = X(x)T(t)$, and substitute it in the differential equation to get

$$XT' = a^2X''T.$$

Dividing by a^2XT to obtain

$$\frac{T'}{a^2T} = \frac{X''}{X}. \tag{3.6}$$

The left hand side depends only on t and the right hand side only depends on x. The only solution of (3.6) is that both sides must be a constant, say λ,

$$\frac{T'}{a^2T} = \frac{X''}{X} = \lambda.$$

(2) **Two related ODEs**

$$T' - a^2\lambda T = 0.$$

$$X'' - \lambda X = 0.$$

(3) **Homogeneous Boundary Conditions**

$$v(0,t) = X(0)T(t) = 0 \Longrightarrow X(0) = 0 \quad \text{since} \quad T(t) \neq 0.$$
$$v(L,t) = X(L)T(t) = 0 \Longrightarrow X(L) = 0 \quad \text{since} \quad T(t) \neq 0.$$

(4) **The X-equation:** $X'' - \lambda X = 0, \quad X(0) = X(L) = 0.$

We take cases for λ:

(i) $\lambda > 0$, let $\lambda = \alpha^2$, $\alpha \neq 0$, $X'' - \alpha^2X = 0$. Let $X = e^{rx}$ be the solution

$$\Longrightarrow r^2e^{rx} - \alpha^2e^{rx} = 0 \Longrightarrow e^{rx}(r^2 - \alpha^2) = 0.$$

Since e^{rx} cannot be zero$\Longrightarrow (r^2 - \alpha^2)$ is equal zero.

$$\Longrightarrow r^2 - \alpha^2 = 0 \Longrightarrow r = \pm\alpha.$$

Hence,

$$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$$

Use boundary conditions to find c_1, c_2.

$$X(0) = 0 \Longrightarrow c_1 + c_2 = 0 \Longrightarrow c_1 = -c_2.$$
$$X(L) = 0 \Longrightarrow c_1 e^{\alpha L} + c_2 e^{-\alpha L} = 0.$$

Solving for c_1 and c_2 to get $c_1 = c_2 = 0$, thus

$$-c_2 e^{\alpha L} + c_2 e^{-\alpha L} = 0.$$
$$\Longrightarrow c_2 \underbrace{(e^{-\alpha L} - e^{\alpha L})}_{\neq 0} = 0 \Longrightarrow c_2 = 0 = c_1$$

Therefore, $X(x) = 0$, the trivial solution.
(ii) $\lambda = 0 \Longrightarrow X'' = 0$, with solution $X(x) = Ax + B$.
The boundary conditions imply

$$X(0) = 0 \Longrightarrow B = 0.$$
$$X(L) = 0 \Longrightarrow AL = 0 \Longrightarrow A = 0.$$

Again, we arrive at $X(x) = 0$, the trivial solution.
(iii) $\lambda < 0, \quad \lambda = -\alpha^2 \Longrightarrow X'' + \alpha^2 X = 0$, the solution is

$$X(x) = A\cos(\alpha x) + B\sin\alpha x.$$

To find A and B use $X(0) = X(L) = 0$,

$$X(0) = 0 \Longrightarrow A = 0.$$

$$X(x) = B\sin(\alpha x).$$

$$X(L)=0 \Longrightarrow B\sin\alpha L = 0, \quad \text{since} \quad B\neq 0,$$
$$\Longrightarrow \sin\alpha L = 0.$$
$$\alpha L = n\pi, \quad n\in\mathbb{N}.$$
$$\alpha_n = \frac{n\pi}{L}, \quad n\in\mathbb{N}.$$

$$X_n = B\sin\frac{n\pi x}{L}, \quad n\in\mathbb{N}.$$

(5) **The T-equation:** $T' + (\frac{an\pi}{L})^2 T = 0.$
Solve it to get

$$T(t) = ce^{-(\frac{an\pi}{L})^2 t}, \quad n\in\mathbb{N}.$$

(6)**Principle of superposition**

$$v_n(x,t) = X_n(x)T_n(t) = c_n\sin(\frac{n\pi x}{L})e^{-(\frac{an\pi}{L})^2 t}, \quad n\in\mathbb{N}.$$

$$v(x,t) = \sum_{n=1}^{\infty} c_n\sin(\frac{n\pi x}{L})e^{-(\frac{an\pi}{L})^2 t}.$$

(6) **Non-Homogeneous condition**

$$v(x,0) = g(x).$$

$$\sum_{n=1}^{\infty} c_n\sin(\frac{n\pi x}{L}) = g(x).$$

Since g is written in terms of sine series, then

$$c_n = \frac{2}{L}\int_0^L \sin(\frac{n\pi x}{l})dx.$$

3.6 Applications

Here, we will mention some applications of the heat equation [19]

1. Particle diffusion.
2. Brownian motion.

3. Frequently utilized in financial mathematics in modeling option such that the renowned Black Scholes model pricing options for the differential equation can be converted into heat equation allowing for comparatively simple solutions from the body common place to mathematics.

4. Describe the diffusion of pressure in a porous medium identical to the figure heat equation.

5. Utilizing at picture analysis.

6. In machine learning as a theory of leadership outside space -wide or Laplacian graph techniques.

Chapter 4

Error Analysis

In this chapter, we will talk about error analysis and clarifying the types of errors, after that, these types of errors are discussed for the heat equation.

4.1 Preface

There are two types of error estimates[2, 11]:

1. An a priori error estimate rely on the exact solution $v(x)$ and not on the approximation $v_h(x)$. In such estimates the error analysis are performed hypothetically and before calculations.

2. An a posteriori error estimate where the error relies upon the residual, i.e., the contract between the left and right hand sides in the equation when the exact solution $v(x)$ is supplanted by the approximate solution $v_h(x)$. A posteriori error evaluations can be determined after that the approximate solution is figured.

Beneath, we first demonstrate a general theorem which shows that the finite element solution is the best approximate solution.

Theorem 4.1. *Assume $v(x)$ is a solution of the Dirichlet boundary value problem.*

$$\begin{aligned} -(a(x)v'(x))' &= g(x), \qquad 0 < x < 1, \\ v(0) = v(1) &= 0, \end{aligned} \tag{4.1}$$

and $v_h(x)$ is the finite element approximation that satisfies

$$\int_0^1 a(x)v_h(x)w'(x)dx = \int_0^1 g(x)w(x)dx, \qquad \forall w \in W_h^L. \tag{4.2}$$

Then we have $\|v - v_h\|_E \leq \|v - w\|_E, \forall w \in W_h^L$. This implies that the finite element solution $v_h \in W_h^L$ is the best approximation of v among the functions of W_h^L, [2, 17].

Proof: Review the VF related to the problem,

$$\int_0^1 a(x)v'(x)w'(x)dx = \int_0^1 g(x)w(x)dx, \qquad \forall w \in H_0^1.$$

Take an arbitrary $w \in W_h^L$, then by definition of the energy norm($\|v\|\_E = (\int_a^b a(x)|v'(x)|^2dx)^{\frac{1}{2}}, a(x) > 0 \forall x \in [a, b]$,

$$\begin{aligned}
\|v - v_h\|_E^2 &= \int_0^1 a(x)(v'(x) - v_h'(x))^2dx \\
&= \int_0^1 a(x)(v'(x) - v_h'(x))(v'(x) \underbrace{-w'(x) + w'(x)}_{=0} - v_h'(x))dx \\
&= \int_0^1 a(x)(v'(x) - v_h'(x))(v'(x) - w'(x))dx + \int_0^1 a(x)(v'(x) - v_h'(x))(w'(x) - v_h'(x))dx.
\end{aligned}$$

Since $w - v_h \in W_h^L \subset H_0^1$, then by the Galerkin orthogonality

$$\int_0^1 a(x)(v'(x) - v_h'(x))w'(x)dx = 0, \qquad \forall w \in W_h^L,$$

we have

$$\|v - v_h\|_E^2 = \int_0^1 a(x)(v'(x) - v_h'(x))(v'(x) - w'(x))dx + \underbrace{\int_0^1 a(x)(v'(x) - v_h'(x))(w'(x) - v_h'(x))dx}_{=0}.$$

Thus,

$$\begin{aligned}\|v-v_h\|_E^2 =& \int_0^1 a(x)(v'(x)-v_h'(x))(v'(x)-w'(x))dx \\ =& \int_0^1 a^{\frac{1}{2}}(x)(v'(x)-v_h'(x))a^{\frac{1}{2}}(x)(v'(x)-w'(x))dx \\ \leq& (\int_0^1 a(x)(v'(x)-v_h'(x))^2dx)^{\frac{1}{2}}(\int_0^1 a(x)(v'(x)-w'(x))^2dx)^{\frac{1}{2}} \\ =& \|v-v_h\|_E\|v-w\|_E,\end{aligned}$$

where, in the last estimate, we utilized Cauchy- Schwarz inequality.
Consequently,

$$\|v-v_h\|_E \leq \|v-w\|_E, \forall w \in W_h^L.$$

Theorem 4.2. *"An a priori error estimate". Let v be the solution of the Dirichlet problem (4.1) and v_h be the corresponding finite element solution. Then there exists an interpolation constant c_i, depending only on $a(x)$ such that*

$$\|v-v_h\|)_E \leq c_i\|hv''\|_a, [2].$$

Proof: As indicated by the theorem beforehand

$$\|v-v_h\|_E \leq \|v-w\|_E, \forall w \in W_h^L.$$

But since $\pi_h v(x) \in W_h^L$ where $\pi_h v$ is the piecewise linear interpolation, then

$$\begin{aligned}\|v-v_h\|_E \leq& \|v-\pi_h v\|_E \\ =& \|v'-(\pi_h v)'\|_a \quad \text{since} \quad \|v\|_E = \|v'\|_a \\ \leq& c_i\|hv''\|_a,\end{aligned}$$

this is because $\|(\pi_h v')-w'\|_{L_p} \leq c_i\|hv''\|_{L_p}$, refers to theorem 2.2.
Below, we will discuss a posteriori error analysis, which depends on rather than the unknown solution $v(x)$, but on the residual of the figured approximate solution v_h.

