Faculty of Graduate Studies

Mathematics Department

Finite Element Method For Convection-Dominated Problems

By:

Shoroq Manasra

Supervisor:

Dr. Hasan Almanasreh

This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of mathematics, College of Graduate Studies and Academic Research, Hebron University, Palestine.

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This thesis was defended successfully on 15/7/2020 and approved by:

Committee Members:		Signature
• Dr. Hasan Almanasreh	Supervisor	....
• Dr.Tareq Amro	Internal Examiner	....
• Dr.Ali Zein	External Examiner	....

Dedication

I dedicate my thesis to my husband, parents, friends and teachers who supported me on each step of the way.

Acknowledgments

Firstly, I would like to thank my thesis advisor Dr. Hasan Almanasreh at Hebron university. The door of Dr. Almanasreh office was always open whenever I ran into a trouble spot or had a question about my research or writting.

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Finally, I must express my very profound gratitude to my husband and to my parents, mother in low, friends and sisters for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of writing this thesis.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان :

طريقة العناصر المتهية للمسائل المحكومة بالحمل

FINITE ELEMENT METHOD FOR CONVECTION-DOMINATED PROBLEMS

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص،
باستثناء ما تمت الإشارة إليه حيثما ورد، و أن هذه الرسالة ككل لم تقدم
من قبل لنيل أية درجه علمية أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية او بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the result of the researcher's work, and has not been submitted elsewhere for any other degree or qualification.

Abstract

In this thesis we will study the finite element method to approximate the solution of differential equations. We are mainly interested in convection-dominated problems where the finite element solution is not stable. According to the finite element method, the differential equations are classified into convection and diffusion dominated problems using some dimensionless parametric measures such as the Peclet and Damkohler numbers.

For diffusion-dominated problems, we imploy the usual finite element method (FEM) but for convection-dominated problems, which are the major concern of this work, stable FEMs are required such as streamline upwind Petrov-Galerkin method (SUPG) and artificial diffusion method(ADM). These two methods provides more accurate and stable solution compared with the usual FEM when applied to convection-dominated problems. The mechanism of SUPG and ADM is to create sufficient diffusion term with size controlled by stability parameters, the diffusion term enhances the numerical solution and omits the spurious oscillations.

الملخص

في هذه الرسالة سوف ندرس طريقة العناصر المتهية لتقديم حلول عددية للمعادلات التفاضلية . في هذا السياق نحن مهتمين بشكل رئيسي بالمشكلات المحكومة بالحمل حيث الحل العددي غير ثابت . بالنسبة لطريقة العناصر المتهية ؛ المعادلات التفاضلية تصنف الى مشكلات محكومة بالحمل ومشكلات محكومة بالانتشار وذلك باستخدام مقاييس بلا أبعاد مثل رقم بكليت و رقم دامكولر .

للمسائل المحكومة بالانتشار نستخدم طريقة العناصر المتهية القياسية وللمسائل المحكومة بالحمل التي سيرتكز عليها عملنا بشكل رئيسي سوف نحتاج لطرق العناصر المتهية المستقرة أو الثابتة مثل طريقة بيتروف قاليريكن و طريقة الانتشار الاصطناعي. هذه الطرق تزودنا بحل أكثر دقة و ثباتية مقارنة بطريقةالعناصرالمتهية القياسيةعندما نطبقها على المسائل المحكومة بالحمل. الآلية لطريقتي بيتروف قاليريكن و طريقة الانتشار الاصطناعي تتمثل بخلق حد انتشار فعال تتحكم بحجمه معاملات الثباتية. حد الانتشار يعزز الحل العددي و يحذف الذبذبات الكاذبةفي الحل التقريبي.

Contents

Chapter 1

Introduction

In this thesis we discuses the finite element method to approximate solutions to differential equations. The main goal is to find stable solutions to differential equations that are dominated by convection terms. In this issue, any partial differential equation is classified either as diffusion-dominated problem or convection-dominated problem. For that, in this work the following problem is considered:

$$\begin{aligned} -\epsilon\Delta u + \beta.\nabla u + cu &= f, && \textbf{in } \Omega, \\ u &= g, && \textbf{on } \Gamma, \end{aligned} \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^n$ with boundary Γ.
ϵ **: is a small parameter.**
$\beta, c \geq 0$**:are a functions depends on** x
f **: is the source function.**
The operators ∇ and Δ are the first and second derivatives respectively.
For the classifications of this problem, dimensionless numbers such as Peclet number: $Pe = \frac{|\beta|h}{2\epsilon}$ and Damkohler number: $Da = \frac{ch}{|\beta|}$ are considered, where h is the element size, to determine whether the problem is convection-dominated or diffusion-dominated.
If $Pe > 1$, then equation (1.1) is convection - dominated problem. On the other hand, if $Pe \leq 1$, then equation (1.1) is diffusion-dominated problem and has a stable finite element solution.
If equation (1.1) is convection dominated, then the finite element solution is not stable and we use the stabilized finite element methods to improve this

numerical solution such as the streamline upwind Petrov-Galerkin method (SUPG) and the artificial diffusion method (ADM). The SUPG is same as of usual finite element method (FEM) but the test function has a small modification in the form $v + \tau v'$, where τ is a small parameter depends on h, called a stability parameter.

The basic idea of the ADM is the addition of an artificial term $\langle u' \, , \, \tau v' \rangle$, where τ is a stability parameter depends on h, different from that of the SUPG. The stability parameter of the SUPG is given by

$$\tau = \frac{h}{2}\big(\coth(Pe) - \frac{1}{Pe}\big),$$

and the stability parameter of the ADM is

$$\tau = (1 - \frac{1}{Pe})\frac{h|\beta|}{2}.$$

In Chapter two we talk about a history of the FEM, advantages and disadvantages of the FEM, and the finite element spaces. Also, types of boundary conditions, and the procedure of the FEM in one and two dimensions are discussed. In Chapter three, diffusion dominated and convection dominated problems are distinguished. Moreover, stability methods such as SUPG and ADM are discussed, and the coth formula of the stabilization parameter τ is derived.

In Chapter four, we use MATLAB software to discuss the numerical solution obtained by the usual FEM and the numerical solution obtained by the stabilized methods.

Chapter 2

The Finite element method (FEM)

2.1 History of the FEM

In this section we talk about the history of the FEM and some of its advantages and disadvantages.
The beginnings of the FEM were actually lunched in the **mid-1950s**, where efforts to solve continuum problems in elasticity using small discrete element to describe the overall behavior of simple elastic bars were devoted.
Technically, Galerkins method is a subset of the general weighted residuals procedure and in the case of Galerkins method, the weights are chosen to be the same as the functions used to define the unknown variables.
Recently, most practitioners of the finite element method now employ Galerkins method to establish the approximations of the governing equations, see [24].

2.1.1 Some advantages of the FEM

The FEM has a lot of usefullness such as boundary conditions can be easily incorporated in it, different types of material properties can be easily accommodated in modeling from element to element or even within an element, higher order elements may be implemented. The FEM is widely popular among engineering community. Besides that, availability of large number of computer software packages and literature makes the FEM a versatile and powerful numerical method, see [19].

2.1.2 Some disadvantages of the FEM

Finite element method has also some of disadvantages, for example large amount of data is required as input for the mesh used in terms of nodal connectivity and other parameters depending on the problem. It requires a digital computer and fairly extensive, and the output result will vary considerably, [19].

2.2 Inner product and the finite element spaces

Definition 2.1 (Inner product). *A function* $(.,.) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is an inner product if:*

- $(x, x) \geq 0,$ *and* $(x, x) = 0 \iff x = 0.$ *(positivity)*
- $(x, y) = (y, x).$ *(symmetry)*
- $(x + y, z) = (x, z) + (y, z).$ *(additivity)*
- $(rx, y) = r(x, y), \quad \forall r \in \mathbb{R}.$ *(homogeneity)*

Example 2.1. *The standard inner product of* x *and* y *in* $\mathbb{R}^n$ *is* $(x, y) = x^T y = \sum x_i y_i.$

Definition 2.2 (Norms). *A function* $||.||$*:* $\mathbb{R}^n \longrightarrow \mathbb{R}$ *is a norm if :*

- $||x|| \geq 0,$ *and* $||x|| = 0 \iff x = 0.$ *(Positivity)*
- $||\alpha x|| = |\alpha| ||x||, \quad \forall \alpha \in \mathbb{R}.$ *(Homogenity)*
- $||x + y|| \leq ||x|| + ||y||.$ *(Triangle inequality)*

Example 2.2.

(1) The 2-norm: $||x|| = \sqrt{\sum_i x_i^2}.$

(2) The 1-norm: $||x||_1 = \sum_i |x_i|.$

(3) The inf-norm: $||x||_\infty = max|x_i|.$

(4) The p-norm: $||x||_p = (\sum_i |x_i|^p)^{\frac{1}{p}}, \quad p \geq 1,$ *see e.g. [6].*

Theorem 2.1 (Cauchy-Schwarz inequality). *For $x, y \in \mathbb{R}^n$,*

$$|(x,y)| \leq ||x||||y||,$$

where $||x|| = \sqrt{(x,x)}$ is the length of x .

Definition 2.3. *A matrix $A \in S^{n\times n}$ is*

- *positive semidefinite if :*
  $x^T Ax \geq 0, \quad \forall x \in \mathbb{R}^n.$

- *positive definite if :*
  $x^T Ax > 0, \quad \forall x \in \mathbb{R}^n, \quad x \neq 0.$

- *negative semidefinite if $-A$ is positive semidefinite.*

- *negative definite if $-A$ is positive definite.*

Theorem 2.2. *If A is symmetric, then*

- *A is positive semidefinite $\iff$ all eigenvalues of A are nonnegative.*

- *A is positive definite $\iff$ all eigenvalues of A are positive, see[4].*

Remark 2.1. *If V is a linear space, then we say that L is a linear form on V if*

(1) $L : V \to R$, *i.e.,* $L(v) \in \mathbb{R} \quad \forall v \in V$.