Theorem 4.3. *"An a posteriori error estimate" There is an interpolation constant c_i depending only on $a(x)$ such that the error in the finite element approximation of the*

Dirichlet boundary value problem (4.1)satisfies

$$\|e(x)\|_E \leq c_i\left(\int_0^1 \frac{1}{a(x)} h^2(x) R^2(v_h(x)) dx\right)^{\frac{1}{2}},$$

where $R(v_h(x)) = g + (a(x)v_h'(x))'$ *is the residual and* $e(x) = v(x) - v_h(x)$*,[2].*

Proof: By the definition of the energy norm we have

$$\begin{aligned}
\|e(x)\|_E^2 &= \int_0^1 a(x)(e'(x))^2 dx \\
&= \int_0^1 a(x)(v'(x) - v_h'(x))e'(x) dx \\
&= \int_0^1 a(x)v'(x)e'(x) dx - \int_0^1 a(x)v_h'(x)e'(x) dx.
\end{aligned}$$

Since $e \in H_0^1$ the VF gives that

$$\int_0^1 a(x)v'(x)e'(x) dx = \int_0^1 g(x)e(x) dx.$$

Thus, we can write

$$\|e(x)\|_E^2 = \int_0^1 g(x)e(x) dx - \int_0^1 a(x)v_h'(x)e'(x) dx.$$

Adding and subtracting the interpolant $\pi_h e(x)$ and its derivative $(\pi_h e)'(x)$ to e and e' respectively in the integrands above returns

$$\|e(x)\|_E^2 = \int_0^1 g(x)(e(x) - \pi_h e(x)) dx + \underbrace{\int_0^1 g(x)\pi_h e(x) dx}_{i}$$

$$- \int_0^1 a(x)v_h'(x)(e'(x) - \pi_h e'(x)) dx - \underbrace{\int_0^1 a(x)v_h'(x)\pi_h e'(x) dx}_{ii}.$$

Since $\pi_h e(x) \in W_h^L$ and $v_h(x)$ is the FEM a solution, then

$$\int_0^1 a(x)v_h'(x)\pi_h e'(x) dx = \int_0^1 g(x)\pi_h e(x) dx \Longrightarrow i - ii = 0.$$

Thus,

$$\begin{aligned}\|e(x)\|_E^2 &= \int_0^1 g(x)(e(x)-\pi_h e(x))dx - \int_0^1 a(x)v_h'(x)(e'(x)-(\pi_h e)'(x))dx \\ &= \int_0^1 g(x)(e(x)-\pi_h e(x))dx - \sum_{k=1}^{m+1}\int_{x_{k-1}}^{x_k} a(x)v_h'(x)(e'(x)-(\pi_h e)'(x))dx.\end{aligned}$$

To proceed, we integrate by parts in the summation above,

$$\begin{aligned}-\int_{x_{k-1}}^{x_k} a(x)v_h'(x)(e'(x)-(\pi_h e)'(x))dx =& -\left(a(x)v_h'(x)(e(x)-(\pi_h e)(x)\right)_{x_{k-1}}^{x_k} + \\ &+\int_{x_{k-1}}^{x_k}(a(x)v_h'(x))'(e(x)-\pi_h e(x))dx.\end{aligned}$$

Using $e(x_k)=\pi_h e(x_k), k=1,...,m+1$, where the x_k's are the interpolation nodes, the boundary terms disappear and therefore we end up with

$$-\int_{x_{k-1}}^{x_k} a(x)v_h'(x)(e'(x)-(\pi_h e)'(x))dx = \int_{x_{k-1}}^{x_k}(a(x)v_h'(x))'(e(x)-\pi_h e(x))dx.$$

Thus, summing over k,to have

$$-\int_0^1 a(x)v_h'(x)(e'(x)-(\pi_h e)'(x))dx = \int_0^1 (a(x)v_h'(x))'(e(x)-\pi_h e(x))dx.$$

where $(a(x)v_h'(x))'$ ought to be explicated locally on each subinterval $[x_{k-1},x_k]$ (since v_h' in general is discontinuous , v_h'' does not exist globally on $[0,1]$).
Consequently ,

$$\begin{aligned}\|e(x)\|_E^2 &= \int_0^1 g(x)(e(x)-\pi_h e(x))dx + \int_0^1 (a(x)v_h'(x))'(e(x)-\pi_h e(x))dx \\ &= \int_0^1 (g(x)+(a(x)v_h'(x))')(e(x)-\pi_h e(x))dx.\end{aligned}$$

Let $R(v_h(x)) = g(x)+(a(x)v_h'(x))'$, i.e., $R(v_h(x))$ is the remainder, which is a well-defined function excluding in the set $\{x_k\}, k=1,...,m+1$; where is undefined.

By Cauchy-Schwarz inequality, the following estimate is obtained

$$\begin{aligned}
\|e(x)\|_E^2 &= \int_0^1 R(v_h(x))(e(x) - \pi_h e(x))dx \\
&= \int_0^1 \frac{1}{\sqrt{a(x)}} h(x)\sqrt{a(x)} R(v_h(x))(\frac{e(x) - \pi_h e(x)}{h(x)})dx \\
&\leq (\int_0^1 \frac{1}{a(x)} h^2(x) R^2(v_h)dx)^{\frac{1}{2}} (\int_0^1 a(x)(\frac{e(x) - \pi_h e(x)}{h(x)})^2 dx)^{\frac{1}{2}}.
\end{aligned}$$

Further, by the definition of the weighted L_2-norm

$$\|\frac{e(x) - \pi_h e(x)}{h(x)}\|_a = (\int_0^1 a(x)(\frac{e(x) - \pi_h e(x)}{h(x)})^2 dx)^{\frac{1}{2}}. \tag{4.3}$$

Equation (4.3) together with $\|\pi_h w - w\|_{Lp} \leq c_i \|hw'\|_{Lp}$ for $e(x)$ yield

$$\|\frac{e(x) - \pi_h e(x)}{h(x)}\|_a \leq c_i \|e'(x)\|_a = c_i \|e(x)\|_E,$$

where c_i as before depends on $a(x)$.
Thus,

$$\|e(x)\|_E^2 \leq (\int_0^1 \frac{1}{a(x)} h^2(x) R^2(v_h)dx)^{\frac{1}{2}} c_i \|e(x)\|_E.$$

Therefore,

$$\|e(x)\|_E \leq c_i (\int_0^1 \frac{1}{a(x)} h^2(x) R^2(v_h)dx)^{\frac{1}{2}}.$$

After the error estimate has been talked about in general, special talk will be devoted to the error in the heat equation.
In the first place we will discuss a priori error estimate of the heat equation.

4.2 A priori Error Estimate Of Heat Equation

The initial boundary value problem (IBVP):

$$\begin{aligned} \dot{v} - \Delta v = 0, \quad & \text{in} \quad \Omega \subset R^2 \quad (\text{or} \quad R^d, d = 1, 2, 3), \\ v = 0, \quad & \text{on} \quad \partial\Omega, \\ v(0, x) = v_0, \quad & \text{for} \quad x \in \Omega, \end{aligned} \tag{4.4}$$

we have the following results.

Theorem 4.4. *The IBVP (4.4) meets the stability estimates:*

1. $\|v\|(t) \le \|v_0\|$.
2. $\displaystyle\int_0^t \|\nabla v\|^2(r)dr \le \frac{1}{2}\|v_0\|^2$, [3].

Proof: Multiply (4.4) by v and integrates over Ω

$$\int_\Omega \dot{v}v dx - \int_\Omega (\Delta v)v dx = \int_\Omega 0(v)dx. \tag{4.5}$$

$$\Longrightarrow \int_\Omega \dot{v}v dx - \int_\Omega (\Delta v)v dx = 0.$$

Using Green's formula

$$-\int_\Omega (\Delta v)v dx = -\int_{\partial\Omega} (\nabla v.n)v ds + \int_\Omega \nabla v.\nabla v dx.$$

Since $v = 0$ on $\partial\Omega$ then $-\displaystyle\int_{\partial\Omega} (\nabla v.n)v ds = 0$, so

$$-\int_\Omega (\Delta v)v dx = \int_\Omega \nabla v.\nabla v dx.$$

Note that

$$\begin{aligned} \frac{d}{dt}(vv) =& v\dot{v} + \dot{v}v. \\ \Longrightarrow \frac{d}{dt}(v^2) =& 2v\dot{v}. \\ \Longrightarrow v\dot{v} =& \frac{1}{2}\frac{d}{dt}(v^2). \end{aligned}$$

Based on the above data , equation(4.5) can be written as

$$\int_\Omega \frac{1}{2}\frac{d}{dt}(v^2)dx + \int_\Omega |\nabla v|^2 dx = 0 \Longleftrightarrow \frac{1}{2}\frac{d}{dt}\|v\|^2 + \|\nabla v\|^2 = 0, \tag{4.6}$$

where $\|.\|$ denotes the $L_2(\Omega)$ norm.
Integrate (4.6) over $r \in (0,t)$ to obtain

$$\int_0^t \frac{1}{2}\frac{d}{dr}\|v\|^2(r)dr + \int_0^t \|\nabla v\|^2(r)dr = 0$$

$$\Longrightarrow \frac{1}{2}\|v\|^2(r)|_0^t + \int_0^t \|\nabla v\|^2(r)dr = 0$$

$$\Longrightarrow \frac{1}{2}\|v\|^2(t) - \frac{1}{2}\|v\|^2(0) + \int_0^t \|\nabla v\|^2(r)dr = 0.$$

Since $v(0) = v_0$, then we have

$$\frac{1}{2}\|v\|^2(t) - \frac{1}{2}\|v_0\|^2 + \int_0^t \|\nabla v\|^2(r)dr = 0.$$

Equivalently,

$$\|v\|^2(t) + 2\int_0^t \|\nabla v\|^2(r)dr = \|v_0\|^2.$$

The last equality implies

- $\|v\|(r) \leq \|v_0\|$ and
- $\int_0^t \|\nabla v\|^2(r)dr \leq \frac{1}{2}\|v_0\|^2.$

Thus, the theorem has been proved.

Theorem 4.5. *For the initial boundary value problem (4.4), the following estimates hold*

1. $\|\nabla v\|(t) \leq \frac{1}{\sqrt{2t}}\|v_0\|.$
2. $(\int_0^t r\|\Delta v\|^2(r)dr)^{\frac{1}{2}} \leq \frac{1}{2}\|v_0\|,$ *[3].*

Proof: Multiplying (4.4) by $-t\Delta v$ and integrate over Ω to get

$$\int_\Omega -t\dot{v}\Delta v dx + \int_\Omega t(\Delta v)^2 dx = 0.$$

$$-t\int_\Omega \dot{v}\Delta v dx + t\int_\Omega (\Delta v)^2 dx = 0. \tag{4.7}$$

Using Green's formula

$$\begin{aligned}\int_\Omega \dot{v}\Delta v dx &= \int_{\partial\Omega}(\nabla v.n)\dot{v}ds - \int_\Omega \nabla v.\nabla\dot{v}dx\\ &= -\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2.\end{aligned}$$

Thus, equation(4.7) can be simplified as

$$\frac{1}{2}t\frac{d}{dt}\|\nabla v\|^2 + t\|\Delta v\|^2 = 0. \tag{4.8}$$

Note that

$$\begin{aligned}\frac{d}{dt}(\frac{1}{2}t\|\nabla v\|^2) =& \frac{1}{2}t\frac{d}{dt}\|\nabla v\|^2 + \frac{1}{2}\|\nabla v\|^2\\ \frac{1}{2}t\frac{d}{dt}\|\nabla v\|^2 =& \frac{1}{2}\frac{d}{dt}(t\|\nabla v\|^2) - \frac{1}{2}\|\nabla v\|^2.\end{aligned}$$

Hence, equation(4.8) can be written as

$$\frac{1}{2}\frac{d}{dt}(t\|\nabla v\|^2) - \frac{1}{2}\|\nabla v\|^2 + t\|\Delta v\|^2 = 0.$$

$$\frac{d}{dt}(t\|\nabla v\|^2) + 2t\|\Delta v\|^2 = \|\nabla v\|^2.$$

Integration over $r \in (0,t)$ to get

$$\int_0^t \frac{d}{dr}(r\|\nabla v\|^2(r))dr + 2\int_0^t r\|\Delta v\|^2(r)dr = \int_0^t \|\nabla v\|^2(r)dr \leq \frac{1}{2}\|v_0\|^2,$$

where we have used Theorem(4.4). thus,

$$r\|\nabla v\|^2(r)|_0^t + 2\int_0^t r\|\Delta v\|^2(r)dr \leq \frac{1}{2}\|v_0\|^2$$

$$\Longrightarrow t\|\nabla v\|^2(t) - 0 + 2\int_0^t r\|\Delta v\|^2(r)dr \leq \frac{1}{2}\|v_0\|^2.$$

Consequently,

$$t\|\nabla v\|^2(t) + 2\int_0^t r\|\Delta v\|^2(r)dr \leq \frac{1}{2}\|v_0\|^2,$$

which implies

$$t\|\nabla v\|^2(t) \leq \frac{1}{2}\|v_0\|^2.$$

To get part 1 of the theorem, t is moved to the second side of the inequality, after which the square root of the two sides is taken to obtain

$$\|\nabla v\|(t) \leq \frac{1}{\sqrt{2t}}\|v_0\|,$$

For the second assertion we have

$$2\int_0^t r\|\Delta v\|^2(r)dr \leq \frac{1}{2}\|v_0\|^2.$$

This implies

$$(\int_0^t r\|\nabla v\|(r)dr)^{\frac{1}{2}} \leq \frac{1}{2}\|v_0\|.$$

Theorem 4.6. *For any $\varepsilon > 0$, the solution of the homogeneous IBVP (4.4) satisfies the estimate*

$$\int_\varepsilon^t \|\dot{v}\|(r)dr \leq \frac{1}{2}\sqrt{\ln\frac{t}{\varepsilon}}\|v_0\|, [3]. \tag{4.9}$$

Proof: It will be proven as follows

$$\begin{aligned}
\int_\varepsilon^t \|\dot{v}\|(r)dr &= \int_\varepsilon^t \|\nabla v\|(r)dr \quad \text{since} \quad \dot{v} - \nabla v = 0 \Longrightarrow \dot{v} = \nabla v. \\
&= \int_\varepsilon^t \frac{1}{\sqrt{r}}\sqrt{r}\|\nabla v\|(r)dr \\
&\leq (\int_\varepsilon^t \frac{1}{r}dr)^{\frac{1}{2}}(\int_\varepsilon^t r\|\nabla v\|^2(r)dr)^{\frac{1}{2}}, \text{ using Caushy Schwartz inequality.}
\end{aligned}$$

Use the previous theorem to get

$$\int_\varepsilon^t \|\dot{v}\|(r)dr \leq (\int_\varepsilon^t \frac{1}{r}dr)^{\frac{1}{2}}(\frac{1}{2}\|v_0\|).$$

The integral $(\int_{\varepsilon}^{t} \frac{1}{r})^{\frac{1}{2}} = (\ln(t) - \ln(\varepsilon))^{\frac{1}{2}}$, hence

$$\int_{\varepsilon}^{t} \|\dot{v}\|(r)dr \leq (\ln(t) - \ln(\varepsilon))^{\frac{1}{2}}(\frac{1}{2}\|v_0\|).$$

To match the final formula, use properties of ln to have

$$\int_{\varepsilon}^{t} \|\dot{v}\|(r)dr \leq \frac{1}{2}\sqrt{\ln \frac{t}{\varepsilon}}(\|v_0\|).$$

Theorem 4.7. *(The estimate for the gradient) For the IBVP (4.4) we have the stability estimate [3].*

$$\|\nabla v(t)\| \leq \|\nabla v_0\|.$$

Proof: To show this, multiply (4.4) by $(-\Delta v)$ and integrate over Ω,

$$\int_{\Omega} -\Delta v \dot{v} dx + \int_{\Omega} \Delta v \Delta v dx = 0.$$

Using Green's formula to obtain

$$\int_{\Omega} -\Delta v \dot{v} dx = -\int_{\partial\Omega} (\nabla v.n)\dot{v} ds + \int_{\Omega} \nabla v.\nabla \dot{v} dx,$$

so,

$$-\int_{\partial\Omega} (\nabla v.n)\dot{v} ds + \int_{\Omega} \nabla v.\nabla \dot{v} dx + \int_{\Omega} |\Delta v|^2 dx = 0. \tag{4.10}$$

Since $v = 0$ on $\partial\Omega$, all derivatives over it are equal 0 at $\partial\Omega \Longrightarrow \int_{\partial\Omega} (\nabla v.n)\dot{v} ds = 0.$
So,

$$\int_{\Omega} \nabla v.\nabla \dot{v} dx + \int_{\Omega} |\Delta v|^2 dx = 0.$$

Note that

$$\begin{aligned} \frac{d}{dt}(\nabla v.\nabla v) =& \nabla \dot{v}.\nabla v + \nabla v.\nabla \dot{v} \\ =& 2\nabla v.\nabla \dot{v} \\ \Longrightarrow \frac{1}{2}\frac{d}{dt}|\nabla v|^2 =& \nabla v.\nabla \dot{v}. \end{aligned}$$

Substitute the last equality in (4.10) to get

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \|\Delta v\|^2 = 0.$$

Integrate over $r \in (0, t)$:

$$\frac{1}{2}\int_0^t \frac{d}{dr}\|\nabla v\|^2(r)dr + \int_0^t \|\Delta v\|^2(r)dr = 0$$

$$\Longrightarrow \frac{1}{2}\|\nabla v\|^2(r)|_0^t + \int_0^t \|\Delta v\|^2(r)dr = 0$$

$$\Longrightarrow \frac{1}{2}\|\nabla v\|^2(t) - \frac{1}{2}\|\nabla v\|^2(0) + \int_0^t \|\Delta v\|^2(r)dr = 0.$$

Since $\|\nabla v\|^2(0) = \|\nabla v_0\|^2$, then

$$\Longrightarrow \frac{1}{2}\|\nabla v\|^2(t) + \int_0^t \|\Delta v\|^2(r)dr = \|\nabla v_0\|^2,$$

$$\Longrightarrow \|\nabla v(t)\|^2 \leq \|\nabla v_0\|^2.$$

Which is the desired estimate after taking the square root.

4.3 A posteriori Error Estimate Of Heat Equation

Theorem 4.8. *(A posteriori error estimates) We have the following a posteriori error estimate for the heat conductivity equation*

$$\begin{aligned} \dot{v} - v'' =& g, \quad 0 < x < 1, \quad t > 0, \\ v(0,t) =& v(1,t) = 0, \quad t > 0, \\ v(x,0) =& v_0(x), \quad 0 < x < 1. \end{aligned}$$

$$\|e(x)\| \leq (2 + \sqrt{\ln \frac{T}{\epsilon}}) \max_{[0,T]} \|(k + h^2)r(V)\|,$$

where $r(V) = g + V'' - \dot{V}$ is the residual, $T = t_m$, k and h are temporal and spatial mesh functions, respectively, and where V is the finite element approximation [2].

Proof : Let $\Psi(x,t)$ be the solution of the dual problem

$$\begin{aligned} -\dot{\Psi} - \Psi'' = 0, \quad &\text{in} \quad \Omega \quad t < T, \\ \Psi = 0, \quad &\text{on} \quad \partial\Omega \quad t < T, \\ \Psi = e(t), \quad &\text{in} \quad \Omega \quad \text{for} \quad t < T, \end{aligned} \tag{4.11}$$

where $e(t) = e(x,T) = v(x,T) - V(x,T)$. Changing of variables, and letting $u(x,s) = \Psi(x, T-s), (s > 0)$, problem (4.11)is rewritten as

$$\begin{aligned} -\dot{u} - u'' = 0, \quad &\text{in} \quad \Omega \quad t > 0, \\ u = 0, \quad &\text{on} \quad \partial\Omega \quad t > 0, \\ u = e, \quad &\text{in} \quad \Omega \quad \text{for} \quad s = 0. \end{aligned}$$

This problem is similar to the two-dimensional problem (4.9).

$$\int_{\varepsilon}^{T} \|\dot{u}ds\| \leq \frac{1}{2}\sqrt{\ln\frac{T}{\varepsilon}}\|e\|,$$

with $s = T - t \Longrightarrow ds = -dt$.
So,

$$\dot{u}(x,s) = -\dot{\Psi}(x, T-s).$$

Since $-\Psi'' = \dot{\Psi}$, we have for Ψ.

$$\int_{0}^{T-\varepsilon} \|\dot{\Psi}\|dt \leq \frac{1}{2}\sqrt{\ln\frac{T}{\varepsilon}}\|e\|,$$

and

$$\int_{0}^{T-\varepsilon} \|\Psi''\|dt \leq \frac{1}{2}\sqrt{\ln\frac{T}{\varepsilon}}\|e\|.$$

Let $v_0 \in W_h^L$, then since $-\dot{\Psi} - \Psi'' = 0$, integration by parts in t and x yileds.

$$\begin{aligned} \|e(T)\|^2 &= \int_{\Omega} e(T).e(T)dx + \int_{0}^{T}\int_{\Omega} e(-\dot{\Psi} - \Psi'')dxdt \\ &= \int_{\Omega} e(T).e(T)dx - \int_{0}^{T} e(T).\Psi(T)dx + \underbrace{\int_{0}^{T} e(0).\Psi(0)dx}_{=0} + \int_{0}^{T}\int_{\Omega}(\dot{e}\Psi - e'\Psi')dxdt. \end{aligned}$$