(2) L *is linear i.e.,* $\forall v, w \in V$ *and* $\beta, \theta \in \mathbb{R}$,
$L(\beta v + \theta w) = \beta L(v) + \theta L(w)$.

Remark 2.2. *We say that $a(.,.)$ is a bilinear form on $V \times V$ if*

(1) $a : V \times V \to \mathbb{R}$ *i.e.,* $a(v,w) \in \mathbb{R}, \forall v, w \in V$

(2) a *is linear in each argument i.e.,* $\forall u, v, w \in V$ *and* $\beta, \theta \in \mathbb{R}$ *we have*

(a) $a(u, \beta v + \theta w) = \beta a(u,v) + \theta a(u,w)$.

(b) $a(\beta u + \theta v, w) = \beta a(u,w) + \theta a(v,w)$.

Also, the bilinear form $a(.,.)$ on $V \times V$ is said to be symmetric if $a(u,v) = a(v,u)$, $\forall v, u \in v$.
A symmetric bilinear form $a(.,.)$ on $V \times V$ is said to be scalar product on V if $a(v,u) > 0, \forall v \in V, \quad v \neq 0$, see [18].
The norm $||.||_a$ associated with a scalar product $a(.,.)$ is defined by

$$||v||_a = (a(v,v))^{\frac{1}{2}}, \qquad \forall v \in V.$$

Definition 2.4. *A sequence $v_1, v_2, v_3, ...$ in the space V with norm $||.||$ is said to be a Cauchy sequence if for all $\epsilon > 0$, there is a natural number $N \in \mathbb{N}$ such that $||v_i - v_j|| < \epsilon, forall\ i, j > N$. Furthermore v_i converges to v if $||v_i - v|| \longrightarrow 0$ as $i \longrightarrow \infty$.*

A linear space V is said to be complete if every Cauchy sequence with respect to $||.||$ is convergent in V.
A linear space V with a scalar product and corresponding norm $||.||$ is said to be a Hilbert space if V is complete.

2.2.1 The Hilbert spaces $L_2(\Omega), H^1(\Omega)$ and $H_0^1(\Omega)$.

When giving variational formulation of boundary value problems for partial differential equations, it is very useful to work with function space V, that are slightly larger than the spaces of continuous functions with piecewise continuous derivatives, see [7], V will be a Hilbert space. In one-dimensional case, if $I = (a,b)$ is an interval, we define the space of square integrable functions on I:

$$L_2(I) = \{v : v \quad \text{is defined on } I \text{ and } \textstyle\int_I v^2\ dx < \infty\}.$$

The space $L_2(I)$ is a Hilbert space, with scalar product$(u,v) = \int_I uv\ dx$ and corresponding norm "$L_2 - norm$":

$$||v||_{L_2(I)} = (\textstyle\int_I v^2.dx)^{\frac{1}{2}} = (v,v)^{\frac{1}{2}}.$$

We introduce the space:

$$H^1(I) = \{v : v \text{ and } v' \text{ belong to } L_2(I)\},$$

which is also a Hilbert space with the scalar product $(u,v)_{H^1(I)} = \int_I (uv + u'v')\ dx$ and corresponding norm :

$$||v||_{H^1(I)} = (\textstyle\int_I (v^2 + (v')^2)\ dx)^{\frac{1}{2}}.$$

Furthermore, the space

$$H_0^1(I) = \{v \in H^1(I) : v(a) = v(b) = 0\}$$

is also a Hilbert space with the same scalar product and norm as for $H^1(I)$.
Now if Ω is a bounded domain in $\mathbb{R}^d, d = 2$ or 3, we define

$$L_2(\Omega) = \{v : v \text{ is defined on } \Omega \text{ and } \int_\Omega v^2 \, dx < \infty\},$$

$$H^1(\Omega) = \{v \in L_2(\Omega) : \tfrac{\partial v}{\partial x_i} \in L_2(\Omega), \quad i = 1, 2, ..., d\}.$$

And the corresponding scalar products and norms :

$$(u, v) = \int_\Omega uvdx.$$

$$||v||_{L_2(\Omega)} = (\int_\Omega v^2 dx)^{\frac{1}{2}}.$$

$$(u, v)_{H^1(\Omega)} = \int_\Omega (uv + \nabla u.\nabla v)dx.$$

$$||v||_{H^1(\Omega)} = (\int_\Omega (v^2 + |\nabla v|^2)dx)^{\frac{1}{2}}.$$

We also define

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\},$$

where $\partial\Omega$ is the boundary of Ω with scalar product and norm as for $H^1(\Omega)$, see [18].

2.2.2 Boundary conditions

There are three classes of boundary conditions:

(a) **Dirichlet boundary condition:**
The value of the dependent variable is specified on the boundary.

(b) **Neumman boundary condition:**
The normal derivative of the dependent variable is specified on the boundary .

(c) **Cauchy boundary condition:**
Both the value and the normal derivative of the dependent variable are specified on the boundary.

(d) **Robin boundary conditions:**
A linear combination of the value of the dependent variable and its normal derivative is specified on the boundary, see [9].

2.3 One-dimension FEM

Modeling:
Let us consider the following mathematical model of a stationary reaction- diffusion process involving a single substance,

$$-(au')' + cu = f, \qquad\qquad x_1 < x < x_n,$$

$$\begin{aligned} a(x_1)u'(x_1) &= \gamma(x_1)\big(u(x_1) - g_D(x_1)\big) + g_N(x_1), \\ -a(x_n)u'(x_n) &= \gamma(x_n)\big(u(x_n) - g_D(x_n)\big) + g_N(x_n), \end{aligned}$$

where $u(x)$, denoting the concentration of the substance, is the unknown function that we wish to compute. The following functions are data to the problem:
$a(x)$: diffusion coefficient. $\big(a(x) > 0\big)$.
$c(x)$: rate coefficient factor. $\big(c(x) \geq 0\big)$.
$f(x)$: source function.
$\gamma(x_1), \gamma(x_n)$: permeability at the end points. $\big(\gamma \geq 0\big)$.
$g_D(x_1), g_D(x_n)$: ambient concentration factor.
$g_N(x_1), g_N(x_n)$: externally induced flux through the boundary.
To solve any boundary value problem (BVP) using the FEM, one rephrases the original boundary value problem in its weak form, we recall this step by 'variational formulation'. The second step is the discretization, where the weak form is discretized in a finite dimensional space.

(1) **Variational formulation**
To derive the variational formulation of the previous boundary value problem, see [22], we multiply it by a test function v and integrate over the domain (x_1, x_n),

$$-\int_{x_1}^{x_n} (au')'v dx + \int_{x_1}^{x_n} cuv dx = \int_{x_1}^{x_n} fv dx. \tag{2.1}$$

Integrate by parts to get

$$-(au')v|_{x_1}^{x_n} + \int_{x_1}^{x_n} au'v'dx + \int_{x_1}^{x_n} cuvdx = \int_{x_1}^{x_n} fvdx.$$

Applying the boundary condition gives

$$\Big(\gamma(x_n)\big(u(x_n) - g_D(x_n)\big) + g_N(x_n)\Big)v(x_n) + \Big(\gamma(x_1)\big(u(x_1) - g_D(x_1)\big) + g_N(x_1)\Big)v(x_1) +$$
$$+ \int_{x_1}^{x_n} au'v'\ dx + \int_{x_1}^{x_n} cuvdx = \int_{x_1}^{x_n} fvdx.$$

Rearranging the terms so that the quantities with the unknown function appear on the left-hand side, and the given data are on the right-hand side,

$$\gamma(x_n)u(x_n)v(x_n) + \gamma(x_1)u(x_1)v(x_1) + \int_{x_1}^{x_n} au'v'dx + \int_{x_1}^{x_n} cuv\ dx$$
$$= \big(\gamma(x_n)g_D(x_n) - g_N(x_n)\big)v(x_n) + \big(\gamma(x_1)g_D(x_1) - g_N(x_1)\big)v(x_1) + \int_{x_1}^{x_n} fvdx.$$

Thus, we have the following variational formulation

$$\text{Find} \quad u(x) \in H^1([x_1, x_n]), \quad \text{such that}$$

$$\gamma(x_n)u(x_n)v(x_n) + \gamma(x_1)u(x_1)v(x_1) + \int_{x_1}^{x_n} au'v'dx + \int_{x_1}^{x_n} cuvdx$$
$$= \big(\gamma(x_n)g_D(x_n) - g_N(x_n)\big)v(x_n) + \big(\gamma(x_1)g_D(x_1) - g_N(x_1)\big)v(x_1) + \int_{x_1}^{x_n} fvdx,$$
$$\forall v \in H^1([x_1, x_n]).$$

(2) **Discretization**

Introduce the vector space V_h of continuous piecewise linear functions on the partition $x_1 < x_2 < ... < x_n$ of $[x_1, x_n]$. To discretize the problem, let $U \in V_h$ be an approximation of u in the variational formulation equation

$$\gamma(x_n)U(x_n)v(x_n) + \gamma(x_1)U(x_1)v(x_1) + \int_{x_1}^{x_n} aU'v'\;dx + \int_{x_1}^{x_n} cUv\;dx \tag{2.2}$$
$$= \big(\gamma(x_n)g_D(x_n) - g_N(x_n)\big)v(x_n) + \big(\gamma(x_1)g_D(x_1) - g_N(x_1)\big)v(x_1) + \int_{x_1}^{x_n} fv\;dx,$$
$$\forall v \in V_h.$$

Now, $U(x)$ can be expressed as a linear combination of the basis $\{\varphi_i\}_{i=1}^n$ of V_h, these basis functions are linear and known as hat functions and defined by $\varphi_i(x_j) = \delta_{ij}, \quad i, j = 1, 2, ..., n$, where δ_{ij} denotes the Kronecker delta function defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and

$$\varphi_i = \begin{cases} \frac{x - x_{i-1}}{h_i} & , x_{i-1} < x < x_i, \\ \frac{x_{i+1} - x}{h_{i+1}} & , x_i < x < x_{i+1}, \\ 0, & \text{o.w.} \end{cases}$$

Where,

$$h_i = x_i - x_{i-1}$$

and

$$h_{i+1} = x_{i+1} - x_i$$

Hence,

$$U(x) = \sum_{j=1}^{n} \xi_j \varphi_j(x), \tag{2.3}$$

where $\xi_j = U(x_j)$ is the unknown value of U at the nodal point x_j,
thus, we seek to determine the coefficient vector

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} U(x_1) \\ U(x_2) \\ \vdots \\ U(x_n) \end{bmatrix}.$$

Now, substitute (2.3) in equation (2.2) to obtain

$$\gamma(x_n)\xi_n v(x_n) + \gamma(x_1)\xi_1 v(x_1) + \sum_{j=1}^{n} \xi_j \left[\int_{x_1}^{x_n} a\varphi_j' v' dx + \int_{x_1}^{x_n} c\varphi_j v dx \right] \tag{2.4}$$
$$= \Big(\gamma(x_n) g_D(x_n) - g_N(x_n)\Big) v(x_n) + \Big(\gamma(x_1) g_D(x_1) - g_N(x_1)\Big) v(x_1) + \int_{x_1}^{x_n} f v dx,$$
$$\forall v \in V_h.$$

Since $\{\varphi_i\}_{i=1}^{n}$ is a basis of V_h, and $v \in V_h$, then we may assume $v = \varphi_i$ in (2.4)

$$\gamma(x_n)\xi_n \varphi_i(x_n) + \gamma(x_1)\xi_1 \varphi_i(x_1) + \sum_{j=1}^{n} \xi_j \left[\int_{x_1}^{x_n} a\varphi_j' \varphi_i' \, dx + \int_{x_1}^{x_n} c\varphi_j \varphi_i \, dx \right] \tag{2.5}$$
$$= \Big(\gamma(x_n) g_D(x_n) - g_N(x_n)\Big) \varphi_i(x_n) + \Big(\gamma(x_1) g_D(x_1) - g_N(x_1)\Big) \varphi_i(x_1) + \int_{x_1}^{x_n} f \varphi_i \, dx,$$
$$i = 1, 2, ..., n,$$

which is a quadratic system of **n** linear equations and **n** unknowns.
In matrix form, this is read as:

$$[A + M_c + R]\xi = b + rv,$$

where
$A = [a_{ij}]$: is the stiffness matrix;

$$a_{ij} = \int_{x_1}^{x_n} a\varphi_j' \varphi_i' dx.$$

$M_c = [m_{c_{ij}}]$: is the mass matrix;

$$m_{c_{ij}} = \int_{x_1}^{x_n} c\varphi_j \varphi_i dx.$$

$b = [b_i]$: is the load vector;

$$b_i = \int_{x_1}^{x_n} f\varphi_i dx.$$

R : contains the boundary contribution to the system matrix.
rv : contains the boundary contribution to the right hand side.

We conclude that the FEM makes use of a spatial discretization and a weighted residual formulation to arrive at a system of matrix equations, where the solution of the matrix equations gives an approximate solution to the original boundary value problem, see, e.g., [5].

Example 2.3. *Consider the following BVP:*

$$\begin{aligned} -u''(x) &= f(x), \qquad\qquad 0 < x < 1, \\ u(0) &= u(1) = 0 \end{aligned}$$

(1) To derive the variational formulation of the previous boundary value problem, we multiply it by a test function v and integrate over $(0,1)$

$$-\int_0^1 u''(x)v(x)\ dx = \int_0^1 f(x)v(x)\ dx.$$

Integration by parts yields

$$-u'(x)v(x)|_0^1 + \int_0^1 u'(x)v'(x)\ dx = \int_0^1 f(x)v(x)\ dx. \tag{2.6}$$

Since u is given on the boundaries, then v is chosen so that

$$v(1) = v(0) = 0.$$

Now, introduce the notation

$$\langle u, v\rangle = \int_0^1 u(x)v(x)dx.$$

Hence equation (2.6) becomes

$$\langle u', v'\rangle = \langle f, v\rangle. \tag{2.7}$$

We also introduce the linear space
$V = \{v : v$ is continuous on $[0,1], v'$ is piecewise continuous and bounded on $[0,1]$ and $v(0) = v(1) = 0\}$.
Consider the linear functional
$F : V \longrightarrow R$ given by:

$$F(v) = \frac{1}{2}\langle v', v'\rangle - \langle f, v\rangle. \tag{2.8}$$

We need to define these two problems

(i) **Minimization problem:**
Find $u \in V$ such that $F(u) \leq F(v)$, for all $v \in V$.

(ii) **Variational formulation:**
Find $u \in V$ such that $\langle u', v' \rangle = \langle f, v \rangle$, for all $v \in V$.

We would like to show that if u is a solution of (ii) then it is also a solution of (i).
Let $v \in V$ and $w = v - u$.
$\Rightarrow v = w + u \quad$ and $\quad w \in V$.
Hence,
$F(v) = F(u + w)$,
$= \frac{1}{2}\langle u' + w', u' + w' \rangle - \langle f, u + w \rangle$,
$= \frac{1}{2}\langle u', u' \rangle - \langle f, u \rangle + \langle u', w' \rangle - \langle f, w \rangle + \frac{1}{2}\langle w', w' \rangle$,
$\geq \frac{1}{2}\langle u', u' \rangle - \langle f, u \rangle + 0 + 0 = F(u)$,
since $\langle u', w' \rangle - \langle f, w \rangle = 0$ and $\langle w', w' \rangle \geq 0$. This means $F(u) \leq F(v) \quad \forall v \in V$, thus $(ii) \Rightarrow (i)$.
We also need to prove that (ii) has a unique solution on V.
Suppose that u_1 and u_2 are solutions of (ii), then

$$\langle u_1', v' \rangle = \langle f, v \rangle, \qquad \text{for all} \quad v \in V, \tag{2.9}$$

. and

$$\langle u_2', v' \rangle = \langle f, v \rangle, \qquad \text{for all} \quad v \in V, \tag{2.10}$$

where
$u_1, u_2 \in V$.
Subtracting equations (2.9) and (2.10) and choosing $v = u_1 - u_2 \in V$, we get

$$\int_0^1 (u_1' - u_2')^2 \, dx = 0,$$
$$\Rightarrow u_1' - u_2' = (u_1 - u_2)' = 0,$$
$$\Rightarrow \big(u_1 - u_2\big)'(x) = 0, \text{ for all } x \in [0, 1],$$
$$\Rightarrow \big(u_1 - u_2\big)(x) \text{ is constant on } [0, 1].$$

Since

$$u_1(0) = u_2(0) = 0,$$

$$\Rightarrow (u_1 - u_2)(x) = 0,$$

$$\Rightarrow u_1(x) = u_2(x), \text{ for all } x \in [0,1].$$

Hence, (ii) has a unique solution on V.

- **Discretization:**

Find $U \in V_h$ where V_h is a subspace of the vector space V, such that

$$\langle U', v' \rangle = \langle f, v \rangle, \text{ for all } v \in V_h.$$

Now, consider $\{\varphi_i\}_{i=1}^{Nnodes}$ to be a basis of the before-defined hat functions of V_h, then

$$U(x) = \sum_{j=1}^{Nnodes} \xi_j \varphi_j(x).$$

By substituting U, equation (2.7) becomes

$$\sum_{j=1}^{Nnodes} \xi_j \langle \varphi_j', v' \rangle = \langle f, v \rangle.$$

But $\{\varphi_i\}_{i=1}^{Nnodes}$ forms a basis for V_h, so we may set $v = \varphi_i, i = 1, 2, ..., N$ nodes,

$$\Rightarrow \sum_{j=1}^{Nnodes} \xi_j \langle \varphi_j', \varphi_i' \rangle = \langle f, \varphi_i \rangle.$$

In matrix form, this reads

$$A\xi = b,$$

where
A : is an $N \times N$ matrix with elements $a_{ij} = \langle \varphi_j', \varphi_i' \rangle$ called a stiffness matrix.
ξ : $[\xi_1, \xi_2, ..., \xi_N]$ is the unknowns vector.
b : $[b_1, b_2, ..., b_N]$ with $b_i = \langle f, \varphi_i \rangle$ called the load vector.

★ **Note that:**

(1) A is a tri-diagonal matrix, i.e. only the elements in the main diagonal and the two adjoining diagonals may be different from zero.

(2) A is symmetric.

(3) A is positive definite, hence the eigenvalues of A are strictly positive, and since a positive definite matrix is non-singular it follows that the linear system $A\xi = b$ has a unique solution.

(4) A is sparse, i.e. only a few elements of A are different from zero.