Using the Galerkin Orthogonality and integration by parts of x, we get for $w \in V_h^L$ (i.e., piecewise constant in time and continuous, piecewise linear in space),

$$\|e(T)\|^2 = \int_0^T \int_\Omega \dot{e}(\Psi - w) + e'(\Psi - w)' dxdt.$$

In 2^{nd} term integration by part of x to get

$$\begin{aligned}
&= \int_0^T \int_\Omega (\dot{e} - e'')(\Psi - w)dxdt + \int_0^T e' \underbrace{(\Psi - w)|_{\partial\Omega}}_{=0} dt. \\
&= \int_0^T \int_\Omega (g - \dot{V} - V'')(\Psi - w)dxdt. \\
&= \int_0^T \int_\Omega e(V)(\Psi - w)dxdt,
\end{aligned}$$

where using $\dot{e} = \dot{v} - \dot{V}$ and $e'' = v'' - V''$ to write $\dot{e} - e'' = \dot{v} - \dot{V} - v'' + V'' = g - \dot{V} + V'' := r(V)$, with mesh variables $h = h(x,t)$ and $k = k(t)$ in x and t respectively .
Can be derived an interpolation estimate of the form :

$$\|\Psi - w\|_{L_2} \leq k\|\dot{\Psi}\|_{L_2} + h^2\|\Psi''\|_{L_2} \leq (k + h^2)\|\dot{\Psi}\|_{L_2} + (k + h^2)\|\Psi''\|_{L_2}.$$

Summing up to have using maximum principle

$$\begin{aligned}
\|e(T)\|^2 &\leq \int_0^T \|(k + h^2)r(V)\|(\|\dot{\Psi}\| + \|\Psi\|). \\
&\leq \max_{[0,T]} \|(k + h^2)r(V)\|[\int_0^{T-\varepsilon} (\|\dot{\Psi}\| + \|\Psi\|) + 2 \max_{[T-\varepsilon,T]} \|\Psi\|]. \\
&\leq \max_{[0,T]} \|(k + h^2)r(V)\|(\sqrt{\ln \frac{T}{\varepsilon}}\|e\| + 2\|e\|).
\end{aligned}$$

The final estimate is

$$\|e(T)\| \leq (2 + \sqrt{\ln \frac{T}{\epsilon}}) \max_{[0,T]} \|(k + h^2)r(V)\|.$$

4.4 Steady State

As defined previously, the steady state assumes $v_t = 0$, thus the equation becomes as follows

$$0 = \alpha \Delta v + \frac{g}{c\rho} n(x, y, z, t).$$

In this way, the equation above becomes elliptic equation, which will be discussed the error analysis estimates to it.
At present, some theorems of error analysis estimates of elliptic equation will be discussed about.

Theorem 4.9. *Consider*

$$\begin{aligned} -\Delta v &= g, \quad in \quad \Omega, \\ v &= 0, \quad on \quad \partial\Omega. \end{aligned} \tag{4.12}$$

The following stability estimates hold [1]

1. $\|v\| \le A_\Omega \|\nabla v\|$.
2. $\|\nabla v\| \le A_\Omega \|g\|$, *where* A_Ω *is constant.*

Proof : Let Ψ be a function such that $\Delta \Psi = 1$ in Ω and $2\|\nabla \Psi\| \le A_\Omega$ in Ω (such a function exists)

$$\|v\|^2 = \int_\Omega v^2 \Delta \Psi dx.$$

Using the Green's formula of $\int_\Omega v^2 \Delta \Psi dx$ to get

$$\begin{aligned} \int_\Omega v^2 \Delta \Psi dx &= \int_{\partial\Omega} v^2 n.\nabla \Psi ds - \int_\Omega \nabla(v^2).\nabla \Psi dx, \quad \text{the first term} = 0 \quad \text{since} \quad v = 0 \quad on \quad \partial\Omega \\ &= - \int_\Omega \nabla v^2 . \nabla \Psi dx \\ &= - \int_\Omega 2v \nabla v . \nabla \Psi dx, \quad \text{since} \quad \nabla(v^2) = \nabla(vv) = (\nabla v)v + v(\nabla v) = 2v\nabla v \\ &\le 2\|\nabla v\| \|v\| \|\nabla \Psi\| \\ &\le A_\Omega \|\nabla v\| \|v\|, \quad \text{since} \quad 2\|\nabla \Psi\| \le A_\Omega. \end{aligned}$$

Therefore,

$$\|v\| \leq A_\Omega \|\nabla v\|.$$

The proof of (1) is done.
To prove the second assertion we multiply Equation (4.12) by v and integrate over Ω to get

$$-\int_\Omega (\Delta v) v dx = \int_\Omega g v dx.$$

Using Green's formula of the first term to obtain

$$-\int_{\partial\Omega} v(n.\nabla v) ds + \int_\Omega \nabla v.\nabla v dx = \int_\Omega g v dx,$$

the first term= 0 since $v = 0$ on $\partial\Omega$.

$$\Longrightarrow \int_\Omega \nabla v.\nabla v dx = \int_\Omega g v dx.$$

$$\Longrightarrow ((\int_\Omega |\nabla v|^2 dx)^{\frac{1}{2}})^2 = \int_\Omega g v dx.$$

$$\Longrightarrow \|\nabla v\|^2 = \int_\Omega g v dx \leq \|g\|\|v\|.$$

Using the first assertion of the theorem $\|v\| \leq A_\Omega \|\nabla v\|$, we have

$$\|\nabla v\|^2 \leq \|g\| A_\Omega \|\nabla v\|.$$

Therefore,

$$\|\nabla v\| \leq A_\Omega \|g\|.$$

Remark 4.1. *Theorem* (4.9) *also implies that* $\|v\| \leq A_\Omega^2 \|g\|$.

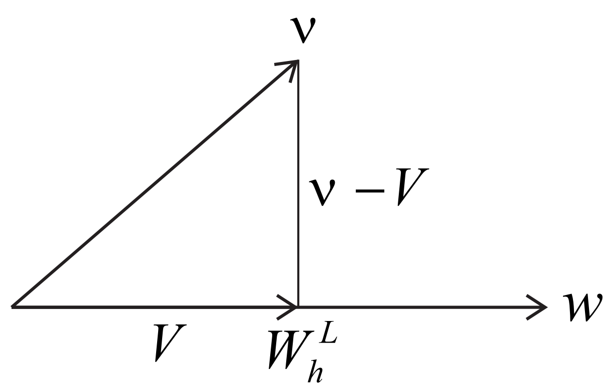
Theorem 4.10. *For problem (4.12), the following stability estimate holds*

$$\|\nabla(v - V)\| \leq \|\nabla(v - w)\|, \forall w \in W_h^L,$$

where v is the finite element solution [1]. This means that V is closer to v than any other $w \in W_h^L$.

Proof : By the Green's formula

$$\int_\Omega \nabla v.\nabla w dx = \int_\Omega g w dx, \forall w \in W_h^L. \tag{4.13}$$



The VF : Find $v(x)$ such that $v(x) = 0$ on $\partial\Omega$ and

$$\int_\Omega \nabla v.\nabla w dx = \int_\Omega g w dx, \forall w \in W_h^L. \tag{4.14}$$

Let $V \in W_h^L$ be an approximation of v, then

$$\int_\Omega \nabla V.\nabla w dx = \int_\Omega g w dx, \forall w \in W_h^L. \tag{4.15}$$

Subtract equation (4.15) from equation (4.14) to get

$$\int_\Omega (\nabla v - \nabla V).\nabla w dx = 0. \tag{4.16}$$

The error $e = v - V \implies \nabla e = \nabla v - \nabla V = \nabla(v - V)$.
From (4.16) we get

$$\int_\Omega (\nabla v - \nabla V).\nabla w dx = \int_\Omega \nabla e.\nabla w dx = 0, \forall w \in W_h^L, \quad \text{(the Galerkin Orthogonality).} \tag{4.17}$$

Now,

$$\begin{aligned} \|\nabla e\|^2 &= \int_\Omega \nabla e.\nabla e dx \\ &= \int_\Omega \nabla e.\nabla(v - V) dx, \quad \text{since} \quad \nabla e = \nabla(v - V) \\ &= \int_\Omega \nabla e.\nabla v dx - \int_\Omega \nabla e.\nabla V dx. \end{aligned}$$

Using (4.17) to get

$$\|\nabla e\|^2 = \int_\Omega \nabla e.\nabla v dx.$$

$$\Longrightarrow \|\nabla e\|^2 = \int_\Omega \nabla e.\nabla v dx - \int_\Omega \nabla e.\nabla w dx,$$

as $\int_\Omega \nabla e.\nabla w dx = 0$ for $w \in W_h^L$.
Taking a common factor from the two integrals to obtain

$$\begin{aligned}\|\nabla e\|^2 =& \int_\Omega \nabla e.\nabla (v-w) dx \\ \leq& \|\nabla e\| \|\nabla (v-w)\|.\end{aligned}$$

Thus,

$$\|\nabla (v-V)\| \leq \|\nabla (v-w)\|, \forall w \in W_h^L. \tag{4.18}$$

Theorem 4.11. *The finite element approximation V satisfies (4.18). In particular, there is a constant A such that*

$$\|\nabla (v-V)\| \leq A\|hD^2 v\|,$$

where $D^2 v = (v_{xx}^2 + v_{yy}^2 + v_{zz}^2)^{\frac{1}{2}}$, [2].

Proof : see [2].
At this moment, we will find an error estimate for $e = v - V$.

Theorem 4.12. *(A priori error estimate for the solution $e = v - v_h$). For a general mesh we have the following a priori error estimate for the solution of the problem (4.12)*

$$\|e\| \leq A^2 A_\Omega \max_\Omega h \|hD^2 v\|. \tag{4.19}$$

where A is a constant (generated twice) [2].

Proof : Assume Ψ be a solution of the dual problem

$$\begin{aligned}-\Delta \Psi &= e, \quad \text{in} \quad \Omega, \\ \Psi &= 0, \quad \text{on} \quad \partial\Omega.\end{aligned}$$

Using Green's formula

$$\begin{aligned}\|e\|^2 =& \int_\Omega e(-\Delta\Psi)dx \\ =& \int_\Omega \nabla e.\nabla\Psi dx \\ =& \int_\Omega \nabla e.\nabla(\Psi - w)dx \\ \leq& \|\nabla e\|\|\nabla(\Psi - w)\|, \forall w \in W_h^L,\end{aligned}$$

where in the last equality we have used the Galerkin orthogonality. We now choose w (e.g. as an interpolant of Ψ) such that

$$\|\nabla(\Psi - w)\| \leq A\|hD^2\Psi\| \leq A(h)\|D^2\Psi\|.$$

By use the regularity lemma which is (let that Ω has no re-intrents, we have for $v \in H^2(\Omega)$ with $v = 0$ or $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$, that

$$\|D^2v\| \leq a_\Omega\|\Delta v\|.$$

), see to proof the lemma in [2]. Applying theorem 4.11 and $\|\Delta v\| = e$ to get

$$\|D^2\Psi\| \leq A_\Omega|\Delta\Psi\| = A_\Omega\|e\|.$$

Finally,

$$\begin{aligned}\|e\| \leq& \|\nabla e\|\|\nabla(\Psi - w)\| \leq \|\nabla e\|Ah\|D^2\Psi\| \\ \leq& \|\nabla e\|A(h)A_\Omega\|e\| \leq A\|hD^2v\|Ah\|D^2\Psi\|.\end{aligned}$$

Thus we have obtained the desired a priori error estimate

$$\|e\| = \|v - V\| \leq A^2A_\Omega \max_\Omega h\|hD^2v\|. \tag{4.20}$$

Corollary 4.1. *(strong stability estimate). Using the regularity Lemma, for a piecewise linear approximation, the a priori error estimate (4.20) can be written as the following* *strong stability estimate,[2, 16]*

$$\|e\| = \|v - V\| \leq A^2 A_\Omega (\max_\Omega h)^2 \|g\|.$$

All previous theorems talked about of a priori error estimate but now will be talked about posteriori error estimate.

Theorem 4.13. *Let v be the solution of the problem*

$$\begin{aligned} -\Delta v &= g, \quad in \quad \Omega, \\ v &= 0, \quad on \quad \partial\Omega, \end{aligned}$$

and V be the finite element approximation, then there is a constant A independent of v and h such that

$$\|e\| \leq A\|h^2 r(v)\|,$$

where $r(v) = g + \Delta V$ is the residual, and h is spatial mesh function, [1].

Proof : Consider the dual problem

$$\begin{aligned} -\Delta\Psi(x) &= e(x), \quad x \in \Omega, \\ \Psi &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Note that $e(x) = 0, \forall x \in \partial\Omega$, since $e = -\Delta\Psi$ and $\Psi = 0$, $\forall x \in \partial\Omega$. By Green's formula

$$\begin{aligned} \|e\|^2 = \int_\Omega e e dx &= -\int_\Omega e(\Delta\Psi) dx \\ &= \underbrace{-\int_{\partial\Omega} e(n.\nabla\Psi) ds}_{=0} + \int_\Omega \nabla e.\nabla\Psi dx \\ &= \int_\Omega \nabla e.\nabla\Psi dx. \end{aligned}$$

Then, by the Galerkin Orthogonality we get

$$\begin{aligned}\|e\|^2 &= \int_{\Omega} \nabla e.\nabla \Psi dx - \int_{\Omega} \nabla e.\nabla w dx, \forall w \in W_h^L \\ &= \int_{\Omega} \nabla e.\nabla(\Psi - w)dx \\ &= \int_{\Omega} (-\Delta e)(\Psi - w)dx (\text{by Green's formula}).\end{aligned}$$

But $-\Delta e = -\Delta(v - V) = -\Delta v + \Delta V = g + \Delta V = r(v)$,
where r is the residual and v is an interpolant of Ψ, so

$$\|e\|^2 \leq \|h^2 r(v)\| \|h^{-2}(\Psi - w)\|.$$

Using the inequality

$$\|(\Psi - v)\| \leq A\|h^2 D^2 \Psi|| \leq AA_{\Omega}\|\Delta\Psi\|,$$

where A and A_{Ω} are constants, we get

$$\begin{aligned}\|e\|^2 \leq & AA_{\Omega}\|h^2 r(v)\| \|\Delta\Psi\| \\ \leq & AA_{\Omega}\|h^2 r(v)\| \|e\|.\end{aligned}$$

Therefore, the final a posteriori error estimate is

$$\|e\| \leq a\|h^2 r(v)\|.$$

Chapter 5

Computation Of Heat Equation

In this chapter, the computation of FEM for heat equation in one and two dimensions is discussed using the MATLAB program.

5.1 One Dimensional Problems

In this section, several examples are discussed to illustrate the use of the FEM for heat equation in one dimension.

Example 5.1. *Consider the IBVP*

$$\begin{aligned} v_t - v_{xx} &= 0, \quad 0 < x < 10, \quad t > 0, \\ v(x,0) &= \sin(\frac{\pi x}{2}) + 5x + 20, \quad 0 < x < 10, \\ v(0,t) &= 20, \quad t > 0, \\ v(10,t) &= 70, \quad t > 0. \end{aligned}$$

Figure 5.1 the approximations with $m = 5$ and $m = 20$ at $mm = 6$, where m is the number of subintervals for the spatial variable, and mm is the number of subintervals for the time. We study the solution of the problem at the last value of time, T, because we care about the final result of the heat transfer. The exact solution $v(x,t) = e^{-(\frac{\pi}{2})^2 t} \sin(\frac{\pi x}{2}) + 5x + 20$ and the approximation are depicted on the same graph. The exact solution (the solid line) and the FEM solution (the stars) are shown in Figure 5.1.

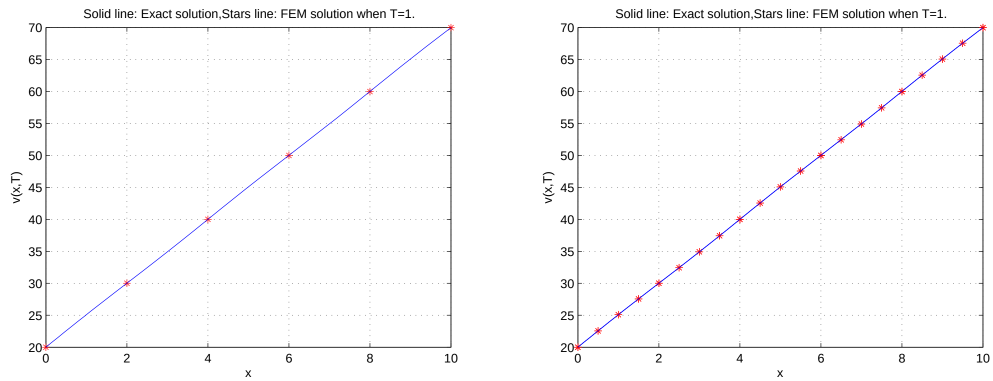


Figure 5.1: (Left) $m = 5$ (Right) $m = 20$ at $mm = 6$ using Backward Euler method

From the Figure 5.1, it seems that the approximation is closed to the exact solution at the nodal points. Figure 5.1 (Right) shows the computation with 20 nodal points. Note that when m increases then the approximate solution becomes more closed to the exact one.

Example 5.2. *Consider the IBVP*

$$\begin{gathered} v_t - 4v_{xx} = -x, \quad 0 < x < 4, \quad t > 0, \\ v(x,0) = \sin(\frac{3\pi x}{4}) + \frac{x^3}{24} + \frac{x}{3} + 2, \quad 0 < x < 4, \\ v(0,t) = 2, v(4,t) = 6, \quad t > 0. \end{gathered}$$

Figure 5.2 is the approximations with $m = 10$, $m = 20$ and $m = 30$ respectively at $mm = 6$ using Backward Euler Method, depicted with the exact solution $v(x,t) = e^{-4\pi^2 t}\sin(\pi x) + \frac{x^3}{24} + \frac{x}{3} + 2$. From these figures it seems that increasing the number of nodal points for the spatial variable gives better approximation using the Back Euler method, but, unfortunately, this is not the case, see Tables 5.1, 5.2, and 5.3.
At some nodal points, with increasing of the number of nodes, the error increases, which should not be. For that we search for a new method to treat the iteration of time instead of the Backward Euler method, the proposed stable method is the Theta method. Figure 5.3 shows the solution of the problem of Example 5.2 using the Theta method for different numbers of nodal points.
While comparing between Figures 5.2 and 5.3, you do not notice the difference between them or the difference in the drawing, but when you check the errors as in Tables 5.1, 5.2,

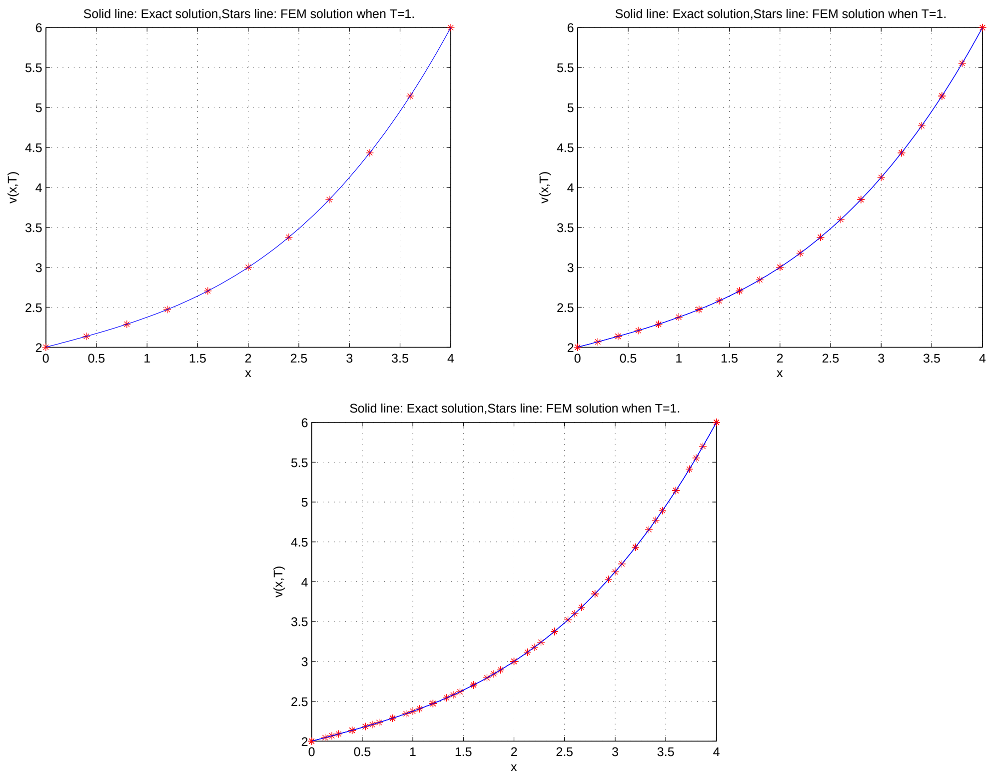


Figure 5.2: (Left) $m = 10$ (Right) $m = 20$ (Center) $m = 30$ at $mm = 6$ using Backward Euler Method

5.3, 5.4, 5.5, and 5.6, you notice that the Theta method is better than the Backward Euler Method. These drawings describe the rate of temperature change in a bar so thin and long that it is possible to turn a blind eye to the transfer of heat in other dimensions as a small result its effect. The equation is given according to the above-mentioned formula, which is derived from Fourier's law and the law of energy conservation. We note that the application of an accurate solution and an approximate solution determines the shape of the drawing whether it is a curve or a straight line, and we also know that the more x increases, v increases by fixing the value of T, meaning that the relationship is direct, as is evident when looking at the drawings.