2.3.1 The element integrals of the finite element matrices

Consider the partition $x_0, x_1, ..., x_n$ of an interval $[a, b]$, consider the linear basis functions on this partition:

$$\varphi_i = \begin{cases} \frac{x-x_{i-1}}{h_i} & , x_{i-1} < x < x_i, \\ \frac{x_{i+1}-x}{h_{i+1}} & , x_i < x < x_{i+1}, \\ 0, & \text{o.w}, \end{cases} \qquad i = 1, 2, ..., n-1,$$

where,

$$h_i = x_i - x_{i-1}$$

and

$$h_{i+1} = x_{i+1} - x_i$$

$$\varphi_n = \begin{cases} \frac{x-x_{n-1}}{h_n} & , x_{n-1} < x < x_n, \\ 0, & \text{o.w}, \end{cases}$$

and

$$\varphi_0 = \begin{cases} \frac{x_1-x}{h_1} & , x_0 < x < x_1, \\ 0, & o.w. \end{cases}$$

★ **Stiffness matrix;** $A = [a_{ij}]$:

The stiffness matrix A with linear basis functions obeys the following

$$a_{ij} = \int_\Omega \varphi_i' \varphi_j' \, dx,$$

$$a_{ij} = 0 \text{ if } |j-i| > 1 \text{ since } \varphi_j \varphi_i = 0 \text{ if } |j-i| > 1,$$

$$a_{ij} \neq 0 \text{ if } |j-i| \leq 1, \text{ i.e.}$$

- $j - i = 0 \quad\quad \to j = i.$
- **Or** $j - i = 1 \quad \to j = i + 1.$
- **Or** $j - i = -1 \to j = i - 1.$

<u>if $j = i$:</u>

$$\begin{aligned} a_{ii} &= \int_{x_{i-1}}^{x_{i+1}} \varphi_i' \varphi_i' \, dx, \\ &= \int_{x_{i-1}}^{x_i} \frac{1}{h_i} \frac{1}{h_i} \, dx + \int_{x_i}^{x_{i+1}} -\frac{1}{h_{i+1}} \frac{-1}{h_{i+1}} \, dx = \frac{1}{h_i} + \frac{1}{h_{i+1}}. \end{aligned}$$

<u>if $j = i + 1$:</u>

$$\begin{aligned} a_{i,i+1} &= \int_\Omega \varphi_{i+1}' \varphi_i' \, dx, \\ &= \int_{x_i}^{x_{i+1}} \frac{1}{h_{i+1}} \frac{-1}{h_{i+1}} \, dx = -\frac{1}{h_{i+1}}. \end{aligned}$$

<u>if $j = i - 1$:</u>

$$\begin{aligned} a_{i,i-1} &= \int_\Omega \varphi_{i-1}' \varphi_i' \, dx, \\ &= \int_{x_{i-1}}^{x_i} -\frac{1}{h_i} \frac{1}{h_i} \, dx = -\frac{1}{h_i}. \end{aligned}$$

Hence, the stiffness matrix A is given by:

$$A = \begin{bmatrix} \frac{1}{h_1} & -\frac{1}{h_2} & \cdots & 0 \\ -\frac{1}{h_2} & \frac{1}{h_1} + \frac{1}{h_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\frac{1}{h_n} \\ 0 & \cdots & -\frac{1}{h_n} & \frac{1}{h_n} \end{bmatrix}.$$

With uniform mesh, $h_i = h$, then A is given as

$$A = \frac{1}{h} \begin{bmatrix} 1 & -1 & \cdots & 0 \\ -1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 0 & \cdots & -1 & 1 \end{bmatrix}.$$

★ **Mass matrix ; $M = [m_{ij}]$:**

$$m_{ij} = \int_\Omega \varphi_i \varphi_j \; dx,$$

When $\underline{j = i:}$

$$\begin{aligned} m_{ii} &= \int_\Omega \varphi_i \varphi_i \; dx, \\ &= \int_{x_{i-1}}^{x_{i+1}} \varphi_i^2 \; dx = \int_{x_{i-1}}^{x_i} \frac{(x - x_{i-1})^2}{h_i^2} \; dx + \int_{x_i}^{x_{i+1}} \frac{(x_{i+1} - x)^2}{h_{i+1}^2} \; dx, \\ &= \frac{h_i^3}{3h_i^2} + \frac{h_{i+1}^3}{3h_{i+1}^2} = \frac{h_i + h_{i+1}}{3}. \end{aligned}$$

When $\underline{j = i + 1:}$

$$\begin{aligned} m_{i,i+1} &= \int_\Omega \varphi_{i+1} \varphi_i \; dx, \\ &= \int_{x_i}^{x_{i+1}} \varphi_{i+1} \varphi_i \, dx = \int_{x_i}^{x_{i+1}} \frac{(x - x_i)}{h_{i+1}} \frac{(x_{i+1} - x)}{h_{i+1}} \; dx, \\ &= \frac{1}{h_{i+1}^2} \int_{x_i}^{x_{i+1}} (x - x_i) \big[(x_{i+1} - x_i) + (x_i - x) \big] \; dx, \\ &= \frac{1}{h_{i+1}^2} \int_{x_i}^{x_{i+1}} \big[h_{i+1}(x - x_i) - (x - x_i)^2 \big] \; dx, \\ &= \frac{1}{h_{i+1}} \frac{(x - x_i)^2}{2} \Big|_{x_i}^{x_{i+1}} - \frac{1}{h_{i+1}^2} \frac{(x - x_i)^3}{3} \Big|_{x_i}^{x_{i+1}}, \\ &= \frac{h_{i+1}}{2} - \frac{h_{i+1}}{3} = \frac{h_{i+1}}{6}. \end{aligned}$$

When $\underline{j = i - 1:}$

$$m_{i,i-1} = \int_\Omega \varphi_{i-1} \varphi_i \; dx = \frac{h_i}{6},$$

Thus, the mass matrix has the form

$$M = \begin{bmatrix} \frac{h_1}{3} & \frac{h_2}{6} & \cdots & 0 \\ \frac{h_2}{6} & \frac{h_1+h_2}{3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{h_n}{6} \\ 0 & \cdots & \frac{h_n}{6} & \frac{h_n}{3} \end{bmatrix}.$$

When $h_i = h_j, \forall i, j$ "Uniform partition", the mass matrix M becomes

$$M = \frac{h}{6}\begin{bmatrix} 2 & 1 & \cdots & 0 \\ 1 & 4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & 2 \end{bmatrix}.$$

★ **The convection matrix $C = [c_{ij}]$:**

$$C_{ij} = \int_{x_i}^{x_{i+1}} \varphi_j' \varphi_i \; dx.$$

When $j = i$:

$$\begin{aligned} C_{ii} &= \int_{x_{i-1}}^{x_{i+1}} \varphi_i' \varphi_i \; dx, \\ &= \int_{x_{i-1}}^{x_i} \varphi_i' \varphi_i \; dx + \int_{x_i}^{x_{i+1}} \varphi_i' \varphi_i \; dx, \\ &= \int_{x_{i-1}}^{x_i} \frac{1}{h_i} \frac{(x - x_{i-1})}{h_i} \; dx + \int_{x_i}^{x_{i+1}} -\frac{1}{h_{i+1}} \frac{(x_{i+1} - x)}{h_{i+1}} \; dx, \\ &= \frac{1}{h_i^2} \frac{(x - x_{i-1})^2}{2} |_{x_{i-1}}^{x_i} + \frac{1}{h_{i+1}^2} \frac{(x_{i+1} - x)^2}{2} |_{x_i}^{x_{i+1}}, \\ &= +\frac{1}{2} - \frac{1}{2} = 0. \end{aligned}$$

When $j = i + 1$:

$$\begin{aligned} C_{i,i+1} &= \int_{x_{i-1}}^{x_{i+1}} \varphi_{i+1}' \varphi_i \; dx, \\ &= \int_{x_i}^{x_{i+1}} \frac{1}{h_{i+1}} \frac{(x_{i+1} - x)}{h_{i+1}} \; dx, \end{aligned}$$

$$= -\frac{1}{h_{i+1}^2}\frac{(x_{i+1}-x)^2}{2}|_{x_i}^{x_{i+1}},$$

$$= \frac{1}{2}.$$

When $\underline{j = i-1:}$

$$\begin{aligned} C_{i,i-1} &= \int_{x_{i-1}}^{x_{i+1}} \varphi'_{i-1}\varphi_i \; dx, \\ &= \int_{x_{i-1}}^{x_i} -\frac{1}{h_i}\frac{(x-x_{i-1})}{h_i} \; dx, \\ &= -\frac{1}{h_i^2}\frac{(x-x_{i-1})^2}{2}|_{x_{i-1}}^{x_i} = -\frac{1}{2}. \end{aligned}$$

Hence,

$$C = \begin{bmatrix} 0 & \frac{1}{2} & \cdots & 0 \\ -\frac{1}{2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{2} \\ 0 & \cdots & -\frac{1}{2} & 0 \end{bmatrix}.$$

★ **<u>The convection matrix $\zeta = [\varsigma_{ij}]$:</u>**

$$\zeta_{ij} = \int_\Omega \varphi_j \varphi'_i \; dx.$$

When $\underline{j = i:}$

$$\begin{aligned} \zeta_{ii} \quad &= \int_{x_{i-1}}^{x_{i+1}} \varphi_i \varphi'_i \; dx, \\ &= \int_{x_{i-1}}^{x_i} \frac{(x-x_{i-1})}{h_i}\frac{1}{h_i} \; dx + \int_{x_i}^{x_{i+1}} \frac{(x_{i+1}-x)}{h_{i+1}} - \frac{1}{h_{i+1}} \; dx, \\ &= +\frac{1}{2} - \frac{1}{2} = 0. \end{aligned}$$

When $\underline{j = i+1:}$

$$\begin{aligned} \zeta_{i,i+1} \qquad &= \int_{x_i}^{x_{i+1}} \varphi_{i+1}\varphi'_i \; dx, \\ &= \int_{x_i}^{x_{i+1}} \Big(\frac{(x-x_i)}{h_{i+1}}. - \frac{1}{h_{i+1}}\Big) \; dx, \end{aligned}$$

$$= -\frac{1}{h_{i+1}^2}\frac{(x-x_i)^2}{2}|_{x_i}^{x_{i+1}},$$

$$= -\frac{1}{2}.$$

When $\underline{j = i-1:}$

$$\zeta_{i,i-1} = \int_{x_{i-1}}^{x_i} \varphi_{i-1}\varphi_i'\, dx,$$

$$= \int_{x_{i-1}}^{x_i} \frac{(x_i - x)}{h_i}\frac{1}{h_i}\, dx,$$

$$= -\frac{1}{h_i^2}\frac{(x_i-x)^2}{2}|_{x_{i-1}}^{x_i} = \frac{1}{2}.$$

Therefore,

$$\zeta = \begin{bmatrix} 0 & -\frac{1}{2} & \cdots & 0 \\ \frac{1}{2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\frac{1}{2} \\ 0 & \cdots & \frac{1}{2} & 0 \end{bmatrix}.$$