Upon testing the Theta method, it turns out that it is better than the Backward Euler method.

Now we will talk a little about it before it is used. Consider the discretized system

$$M\dot{\xi} + K\xi = 0, \tag{5.1}$$

x	exact solution	approximation solution	$\|exact - approximation\|$
0.4	2.136	2.136010615515819	0.000010615515819
0.8	2.288	2.288012479287289	0.000012479287289
1.2	2.472	2.472004054766239	0.000004054766239
1.6	2.704	2.703992287376309	0.000007712623692
2	3	2.999986878500841	0.000013121499159
2.4	3.376	3.375992287376309	0.000007712623691
2.8	3.848	3.848004054766240	0.000004054766240
3.2	4.432	4.432012479287290	0.000012479287290

Table 5.1: Absolute error with $m = 10$, $mm = 6$ using the Backward Euler Method

x	exact solution	approximation solution	$\|exact - approximation\|$
0.4	2.136	2.136014397040309	0.000014397040309
0.8	2.288	2.288016924735940	0.000016924735940
1.2	2.472	2.472005499180056	0.000005499180056
1.6	2.704	2.703989539937930	0.000010460062070
2	3	2.999982204279493	0.000017795720507
2.4	3.376	3.375989539937927	0.000010460062073
2.8	3.848	3.848005499180053	0.000005499180053
3.2	4.432	4.432016924735935	0.000016924735935

Table 5.2: Absolute error with $m = 20$, $mm = 6$ using the Backward Euler Method

x	exact solution	approximation solution	$\|exact - approximation\|$
0.4	2.136	2.136015236143897	0.000015236143897
0.8	2.288	2.288017911161377	0.000017911161378
1.2	2.472	2.472005819689138	0.000005819689138
1.6	2.704	2.703988930293547	0.000011069706453
2	3	2.999981167090494	0.000018832909506
2.4	3.376	3.375988930293564	0.000011069706435
2.8	3.848	3.848005819689167	0.000005819689167
3.2	4.432	4.432017911161406	0.000017911161406

Table 5.3: Absolute error with $m = 30$, $mm = 6$ using the Backward Euler Method

where ξ is a vector function (unknowns) and $\dot{\xi}$ is the time derivative of ξ. For the time discretization of the system of Ordinary Differential Equation (5.1) we apply the well-known Theta-method, which results in the equation

$$M\frac{\xi^{m+1} - \xi^{m}}{\Delta t} + K(\theta\xi^{m+1} + (1-\theta)\xi^{m}) = 0.$$

Obviously, this is a system of linear algebraic equations, the unknown vector ξ^{m+1} being the approximation of the temperature at the new time-level. Here, the parameter θ is related to the applied numerical method and it is arbitrary on the interval $[0, 1]$, for more details see [22].

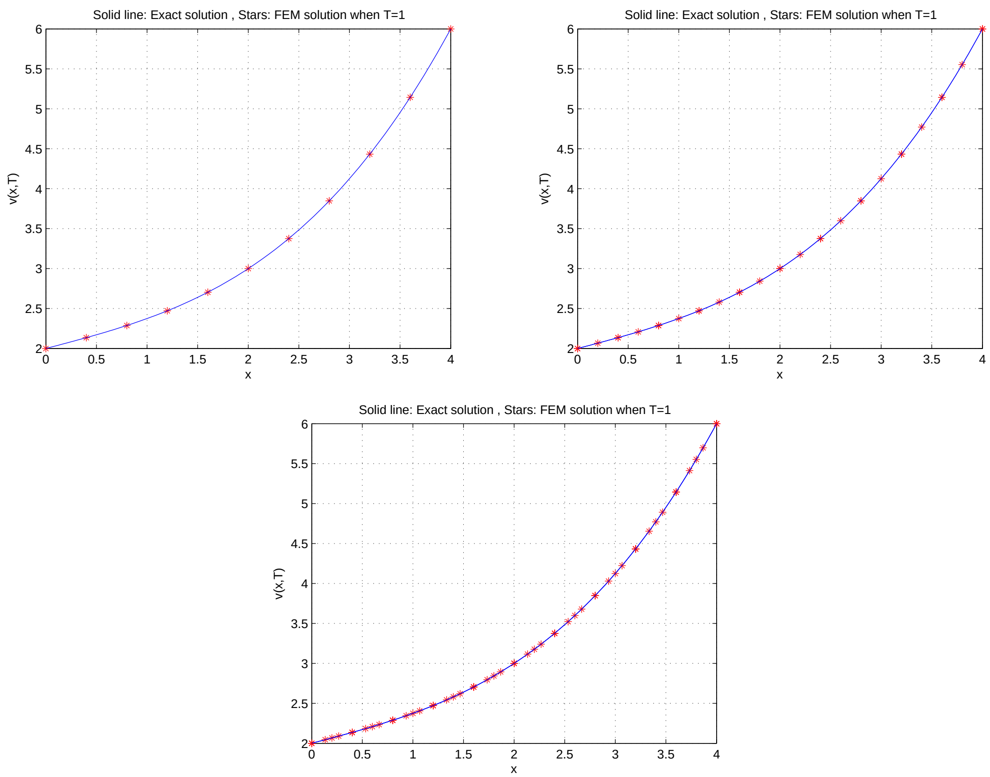


Figure 5.3: (Left) $m = 10$ (Right) $m = 20$ (Center) $m = 30$ at $mm = 6$ using Theta Method in the FEM, $\theta = 0.7$

x	exact solution	approximation solution	$\lvert exact - approximation \rvert$
0.4	2.136	2.135999999232892	0.000000000767108
0.8	2.288	2.287999999098211	0.000000000901789
1.2	2.472	2.471999999706990	0.000000000293010
1.6	2.704	2.704000000557336	0.000000000557336
2	3	3.000000000948196	0.000000000948196
2.4	3.376	3.376000000557335	0.000000000557335
2.8	3.848	3.847999999706989	0.000000000293011
3.2	4.432	4.431999999098209	0.000000000901792

Table 5.4: Absolute error with $m = 10$, $mm = 6$ using Theta Method in the FEM, $\theta = 0.7$

x	exact solution	approximation solution	$\|exact - approximation\|$
0.4	2.136	2.135999999936535	0.000000000063465
0.8	2.288	2.287999999925393	0.000000000074607
1.2	2.472	2.471999999975759	0.000000000024241
1.6	2.704	2.704000000046112	0.000000000046112
2	3	3.000000000078449	0.000000000078449
2.4	3.376	3.376000000046111	0.000000000046112
2.8	3.848	3.847999999975761	0.000000000024239
3.2	4.432	4.431999999925394	0.000000000074606

Table 5.5: Absolute error with $m = 20$, $mm = 6$ using Theta Method in the FEM, $\theta = 0.7$

x	exact solution	approximation solution	$\|exact - approximation\|$
0.4	2.136	2.135999999964937	0.000000000035063
0.8	2.288	2.287999999958788	0.000000000041212
1.2	2.472	2.471999999986628	0.000000000013372
1.6	2.704	2.704000000025510	0.000000000025510
2	3	3.000000000043380	0.000000000043380
2.4	3.376	3.376000000025505	0.000000000025505
2.8	3.848	3.847999999986622	0.000000000013378
3.2	4.432	4.431999999958785	0.000000000041216

Table 5.6: Absolute error with $m = 30$, $mm = 6$ using Theta Method in the FEM, $\theta = 0.7$

We notice that when holding m and increasing the value of mm gives very little error compared to increasing the value of m and holding the value of mm as shown in Tables 5.4-5.9.

x	exact solution	approximation solution	$\|exact - approximation\|$
0.4	2.136	2.135959353049814	0.000040646950186
0.8	2.288	2.287952216644265	0.000047783355735
1.2	2.472	2.471984474246577	0.000015525753423
1.6	2.704	2.704029531737956	0.000029531737956
2	3	3.000050242393522	0.000050242393522
2.4	3.376	3.376029531737959	0.000029531737959
2.8	3.848	3.847984474246582	0.000015525753418
3.2	4.432	4.431952216644268	0.000047783355733

Table 5.7: Absolute error with $mm = 4$, $m = 30$ using Theta Method in the FEM, $\theta = 0.7$

We also know that the value of θ ranges between 0 and 1 and at $\theta = 0$ is called the

x	exact solution	approximation solution	$\|exact - approximation\|$
0.4	2.136	2.135999999964937	0.000000000035063
0.8	2.288	2.287999999958788	0.000000000041212
1.2	2.472	2.471999999986628	0.000000000013372
1.6	2.704	2.704000000025510	0.000000000025510
2	3	3.000000000043380	0.000000000043380
2.4	3.376	3.376000000025505	0.000000000025505
2.8	3.848	3.847999999986622	0.000000000013378
3.2	4.432	4.431999999958785	0.000000000041216

Table 5.8: Absolute error with $mm = 6$, $m = 30$ using Theta Method in the FEM, $\theta = 0.7$

x	exact solution	approximation solution	$\|exact - approximation\|$
0.4	2.136	2.136000000003277	0.000000000003277
0.8	2.288	2.288000000003861	0.000000000003861
1.2	2.472	2.472000000001280	0.000000000001280
1.6	2.704	2.703999999997667	0.000000000002333
2	3	2.999999999996008	0.000000000003992
2.4	3.376	3.375999999997672	0.000000000002328
2.8	3.848	3.848000000001286	0.000000000001286
3.2	4.432	4.432000000003868	0.000000000003867

Table 5.9: Absolute error with $mm = 8$, $m = 30$ using Theta Method in the FEM, $\theta = 0.7$

	error			
x	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$	$\theta = 0.8$
0.4	0.003838578317333	0.000138708277230	0.0000007992154924	0.000000000000481
0.8	0.004512519449396	0.000163061359455	0.000000939534159	0.000000000000566
1.2	0.001466206448743	0.000052981847381	0.000000305273153	0.000000000000183
1.6	0.002788890394621	0.000100777462394	0.0000005806640444	0.000000000000351
2	0.004744743737180	0.000171452859699	0.000000987884676	0.000000000000596
2.4	0.002788890394621	0.000100777462394	0.000000580664044	0.000000000000349
2.8	0.001466206448744	0.000052981847383	0.000000305273153	0.000000000000184
3.2	0.004512519449396	0.000163061359456	0.000000939534157	0.000000000000566

Table 5.10: Study of θ and its effect, where $m = 10, mm = 5$

Forward Euler method, and at $\theta = 1$ it is called the Backward Euler method. Therefore, a study for the values of θ between 0 and 1 is considered as shown in Table 5.10. Several values of θ are tested to reach the optimal value which makes the error as minimum as possible. It is found that the best values of θ should live in the interval $[0.65, 0.8]$.