Example 2.4. *Let α and β be positive constants. Give the piecewise linear finite element approximation procedure and derive the corresponding stiffness matrix, mass matrix, and load vector using the uniform mesh size $h = \frac{1}{4}$ for the problem:*

$$-u''(x) + u(x) = 1, \qquad x \in (0,1), \tag{2.11}$$

$$u(0) = \alpha, \quad u'(1) = \beta.$$

<u>Solution:</u>

Multiply equation (2.11) by a test function v, such that $v(0) = 0$, and integrate over $\Omega = (0,1)$.

$$-u'v|_0^1 + \int_0^1 u'v'dx + \int_0^1 uvdx = \int_0^1 vdx,$$

$$-u'(1)v(1) + \int_0^1 u'v'dx + \int_0^1 uvdx = \int_0^1 vdx,$$

$$\Rightarrow \int_0^1 u'v'dx + \int_0^1 uvdx = \int_0^1 vdx + \beta v(1). \tag{2.12}$$

Find $u \in V = \{w : \int_0^1 (w^2 + w'^2)dx < \infty \text{ and } w(0) = \alpha\}$, such that equation (2.12) holds $\forall v \in V^0$, where

$V^0 = \{v : \int_0^1 (v^2 + v'^2)\ dx < \infty\ and\ v(0) = 0\}$.
Let $V_h \subset V$ be a finite subspace of linear functions spanned by the linear basis functions on the partition $x_j = jh = \frac{j}{4}, j = 0, 1, 2, 3, 4$.

$$U = \sum_{j=0}^{4} \xi_j \varphi_j = \xi_0 \varphi_0 + \sum_{j=1}^{4} \xi_j \varphi_j,$$

$$= \alpha \varphi_0 + \sum_{j=1}^{4} \xi_j \varphi_j.$$

Consider an approximation $U \in V_h$ of u, thus

$$\text{Substitute } U = \alpha\varphi_0 + \sum_{j=1}^{4} \xi_j \varphi_j \text{ in equation (2.12) and take } v = \varphi_i.$$

$$\Rightarrow \int_0^1 (\alpha\varphi_0' + \sum_{j=1}^{4} \xi_j \varphi_j')\varphi_i' dx + \int_0^1 (\alpha\varphi_0 + \sum_{j=1}^{4} \xi_j \varphi_j)\varphi_i dx = \int_0^1 \varphi_i dx + \beta\varphi_i(1),$$

$$\Rightarrow \sum_{j=1}^{4} (\int_0^1 \varphi_j' \varphi_j' dx)\xi_j + \sum_{j=1}^{4} (\int_0^1 \varphi_j \varphi_j dx)\xi_j = -\alpha \int_0^1 \varphi_0' \varphi_i' dx - \alpha \int_0^1 \varphi_0 \varphi_i dx + \int_0^1 \varphi_i dx + \beta\varphi_i(1). \quad (2.13)$$

The integrals $\int_\Omega \varphi_i' \varphi_j' dx, i, j = 1, 2, 3, 4$ is a 4×4 stiffness matrix given by

$$A = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & -4 & 4 \end{bmatrix}.$$

The integrals $\int_\Omega \varphi_i \varphi_j dx, i, j = 1, 2, 3, 4$ is a 4×4 mass matrix given by

$$M = \frac{1}{24} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

The load vector b is

$$b = \begin{bmatrix} -\alpha\int_0^1 \varphi_0'\varphi_1' dx - \alpha\int_0^1 \varphi_0\varphi_1 dx + \int_0^1 \varphi_1 dx \\ \int_0^1 \varphi_2 dx \\ \int_0^1 \varphi_3 dx \\ \int_0^1 \varphi_4 dx + \beta\varphi_4(1) \end{bmatrix},$$

$$= \begin{bmatrix} -\alpha(-\frac{1}{h}) - \alpha.\frac{h}{6} + \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{8} + \beta \end{bmatrix},$$

$$= \begin{bmatrix} -\alpha - \frac{\alpha}{24} + \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{8} + \beta \end{bmatrix}.$$

Note that φ_4 is a half basis, thus

$$\int_0^1 \varphi_4 \; dx = \frac{1}{2}\int_0^1 \varphi_1 dx = \frac{1}{2}.\frac{1}{4} = \frac{1}{8}.$$

Using the above matrices, equation (2.13) is written in the following form,

$$[A+M]\xi = b.$$

where $\xi = [\boldsymbol{\xi_1}, \boldsymbol{\xi_2}, \boldsymbol{\xi_3}, \boldsymbol{\xi_4}]^{\mathbf{T}}$.

2.4 Two-dimension finite element method

Before starting, the following formula, [9] is considered as the corresponding rule of integrating by part in one dimension:

★ **<u>Green's Formula:</u>**

$$\int_\Omega \Delta w v dx = \int_{\partial\Omega} \frac{\partial w}{\partial n} v ds + \int_\Omega \nabla w . \nabla v dx,$$

where

$$\frac{\partial w}{\partial n} = \nabla w.n = (\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}).(n_1(x_1, x_2), n_2(x_1, x_2)) = \frac{\partial w}{\partial x_1} n_1 + \frac{\partial w}{\partial x_2} n_2,$$

and
$n = (n_1, n_2)$ is the outword unite normal to $\partial\Omega$ at the point (x_1, x_2).

• **Modeling Problem:**

As an example, we consider the following mathematical model of a stationary reaction-diffusion process involving a single substance

$$-\nabla.(a\nabla u) + cu = f, \qquad x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2,$$

$$-n.(a\nabla u) = \gamma(u - g_D) + g_N, \qquad x = (x_1, x_2) \in \partial\Omega,$$

where
$u = u(x_1, x_2)$, denoting the concentration of the substance, is the unknown function that we wish to compute.
The following functions are data to the problem
$a(x_1, x_2):$ $\Omega \to \mathbb{R}$ diffusion coefficient. $\big(a(x_1, x_2) > 0\big)$
$c(x_1, x_2):$ $\Omega \to \mathbb{R}$ rate coefficient. $\big(c(x_1, x_2) \geq 0\big)$
$f(x_1, x_2):$ $\Omega \to \mathbb{R}$ source cofactor.
$\gamma(x_1, x_2):$ $\partial\Omega \to \mathbb{R}$ permeability of the boundary. $\big(\gamma(x_1, x_2) \geq 0\big)$
$g_D(x_1, x_2):$ $\partial\Omega \to \mathbb{R}$ ambient concentration.
$g_N(x_1, x_2):$ $\partial\Omega \to \mathbb{R}$ externally induced flux through the boundary.

- To derive the variational formulation of the above equation, we multiply the differential equation by a test function $v = v(x_1, x_2)$ and integrate over Ω

$$-\iint_\Omega \nabla.(a\nabla u) v dx_1 dx_2 + \iint_\Omega cuv dx_1 dx_2 = \iint_\Omega fv dx_1 dx_2,$$

$$-\iint_\Omega \frac{\partial}{\partial x_1}(a\frac{\partial u}{\partial x_1}) + \frac{\partial}{\partial x_2}(a\frac{\partial u}{\partial x_2}) v dx_1 dx_2 + \iint_\Omega cuv dx_1 dx_2 = \iint_\Omega fv dx_1 dx_2.$$

Employing the Green's formula to get

$$-\int_{\partial\Omega}(a\frac{\partial u}{\partial x_1}n_1+a\frac{\partial u}{\partial x_2}n_2)vds+\iint_{\Omega}(a\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}+a\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2})vdx_1dx_2+$$
$$+\iint_{\Omega}cuvdx_1dx_2=\iint_{\Omega}fvdx_1dx_2,$$

or,

$$-\int_{\partial\Omega}\big(n.(a\nabla u)v\big)ds+\iint_{\Omega}a\nabla u.\nabla v.dx_1\ dx_2+\iint_{\Omega}cuvdx_1dx_2$$
$$=\iint_{\Omega}fvdx_1dx_2.$$

Use the boundary condition $-n.(a\nabla u)=\gamma(u-g_D)+g_N$. To obtain

$$\int_{\partial\Omega}\gamma uvds+\iint_{\Omega}a\nabla u.\nabla vdx_1dx_2+\iint_{\Omega}cuvdx_1dx_2=\int_{\partial\Omega}(\gamma g_D-g_N)vds+$$
$$+\iint_{\Omega}fvdx_1dx_2.$$

Now, find $u\in V$ such that

$$\int_{\partial\Omega}\gamma uvds+\iint_{\Omega}a\nabla u.\nabla vdx_1dx_2+\iint_{\Omega}cuvdx_1dx_2=\int_{\partial\Omega}(\gamma g_D-g_N)vds+$$
$$+\iint_{\Omega}fvdx_1dx_2,$$

$\forall v\in V$,
where V is some admissible vector space of functions that are sufficient to the integrals above to exist.

- **Discretization :**
Find $U\in V_h$ such that

$$\int_{\partial\Omega} \gamma U v ds + \iint_{\Omega} a \nabla U . \nabla v dx_1 dx_2 + \iint_{\Omega} c U v dx_1 dx_2 \tag{2.14}$$
$$= \int_{\partial\Omega} (\gamma g_D - g_N) v ds + \iint_{\Omega} f v dx_1 dx_2,$$

$\forall v \in V_h$, where $V_h \subset V$ be a finite subspace of linear functions spanned by the linear basis functions.
Let $\{\varphi_i\}_{i=1}^{Nnodes}$ be the basis of linear functions (tent functions) of V_h defined by

$$\varphi_i(N_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Now, $U \in V_h$ means that

$$\Rightarrow U(x_1, x_2) = \sum_{j=1}^{N} \xi_j \varphi_j(x_1, x_2), \tag{2.15}$$

where N is the number of mesh nodes. Substitute (2.15) in equation (2.14) to get

$$\sum_{j=1}^{N} \xi_j \big[\int_{\partial\Omega} \gamma \varphi_j v ds + \iint_{\Omega} a \nabla \varphi_j . \nabla v dx_1 dx_2 + \iint_{\Omega} c \varphi_j v dx_1 dx_2 \big]$$
$$= \int_{\partial\Omega} (\gamma g_D - g_N) v ds + \iint_{\Omega} f v dx_1 dx_2.$$

Since $\{\varphi_i\}_{i=1}^{Nnodes}$ is a basis of V_h, then we may assume $v = \varphi_i, i = 1, 2, ..., N$, in the above equation.

$$\sum_{j=1}^{N} \xi_j \big[\int_{\partial\Omega} \gamma \varphi_j \varphi_i ds + \iint_{\Omega} a \nabla \varphi_j . \nabla \varphi_i dx_1 dx_2 + \iint_{\Omega} c \varphi_j \varphi_i dx_1 dx_2 \big]$$
$$= \int_{\partial\Omega} (\gamma g_D - g_N) \varphi_i ds + \iint_{\Omega} f \varphi_i dx_1 dx_2.$$

In matrix form, this reads

$$(R + A + M)\xi = rv + b,$$

where
$R \quad = [r_{ij}]$: contains the boundary contributions to the system matrix.
$A \quad = [a_{ij}]$: is the stiffness matrix.
$M \quad = [m_{ij}]$: is the mass matrix.
$rv \quad = [rv_i]$: contains the boundary contributions to the right-hand sides.
$b \quad = [b_i]$: is the load vector.
Each of these matrices are defined below

$$r_{ij} = \int_{\partial\Omega} \gamma\varphi_j\varphi_i ds.$$

$$a_{ij} = \iint_{\Omega} a\nabla\varphi_j\nabla\varphi_i dx_1 dx_2.$$

$$m_{ij} = \iint_{\Omega} c\varphi_j\varphi_i dx_1 dx_2.$$

$$rv_i = \int_{\partial\Omega} (\gamma g_D - g_N)\varphi_i ds.$$

$$b_i = \iint_{\Omega} f\varphi_i \ dx_1 \ dx_2.$$

Example 2.5 (Poisson equation). *Let us consider the Poisson equation with homogenous Dirichlet boundary condition :*

$$\begin{aligned} -\Delta u(x) &= f(x), \quad && \text{for } x \in \Omega, \\ u(x) &= 0, && \text{for } x \in \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$.

- **Variational formulation :**
  Multiply the differential equation by a test function $v = v(x_1, x_2)$ and integrate over Ω :

$$\iint_{\Omega} fv dx_1 dx_2 = -\iint_{\Omega} \Delta u.v dx_1 dx_2.$$

  Use the Green's formula to get

$$\iint_{\Omega} fv dx_1 dx_2 = -\iint_{\Omega} \Delta u.v dx_1 dx_2,$$

$$= -\int_{\partial\Omega} (\frac{\partial u}{\partial x_1} n_1 + \frac{\partial u}{\partial x_2} n_2) v ds + \iint_\Omega \nabla u . \nabla v dx_1 dx_2.$$

Since u is given on $\partial\Omega$, then $v = 0$ on $\partial\Omega$, and the above equation becomes

$$\iint_\Omega f v dx_1 dx_2 = -\iint_\Omega \nabla u . \nabla v dx_1 dx_2.$$

Thus, the variational formulations is to find $u \in V$ such that

$$\iint_\Omega f v dx_1 dx_2 = -\iint_\Omega \nabla u . \nabla v dx_1 dx_2, \quad \forall v \in V,$$

where

$$V = \{v : \int_\Omega (|\nabla v|^2 + v^2)\ dx < \infty \text{ and } v = 0 \text{ on } \partial\Omega\}.$$

In scalar product notation, find $u \in V$ such that

$$\langle \nabla u, \nabla v \rangle = \langle f, v \rangle, \qquad \forall v \in V,$$

where

$$\langle \nabla u, \nabla v \rangle = \iint_\Omega \nabla u . \nabla v . dx_1 dx_2.$$

$$\langle f, v \rangle = \iint_\Omega f v dx_1 dx_2.$$

- **Discretization:**
  Find $U \in V_h$ such that

$$\langle \nabla U, \nabla v \rangle = \langle f, v \rangle, \qquad \forall v \in V_h, \tag{2.16}$$

where $V_h \subset V$ be a finite subspace of linear functions spanned by the linear basis functions.
As before, let $\{\varphi_i\}_{i=1}^M$ be a basis of V_h, M is the number of internal nodes.

If $U \in V_h$, then $U = \sum_{j=1}^{M} \xi_j \varphi_j$. Substitute U in equation (2.16) to get

$$\sum_{j=1}^{M} \xi_j \langle \nabla\varphi_j, \nabla v \rangle = \langle f, v \rangle, \qquad \forall v \in V_h.$$

Choosing $v = \varphi_i, i = 1, 2, ..., M$, yields

$$\sum_{j=1}^{M} \langle \nabla\varphi_j, \nabla\varphi_i \rangle \xi_j = \langle f, \varphi_i \rangle.$$

This is equivalent to the linear system of equations

$$A\xi = b,$$

where
$A = [a_{ij}]$ is the stiffness matrix with elements $a_{ij} = \langle \nabla\varphi_j, \nabla\varphi_i \rangle$.
$b = [b_i]$ is the load vector with elements $b_i = \langle f, \varphi_i \rangle$.
$\xi = [\xi_i]$ is the unknown vector with elements $\xi = [\xi_1, \xi_2, ..., \xi_n]$. That we want to solve for.

Chapter 3

Stabilized methods for convection dominated problems

3.1 Diffusion dominated and convection dominated problems

We begin by considering the stationary scalar linear convection-diffusion problem of the form:

$$\begin{aligned} -\epsilon\Delta u + \beta.\nabla u + cu = f, & \qquad \text{in } \Omega, \\ u = g, & \qquad \text{on } \Gamma, \end{aligned} \tag{3.1}$$

where
Ω : is a boundary domain in $\mathbb{R}^n$ with boundary Γ .
ϵ : is a small parameter.
$\beta, c \geq 0$: are a functions depends on x.
f : is a source function.
The relative size of ϵ and β govern the qualitative nature of equation (3.1).

- If $\dfrac{\epsilon}{|\beta|}$ is small ($\epsilon << \beta$), then equation (3.1) is convection-dominated and has hyperbolic character.
- If $\dfrac{\epsilon}{|\beta|}$ is not small ($\beta << \epsilon$), then equation (3.1) is diffusion-dominated and has elliptic character, see [10].

If equation (3.1) is diffusion-dominated, then its numerical solution is stable, and when it is convection-dominated then the numerical solution is not stable, see [17, 2].
In this situation it is sufficient to study some fundamental dimensionless numbers that characterize the solution such as Peclet and Damkohler numbers, where

$$Pe = \frac{|\beta|h}{2\epsilon},$$

$$Da = \frac{ch}{|\beta|},$$

where h is the element size, see [3].
when $Pe > 1$, then equation (3.1) is convection-dominated problem, otherwise, if $Pe \leq 1$, then equation (3.1) is diffusion-dominated problem , see, e.g., [20].

Example 3.1.

$$-(5u')'(x) = 0, \qquad\qquad x \in (0,1), \tag{3.2}$$

$$-5u'(0) + 3(u(0) - 2) = 0,$$

$u(1) = 0$.

Note that, $Pe = \frac{|\beta|.h}{2\epsilon} = \frac{0.h}{2(5)} = 0$. Since $Pe \leq 1$, then this problem is diffusion-dominated.
Let us calculate the usual FEM approximation U for the problem with $n = 3$. Firstly, multiplying equation (3.2) by a test function v such that $v(1) = 0$ and integrating over $\Omega = (0,1)$ yields

$$-\int_0^1 (5u')'v \; dx = \int_0^1 0(v) \; dx,$$

$$-(5u')v|_0^1 + \int_0^1 5u'v' \; dx = 0,$$

$$-5u'(1)v(1) + 5u'(0)v(0) + \int_0^1 5u'v' \; dx = 0,$$

$$3(u(0) - 2)v(0) + \int_0^1 5u'v' \; dx = 0. \tag{3.3}$$

The variational formulation is to find $u \in H^1$ such that $u(1) = 0$ and equation (3.3) holds $\forall v \in H^1$ with $v(1) = 0$.