Theta method in the FEM is more stable than the Backward Euler method: Increasing number of nodal points provides better convergence to the exact solution, i.e., less error, but this is not the case in this example (Example 5.2) with Backward Euler method. For that we get used of the Theta method in the FEM which gives the desired results.

error		
x	Theta Method in the FEM	Backward Euler Method
0.4	0.000015622268134	0.000021724166051
0.8	0.000018365077633	0.000025538288845
1.2	0.000005967175445	0.000008297893051
1.6	0.000011350242185	0.000015783530526
2	0.000019310185378	0.000026852545998
2.4	0.000011350242184	0.000015783530525
2.8	0.000005967175446	0.000008297893054
3.2	0.000018365077634	0.000025538288846

Table 5.11: Comparison between Backward Euler Method and Theta method in the FEM, where $m = 10, mm = 5$, and $\theta = 0.65$.

When digging deep behind the reason of that, we find that the number of subintervals for the time, Δt, affects the solution of heat equation, see [22] with homogeneous Dirichlet Boundary Conditions.

Also, there is a similar study for diffusion problems see [18], [23]. This means that, careful study should be devoted for the suitable time discretization.

For Examples 1 and 2, the MATLAB codes are appended to the thesis, see the appendix.

5.2 Two Dimensional Problems

In this section, we discuss the computation of the two dimensional problems. Consider the following time dependent model problem

$$\begin{aligned} v_t - \nabla.(a\nabla v) = g, &\quad x = (x_1, x_2) \in \Omega \times (0, T], \\ v = 0, &\quad \text{on} \quad \partial\Omega \times (0, T], \\ v(., 0) = v_0(x), &\quad \text{on} \quad \Omega, \end{aligned} \tag{5.2}$$

where $v(x, t) = v(x_1, x_2, t)$ is the unknown function that we wish to compute, the functions $a = a(x, t)$, and $g = g(x, t)$ are data to the problem.

Multiply the differential equation (5.2) by a test function $w(x_1, x_2)$ such that $w = 0$ on $\partial\Omega$ and integrate over Ω,

$$\iint_\Omega v_t w dx_1 dx_2 - \iint_\Omega \nabla.(a\nabla v) w dx_1 dx_2 = \iint_\Omega f w dx_1 dx_2, \quad 0 < t \leq T.$$

Integrates by parts using Green's formula to get

$$\iint_\Omega v_t w dx_1 dx_2 - \int_{\partial\Omega} (n.(a(x,t)\nabla v) w ds + \iint_\Omega (a\nabla v).\nabla w dx_1 dx_2 = \iint_\Omega f w dx_1 dx_2, \quad 0 < t \leq T.$$

As v is given on $\partial\Omega$, then $w = 0$ on $\partial\Omega$, thus

$$\iint_\Omega v_t w dx_1 dx_2 + \iint_\Omega (a(x,t)\nabla v).\nabla w dx_1 dx_2 = \iint_\Omega f w dx_1 dx_2, \quad 0 < t \leq T. \tag{5.3}$$

VF: Find $v(x_1, x_2, t)$ such that for every fixed t : $v(x_1, x_2, t) \in H_0^1(\Omega)$ and (2.22) holds $\forall w \in H_0^1(\Omega)$.

Discretization: Introducing the vector space W_h^L of continuous piecewise linear functions on a triangulation of Ω, see [4, 14, 16, 20, 21].

We construct a set of basis which called the tent functions $\{\Psi_j\}_{j=1}^n \subset W_h^L$, where

$$\Psi_j(N_i) = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

for $i, j = 1, ..., n$, N_i are the nodes in the generated triangulation, and n refers to the number of the interior nodes in the given triangulation.

Now, $\forall v(x_1, x_2) \in W_h^L$, $v(x_1, x_2)$ can be written as a unique linear combination of Ψ_j's, i.e., $v(x_1, x_2, t) = \sum_{j=1}^n \xi_j(t)\Psi_j(x_1, x_2)$.

For the construction of the discrete system of linear equations, we substitute $v(x_1, x_2, t) = \sum_{j=1}^n \xi_j(t)\Psi_j(x_1, x_2)$ and take $w = \Psi_i, i = 1, ..., n$ in (5.3)

$$\begin{aligned} &\sum_{j=1}^n \dot{\xi}_j(t)(\iint_\Omega \Psi_j(x_1, x_2)\Psi_i(x_1, x_2) dx_1 dx_2) + \\ &+ \textstyle\sum_{j=1}^n \xi_j(\iint_\Omega a(x,t)\nabla\Psi_j(x_1, x_2).\nabla\Psi_i(x_1, x_2) dx_1 dx_2) \\ &= \iint_\Omega f(x,t)\Psi_i(x_1, x_2) dx_1 dx_2, \quad i = 1, ..., n, \quad 0 < t \leq T. \end{aligned} \tag{5.4}$$

Introduce the notation

$$M = (m_{ij})_{ij=1}^{n}, \text{where} \quad m_{ij} = \iint_{\Omega} \Psi_j(x_1, x_2)\Psi_i(x_1, x_2)dx_1dx_2.$$

$$A = (a_{ij})_{ij=1}^{n}, \text{where} \quad a_{ij}(t) = \iint_{\Omega} a(x,t)\nabla\Psi_j(x_1, x_2).\nabla\Psi_i(x_1, x_2)dx_1dx_2.$$

$$D = (d_i(t))_{i=1}^{n}, \text{where} \quad d_i(t) = \iint_{\Omega} f(x,t)\Psi_i(x_1, x_2)dx_1dx_2.$$

In the system of matrices, equation (5.4) can be written as

$$M\dot{\xi}(t) + A(t)\xi(t) = D(t), \quad 0 < t \leq T. \tag{5.5}$$

Approximating equation (5.5) using Backward Euler method

$$M(\xi^m - \xi^{m-1}) + A(t_m)\xi^m k_m = D(t_m)k_m,$$

where ξ^m is $\xi(t_m)$. Simplify the above equation provides

$$\xi^m = \frac{D_m k_m + M\xi^{m-1}}{M + k_m A_m}, \tag{5.6}$$

where $A_m = A(t_m)$ and $D_m = D(t_m)$.
Now we will give a practical example and explain the solution through the drawings below.
Relating to equation(5.2), let $\Omega = [0,1] \times [0,1]$, $T = \frac{\pi}{2}$, $x = (x_1, x_2)$, $v_0 = 0$, and $f(x,t) = x_1(1-x_1)x_2(1-x_2)cos(t) + 2(x_1(1-x_1) + x_2(1-x_2))sin(t)$. The corresponding exact solution is $v(x,t) = x_1(1-x_1)x_2(1-x_2)sin(t)$.

It is clear from the Figures 5.5 (Right) and 5.7 (Right) that refining the mesh gives better approximation mation. This is reasonable because the more mesh there are, the less error. Note that the infinity norm is defined by

$$\|e(T)\|_\infty = \|V(T) - v(T)\|_\infty = \max_{x\in\Omega} |V(x,T) - v(x,T)|.$$

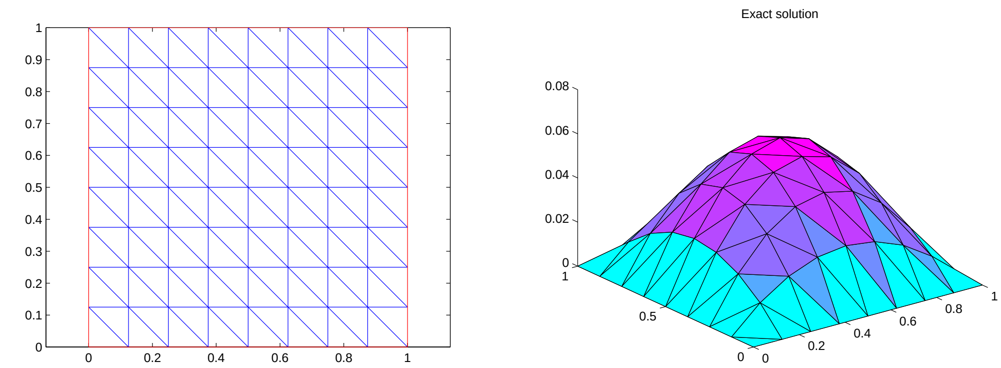


Figure 5.4: (Left) The mesh plot (triangulation) of the region (Right) The exact solution at the nodes of the mesh

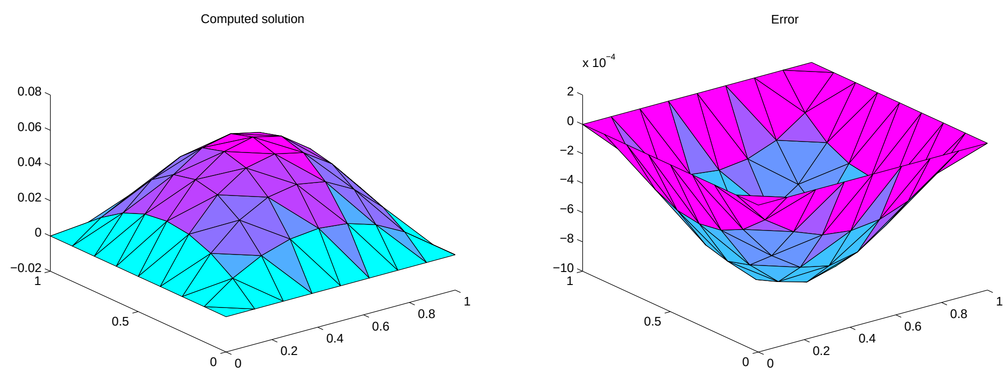


Figure 5.5: (Left) The approximation (Right) The corresponding error using infinity norm