To discretize the problem, let $U \in V_h \subset H^1$, where V_h is a finite subspace on the partition $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}$, and $x_3 = 1$, spanned by $\{\varphi_i\}_{i=0}^{3}$, hence

$$U = \sum_{j=0}^{3} \xi_j \varphi_j,$$

$$= \sum_{j=0}^{2} \xi_j \varphi_j + \xi_3 \varphi_3,$$

$$= \sum_{j=0}^{2} \xi_j \varphi_j,$$

this because $\xi_3 = U(x_3) = U(1) = 0$. Take $v = \varphi_i, i = 0, 1, 2$, in equation (3.3)

$$3(\xi_0 - 2)\varphi_i(x_0) + 5\sum_{j=0}^{2} \xi_j \int_0^1 \varphi_i' \varphi_j' \, dx = 0.$$

In matrix form, we have

$$\begin{bmatrix} 3\xi_0 - 6 \\ 0 \\ 0 \end{bmatrix} + 5.\frac{1}{\frac{1}{3}}. \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$3\xi_0 - 6 + 15\xi_0 - 15\xi_1 = 0. \tag{3.4}$$

$$-15\xi_0 + 30\xi_1 - 15\xi_2 = 0. \tag{3.5}$$

$$-15\xi_1 + 30\xi_2 = 0. \tag{3.6}$$

The solution of this system of linear equation is

$$\xi_0 = \frac{3}{4}, \quad \xi_1 = \frac{1}{2}, \quad \xi_2 = \frac{1}{4}.$$

$$\xi = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ \frac{1}{4} \\ 0 \end{bmatrix}.$$

Hence, the FEM solution is

$$U = \frac{3}{4}\varphi_0 + \frac{1}{2}\varphi_1 + \frac{1}{4}\varphi_2.$$

But, what about the convection-dominated problems case ?
When the Peclet number is greater than one, then the usual FEM produces oscillations in the approximated solution, and thus the results are not accurate.
Therefore, the usual FEM can not be applied to solve a convection-dominated problem, and we turn to use other methods to solve such type of problems. To this end, to understand the instability of the usual FEM solution to the convection-dominated problem, and to figure out how the stabilized method provide much better solution, we consider the following fundamental example

$$\begin{aligned} -ku'' + pu' + qu &= x, \qquad x \in (0,1), \\ u(0) &= u(1) = 0. \end{aligned} \tag{3.7}$$

where k, p and q are arbitrary nonzero constants.
The values of k, p and q are choosen to be arbitrary nonzero constants in order to vary Problem (3.7) to be convection-dominated or diffusion-dominated, also the value of the source function is $f(x) = x$ on the interval $(0,1)$ for simplicity.

3.2 Exact solution of the fundamental example

In this section we will discuse the exact solution of example (3.7) to compare it with the numerical solution obtained by the usual FEM and the stability methods.
★**Solution:**
Let $y(x) = y_h + y_p$, where
y_h: is the homogenous solution, i.e., the solution of $-ku'' + pu' + qu = 0$.
y_p: is the particular solution, i.e., the solution of $-ku'' + pu' + qu = x$.
The homogeneous solution is $y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$, where r_1, r_2 are the solutions of the equation $-kr^2 + pr + q = 0$. Using the quadratic formula the solutions of this equation are

$$r_{1,2} = \frac{-p \pm \sqrt{p^2 - 4(-k)q}}{-2k} = \frac{p \pm \sqrt{p^2 + 4kq}}{2k},$$

$$\Rightarrow r_1 = \frac{p + \sqrt{p^2 + 4kq}}{2k} \quad \text{and} \quad r_2 = \frac{p - \sqrt{p^2 + 4kq}}{2k}.$$

Hence,

$$y_h(x) = c_1 e^{\frac{(p+\sqrt{p^2+4kq})x}{2k}} + c_2 e^{\frac{(p-\sqrt{p^2+4kq})x}{2k}}.$$

To find the particular solution of equation (3.7), there are two cases to be considered :

Case(1): if the problem is diffusion-dominated, then the diffusion term k is larger than the convection term p, hence, the term $4kq$ is much larger than p, i.e., the term $4kq$ can not be ignored. Thus, $\sqrt{p^2+4kq}$ can not be closed to p, so, $p-\sqrt{p^2+4kq}$ can not be closed to zero. Therefore, $\frac{p-\sqrt{p^2+4kq}}{2k}$ and $\frac{p+\sqrt{p^2+4kq}}{2k}$ are far from zero, this means that zero is not a root for the auxiliary equation.
In this case we choose a linear form for the particular solution,

$$y_p = Ax + B,$$

substituting y_p in (3.7) yields the values of A and B, thus

$$y_p = \frac{1}{q}x - \frac{p}{q^2}.$$

Therefore, the solution of the fundamental example in the case that the problem is diffusion-dominated is:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \frac{1}{q}x - \frac{p}{q^2},$$

where $r_1 = \frac{p+\sqrt{p^2+4kq}}{2k}$ and $r_2 = \frac{p-\sqrt{p^2+4kq}}{2k}$.
Employing the boundary condition $u(0) = u(1) = 0$,

$$u(0) = 0 \Rightarrow c_1 + c_2 = \frac{p}{q^2}, \tag{3.8}$$

and

$$u(1) = 0 \Rightarrow c_1 e^{r_1} + c_2 e^{r_2} + \frac{1}{q} - \frac{p}{q^2} = 0. \tag{3.9}$$

Solving these two equation for c_1 and c_2 provides

$$c_1 = \frac{\frac{p}{q^2}(1-e^{r_2}) - \frac{1}{q}}{e^{r_1} - e^{r_2}}.$$

$$c_2 = \frac{p}{q^2} - \frac{\frac{p}{q^2}(1-e^{r_2}) - \frac{1}{q}}{e^{r_1} - e^{r_2}}$$

Case(2): If the problem is convection-dominated, then the convection term p is larger than the diffusion term k, this means that the term p^2 is much larger than the term $4kq$. Accordingly, the term $4kq$ is too small when we compare it to the term p^2, in conclusion, the term $4kq$ can be ignored:

$$\begin{aligned}
&\Rightarrow (4kq \longrightarrow 0),\\
&\Rightarrow \sqrt{p^2+4kq} \longrightarrow \sqrt{p^2+0} = |p|,\\
&\Rightarrow \text{either} \quad \tfrac{p-\sqrt{p^2+4kq}}{2k} \quad \text{or} \quad \tfrac{p+\sqrt{p^2+4kq}}{2k} \quad \text{approaches zero},\\
&\Rightarrow \text{zero is nearly a root for the auxiliary equation.}
\end{aligned}$$

In this case, the particular solution is:

$$y_p = x(Ax+B) = Ax^2 + Bx.$$

After substituting this particular solution in the corresponding differential equation, the constants A and B are given by:

$$A = \frac{p}{2p^2+2kq}.$$

$$B = \frac{k}{p^2+kq}.$$

Summing up, the solution of the fundamental example where the convection term dominates the differential equation is

$$y(x) = y_h(x) + y_p(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \frac{p}{2p^2+2kq}x^2 + \frac{k}{p^2+kq}x.$$

To determine the values of c_1 and c_2, substitute the boundary condition $u(0) = u(1) = 0$

$$u(0) = 0 \Rightarrow c_1 + c_2 = 0,$$

$$\Rightarrow c_1 = -c_2. \tag{3.10}$$

and

$$u(1) = 0 \Rightarrow c_1 e^{r_1} + c_2 e^{r_2} + \frac{p}{2p^2+2kq} + \frac{k}{p^2+kq} = 0.$$

Solving these equations for c_1 and c_2,

$$c_1 = \frac{-\frac{p}{2p^2+2kq} - \frac{k}{p^2+kq}}{e^{r_1} - e^{r_2}},$$

and

$$c_2 = \frac{\frac{p}{2p^2+2kq} + \frac{k}{p^2+kq}}{e^{r_1} - e^{r_2}}.$$

3.3 Solution of the fundamental example by the FEM

In this section we will find the numerical solution of example (3.7) using the FEM and compare it with the numerical solutions by the stabilized methods discussed later.
Let u be the solution to (3.7), and the interval $I = (0,1)$ be divided into a uniform mesh with $h = \frac{1}{n}$, we would like to calculate the finite element approximation U for any value of n .
★ **Solution:**

(1) Variational formulation:
multiply equation (3.7) by a test function v and integrate over $(0,1)$, to get

$$\langle -ku'', v\rangle + \langle pu', v\rangle + \langle qu, v\rangle = \langle x, v\rangle,$$

integrate by parts and use the boundary condition to obtain

$$\langle -ku', v\rangle + \langle pu', v\rangle + \langle qu, v\rangle = \langle x, v\rangle.$$

Now, state the following variational formulation :
Find $u(x) \in H_0^1([0,1])$ such that

$$\langle -ku', v\rangle + \langle pu', v\rangle + \langle qu, v\rangle = \langle x, v\rangle,$$

$\forall v \in H_0^1([0,1])$.

(2) Discretization:
Find $U(x) \in V_h$, where V_h is a finite dimensional vector space on the partition $x_j = 0 + jh, j = 0, 1, ..., n+1$, spanned by the linear basis functions,

$$\langle -kU', v\rangle + \langle pU', v\rangle + \langle qU, v\rangle = \langle x, v\rangle, \tag{3.11}$$

$\forall v \in V_h$.

But $U(x)$ can be written as a linear combination of the basis elements of V_h,

$$U(x) = \sum_{j=0}^{n+1} \xi_j \varphi_j(x) = \sum_{j=1}^{n} \xi_j \varphi_j(x), (\text{ since } \xi_0 = \xi_{n+1} = 0),$$

and seek to determine the coefficient vector

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} U(x_1) \\ U(x_2) \\ \vdots \\ U(x_n) \end{bmatrix}.$$

Now, substitute $U(x) = \sum_{j=1}^{n} \xi_j \varphi_j(x)$ in equation (3.11). and $v = \varphi_i, i = 1, 2, ..., n$,

$$\langle -k \sum_{j=1}^{n} \xi_j \varphi_j', \varphi_i' \rangle + \langle p \sum_{j=1}^{n} \xi_j \varphi_j', \varphi_i \rangle + \langle q \sum_{j=1}^{n} \xi_j \varphi_j, \varphi_i \rangle = \langle x, \varphi_i \rangle.$$

Equivalently,

$$\sum_{j=1}^{n} \xi_j \big[-k \langle \varphi_j', \varphi_i' \rangle + p \langle \varphi_j', \varphi_i \rangle + q \langle \varphi_j, \varphi_i \rangle \big] = \langle x, \varphi_i \rangle, i = 1, 2, ..., n,$$

which is a system of n linear equations and n unknowns.
In matrix form, this system can be written in the form

$$\big[-kA + pC + qM \big] \xi = b,$$

where
A : is the stiffness matrix.
C : is the convection matrix.
M : is the mass matrix.
b : is the load vector.