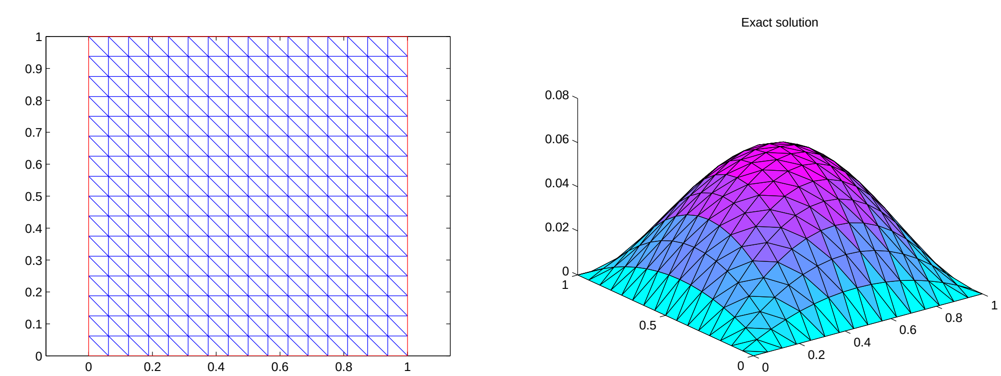


Figure 5.6: (Left) The refined mesh (Right) The exact solution at the nodes of the refined mesh

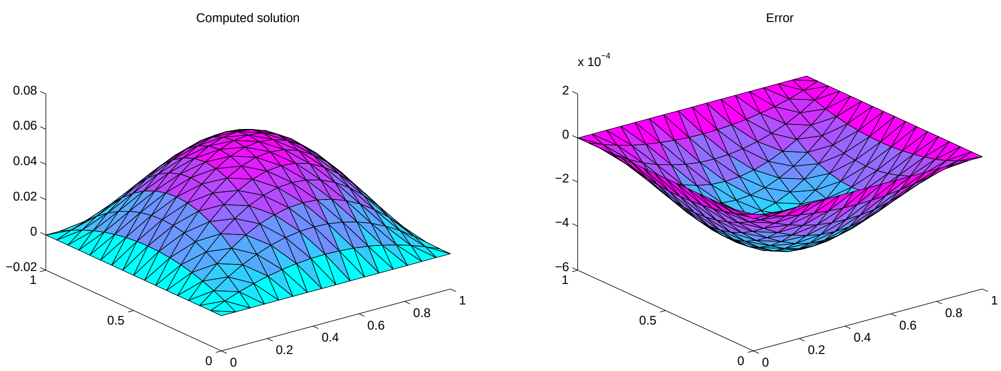


Figure 5.7: (Left) The approximation (Right) The corresponding error using infinity norm

Conclusion

In this thesis we reviewed some basic and general theory of the FEM. We also discussed the variational formulation and discretization of one and two dimensional problems. After that, the error estimation in its both types, a posteriori and a priori is explained.
The main goal of this thesis is to discuss the FEM for heat equation and errors estimates for the method. At the end, we talked about numerical examples in one and two dimensions using MATLAB software, where figures for the exact and approximate solutions with the corresponding errors are provided and explained.

Appendix

Matlab code for example 5.1

```
%The program solves the heat eq. p*du/dt- k*u''=f(x,t),
%u(0,t)=20, and u(10,t)=70,
%u(x,0)=sin(pi*xx/2)+5*xx+20,
%f(x,t)=0,
%k=1 and p=1.

clear all
clc
format long
k=1;
p=1;

x_Int=[0 10]; %Space interval
t_Int=[0 1]; % time interval

n=20; % n is the number of subintervals for the spatial variable
a=x_Int(1);
b=x_Int(2);
h=(b-a)/n;
xx=a:h:b;

% The partition of the time interval
```

```
nn=5; % nn is the number of subintervals for the time
aa=t_Int(1);
bb=t_Int(2);
hh=(bb-aa)/nn;
tt=aa:hh:bb;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Computing the mass matrix M000
M000=zeros(n-1,n-1);
for i=1:n-1
    M000(i,i)=4;
end
for i=1:n-2
    M000(i,i+1)=1;
    M000(i+1,i)=1;
end
M000=(h/6)*M000;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%computing the stiffness matrix M110
M110=zeros(n-1,n-1);
for i=1:n-1
    M110(i,i)=2;
end
for i=1:n-2
    M110(i,i+1)=-1;
    M110(i+1,i)=-1;
end
M110=(1/h)*M110;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Computing the loud vector bb00
bb00=zeros(n-1,1);
bb00(1,1)=k*20/h;
bb00(n-1,1)=k*70/h;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
u_initial=sin(pi*xx/2)+5*xx+20; %initial data
xi(:,1)=u_initial(2:n);

for i=1:length(tt)
    % Usual FEM:
  %xi(:,i+1)=(p*M000+hh*k*M110)\(p*M000*xi(:,i)+hh*bb00);
  % Theta Method in the FE
  theta=0.7; %This parameter is for Theta method in the FEM,  theta in [0 1].
 xi(:,i+1)=(p*M000+theta*hh*k*M110)\((p*M000-(1-theta)*hh*k*M110)*xi(:,i)+hh*bb00);
end

plot(xx,[20 xi(:,length(tt)+1)' 70],'*r')
hold on
fplot('exp(-pi^2*1/4)*sin(pi*x/2)+5*x+20',[0,10])
grid
% Computing the absolute error
exact=exp(-pi^2*1/4).*sin(pi*xx/2)+5*xx+20;
approx=[20 xi(:,length(tt)+1)' 70];
abs_err=abs(exact-approx);
T=ones(length(xx),1);
disp('                  xx                            Exact                     Approximation
disp('                 ==                            =====                    ===========
disp([xx'    exact'       approx'        abs_err'])
```

Matlab code for example 5.2

```
%The program solves the heat eq. p*du/dt- k*u''=f(x,t),
%u(0,t)=2, and u(4,t)=6,
%u(x,0)=sin(3*pi*xx/4)+(xx.^3)/24+xx/3+2,
%f(x,t)=-x,
% k=4 and p=1.

clear all
clc
format long
k=4;
p=1;

x_Int=[0 4]; % space interval
t_Int=[0 1]; % time interval

n=10; % n is the number of subintervals for the spatial variable
a=x_Int(1);
b=x_Int(2);
h=(b-a)/n;
xx=a:h:b;

% The partition of the time interval
nn=5; % nn is the number of subintervals for the time
aa=t_Int(1);
bb=t_Int(2);
hh=(bb-aa)/nn;
tt=aa:hh:bb;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Computing the mass matrix M000
M000=zeros(n-1,n-1);
```

```
for i=1:n-1
    M000(i,i)=4;
end
for i=1:n-2
    M000(i,i+1)=1;
    M000(i+1,i)=1;
end
M000=(h/6)*M000;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%computing the stiffness matrix M110
M110=zeros(n-1,n-1);
for i=1:n-1
    M110(i,i)=2;
end
for i=1:n-2
    M110(i,i+1)=-1;
    M110(i+1,i)=-1;
end
M110=(1/h)*M110;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Computing the loud vector bb00
h=h*ones(size(xx));
d1=1;
d2=0;
c4=[0.347854845137454 0.652145154862546 0.652145154862546 0.347854845137454];
x4=[-0.861136311594053 -0.339981043584856 0.339981043584856 0.861136311594053];
for i=d1:n-1+d2
    dd1=(xx(i+1)-xx(i))/2;
    dd2=(xx(i+1)+xx(i))/2;
    bb00_1(i)=dd1*(c4(1)*f1(dd1*x4(1)+dd2,xx(i+1),h(i+1))*f(dd1*x4(1)+dd2) ...
                          +c4(2)*f1(dd1*x4(2)+dd2,xx(i+1),h(i+1))*f(dd1*x4(2)+dd2) ...
                          +c4(3)*f1(dd1*x4(3)+dd2,xx(i+1),h(i+1))*f(dd1*x4(3)+dd2) ...
```

```
                            +c4(4)*f1(dd1*x4(4)+dd2,xx(i+1),h(i+1))*f(dd1*x4(4)+dd2));
    dd1=(xx(i+2)-xx(i+1))/2;
    dd2=(xx(i+2)+xx(i+1))/2;
    bb00_2(i)=dd1*(c4(1)*f2(dd1*x4(1)+dd2,xx(i+1),h(i+2))*f(dd1*x4(1)+dd2) ...
                            +c4(2)*f2(dd1*x4(2)+dd2,xx(i+1),h(i+2))*f(dd1*x4(2)+dd2) ...
                            +c4(3)*f2(dd1*x4(3)+dd2,xx(i+1),h(i+2))*f(dd1*x4(3)+dd2) ...
                            +c4(4)*f2(dd1*x4(4)+dd2,xx(i+1),h(i+2))*f(dd1*x4(4)+dd2));
    bb00(i-d1+1)=bb00_1(i)+bb00_2(i);
end
bb00=bb00';
bb00(1,1)=bb00(1,1)+k*2/h(1);
bb00(n-1,1)=bb00(n-1,1)+k*6/h(1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
u_initial=sin(3*pi*xx/4)+(xx.^3)/24+xx/3+2; %initial data
xi(:,1)=u_initial(2:n);

for i=1:length(tt)
    % Usual FEM:
  % xi(:,i+1)=(p*M000+hh*k*M110)\(p*M000*xi(:,i)+hh*bb00);
  % Theta Method in the FE
  theta=0.8; %This parameter is for Theta method in the FEM,  theta in [0 1].
xi(:,i+1)=(p*M000+theta*hh*k*M110)\((p*M000-(1-theta)*hh*k*M110)*xi(:,i)+hh*bb00);
end

plot(xx,[2 xi(:,length(tt)+1)' 6],'*r')
hold on
fplot('exp(-9*pi^2*1)*sin(3*pi*x/4)+(x^3)/24+x/3+2',[0,4])
grid on
% Computing the absolute error
exact=exp(-9*pi^2*1).*sin(3*pi*xx/4)+(xx.^3)/24+xx/3+2;
approx=[2 xi(:,length(tt)+1)' 6];
abs_err=abs(exact-approx);
```

```
T=ones(length(xx),1);
disp('                    xx                            Exact                         Approximation
disp('                   ==                            =====                        ===========
disp([xx'    exact'       approx'        abs_err'])
```

```
--------------------------------------------------------------------------------------------------------------
function y=f(x)
y=-x;

--------------------------------------------------------------------------------------------------------------
function y=f1(x,xi,hi)
y=(x-(xi-hi))/hi;


--------------------------------------------------------------------------------------------------------------
function y=f2(x,xi,hhi)
y=(xi+hhi-x)/hhi;

--------------------------------------------------------------------------------------------------------------
function y=difff1(x,xi,hi)
y=1/hi;


--------------------------------------------------------------------------------------------------------------
function y=difff2(x,xi,hhi)
y=-1/hhi;
```

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