3.4 Stability methods

In this section we study some of stability methods, see[12, 11], to solve convection-dominated problems:

(1) Streamline upwind Petrov-Galerkin method (SUPG).

(2) Artificial Diffusion method (ADM).

3.4.1 SUPG solution

Recall the fundamental example

$$-ku'' + pu' + qu = x, \qquad x \in (0,1),$$

$$u(0) = u(1) = 0,$$

where k, p and q are arbitrary nonzero constants.
We will discuss a general steps to solve this problem by using SUPG method which is similar to the usual FEM in the formulation, but the test function v is different, see [16]. To solve this problem using SUPG method, we set below the corresponding variational formulation and discretization.

To derive the SUPG variational formulation of the equation above, we multiply it by a test function $(v + \tau v')$, where τ is a small parameter depends on h and then integrate over the domain $(0,1)$,

$$\langle -ku'' + pu' + qu, v + \tau v' \rangle = \langle x, v + \tau v' \rangle,$$

Equivalently

$$-k\langle u'', v\rangle + p\langle u', v\rangle + q\langle u, v\rangle - k\tau\langle u'', v'\rangle + p\tau\langle u', v'\rangle + q\tau\langle u, v'\rangle = \langle x, v\rangle + \tau\langle x, v'\rangle.$$

Integrate by parts and use $u = 0$ on Γ, to get

$$-k\langle u', v'\rangle + p\langle u', v\rangle + q\langle u, v\rangle + \tau\big[-k\langle u'', v'\rangle + p\langle u', v'\rangle + q\langle u, v'\rangle\big] = \langle x, v\rangle + \tau\langle x, v'\rangle. \quad (3.12)$$

Note that if $\tau = 0$, then the solution of the previous formula (3.12) is the same as the usual FEM.

Now, state the following variational formulation, find $u(x) \in H_0^1([0,1])$ such that

$$-k\langle u', v'\rangle + p\langle u', v\rangle + q\langle u, v\rangle + \tau\big[-k\langle u'', v'\rangle + p\langle u', v'\rangle + q\langle u, v'\rangle\big] = \langle x, v\rangle + \tau\langle x, v'\rangle.$$

$\forall v \in H_0^1([0,1])$.
To discretize the variational formulation, we search for an approximation $U(x) \in V_h$, where V_h is a finite dimensional vector space on the partition, $x_j = a + jh$, $j = 0, 1, 2, ..., n+1$, spanned by the linear basis functions, such that $\forall v \in V_h$ the following holds

$$-k\langle U', v'\rangle + p\langle U', v\rangle + q\langle U, v\rangle + \tau\big[-k\langle U'', v'\rangle + p\langle U', v'\rangle + q\langle U, v'\rangle\big] = \langle x, v\rangle + \tau\langle x, v'\rangle.$$

Now, let $U(x) = \sum_{j=1}^{n} \xi_j \varphi_j(x)$ and $v = \varphi_i$, in the above equation to have

$$\sum_{j=1}^{n} \xi_j \big[-k\langle \varphi_j', \varphi_i'\rangle + p\langle \varphi_j', \varphi_i\rangle + q\langle \varphi_j, \varphi_i\rangle + \tau\big[-k\langle \varphi_j'', \varphi_i'\rangle + p\langle \varphi_j', \varphi_i'\rangle + q\langle \varphi_j, \varphi_i'\rangle\big]$$
$$= \langle x, \varphi_i\rangle + \tau\langle x, \varphi_i'\rangle, \text{ where } i = 1, 2, ..., n.$$

But $\langle \varphi_j'', \varphi_i'\rangle = 0$, since $\varphi_j'' = 0$ (linear functions), thus the previous equation become

$$\sum_{j=1}^{n} \xi_j \big[-k\langle \varphi_j', \varphi_i'\rangle + p\langle \varphi_j', \varphi_i\rangle + q\langle \varphi_j, \varphi_i\rangle + \tau\big[p\langle \varphi_j', \varphi_i'\rangle + q\langle \varphi_j, \varphi_i'\rangle\big] = \langle x, \varphi_i\rangle + \tau\langle x, \varphi_i'\rangle,$$

where
$\langle \varphi_j', \varphi_i'\rangle$: is the stiffness matrix.
$\langle \varphi_j', \varphi_i\rangle$: is a convection matrix.
$\langle \varphi_j, \varphi_i\rangle$: is the mass matrix.
$\langle \varphi_j, \varphi_i'\rangle$: is a convection matrix.

3.4.2 ADM solution

We are interested in applying new stabilized finite element method called the artificial diffusion method (ADM). The basic idea of this method is the addition of an artificial term $\langle u' \ , \ \tau v'\rangle$ to the left hand side, where τ is a small parameter depends on h called a stability parameter.
Consider again the fundamental example

$$-ku'' + pu' + qu = x, \qquad x \in (0,1),$$

$$u(0) = u(1) = 0.$$

where u is the unknown function,k, p and q are arbitrary nonzero constants.
To derive the variational formulation of the above equation, multiply it by a test function v, and integrate over $(0,1)$, to get

$$-k\langle u'', v\rangle + p\langle u', v\rangle + q\langle u, v\rangle = \langle x, v\rangle.$$

The ADM is formalized by adding the artificial term $\langle u', \tau v'\rangle$ to the left hand side as follows:

$$-k\langle u'', v\rangle + p\langle u', v\rangle + q\langle u, v\rangle + \langle u', \tau v'\rangle = \langle x, v\rangle.$$

Integrate by parts and use $u = 0$ on Γ to have

$$-k\langle u'', v\rangle + p\langle u', v\rangle + q\langle u, v\rangle + \langle -\tau u'', v\rangle = \langle x, v\rangle,$$

$$\Rightarrow -k\langle u'', v\rangle + p\langle u', v\rangle + q\langle u, v\rangle - \tau\langle u'', v\rangle = \langle x, v\rangle,$$

$$\Rightarrow -(k+\tau)\langle u'', v\rangle + p\langle u', v\rangle + q\langle u, v\rangle = \langle x, v\rangle. \tag{3.13}$$

Note that, if $\tau = 0$, equation (3.13) becomes

$$\langle -ku'', v\rangle + \langle pu', v\rangle + \langle qu, v\rangle = \langle x, v\rangle,$$

which is the usual finite element formulation of the given problem. Thus,

$$\langle -ku'', v\rangle + \langle pu', v\rangle + \langle qu, v\rangle = \langle x, v\rangle,$$

$$\langle -ku'', v\rangle + \langle pu', v\rangle + \langle qu, v\rangle - \langle x, v\rangle = 0,$$

$$\Rightarrow \langle -ku'' + pu' + qu - x\ ,\ v\rangle = 0, \quad \forall v.$$

$$\Rightarrow -ku'' + pu' + qu - x = 0.$$

$$\Rightarrow -ku'' + pu' + qu = x,$$

which is the original boundary value problem.
Now, if $\tau \neq 0$, then

$$\langle ku'' + pu' + qu\ ,\ v\rangle + \langle -\tau u''\ ,\ v\rangle = \langle x\ ,\ v\rangle.$$

$$\Rightarrow \langle -ku'' + pu' + qu - \tau u'' \ , \ v\rangle - \langle x \ , \ v\rangle = 0.$$

$$\Rightarrow \langle -ku'' + pu' + qu - \tau u'' - x \ , \ v\rangle = 0, \quad \forall v.$$

$$\Rightarrow -ku'' + pu' + qu - \tau u'' - x = 0.$$

$$\Rightarrow -(k+\tau)u'' + pu' + qu - x = 0.$$

$$\Rightarrow -(k+\tau)u'' + pu' + qu = x,$$

which is not the same as of the original problem .Clearly, this is not the original problem, but a modification to it.
To proceed in the ADM formulation, integrate by parts and using $u = 0$ on Γ, to get

$$(k+\tau)\langle u', v'\rangle + p\langle u', v\rangle + q\langle u, v\rangle = \langle x, v\rangle.$$

Now, we state the following variational formulation:
Find $u(x) \in H_0^1$ such that

$$(k+\tau)\langle u', v'\rangle + p\langle u', v\rangle + q\langle u, v\rangle = \langle x, v\rangle,$$

$\forall v \in H_0^1$.

To discretize the ADM variational formulation, we find $U(x) \in V_h$ such that

$$(k+\tau)\langle U', v'\rangle + p\langle U', v\rangle + q\langle U, v\rangle = \langle x, v\rangle,$$

$\forall v \in V_h$.
Since $U \in V_h$, then $U(x) = \sum_{j=1}^{n} \xi_j \varphi_j(x)$, also $v \in V_h$, then we may let $v = \varphi_i, i = 1, 2, ..., n$,
Therefore,

$$\sum_{j=1}^{n} \xi_j \Big[(k+\tau)\langle \varphi_j', \varphi_i'\rangle + p\langle \varphi_j', \varphi_i\rangle + q\langle \varphi_j, \varphi_i\rangle\Big] = \langle x, \varphi_i\rangle.$$

In matrix form

$$\Big[(k+\tau)A + pC + qM\Big]\xi = b,$$

where
A : is the stiffness matrix.
C : is the convection matrix.

M : is the mass matrix.
b : is the load vector.

3.5 Stability parameter τ and the coth-formula

In the previous section, we identified two methods for solving the convection-diffusion problems, the first one is the SUPG method which is summarized by using the test function to be in the form $v + \tau v'$ and we mentioned that τ is a small parameter depends on h .
The second one is the ADM, in this method we add an artificial term $\langle u', \tau v' \rangle$, where τ is a small parameter depends on h.
From now on, we will call this parameter by the " stability parameter ", and the goal of this section is to determine the stability parameter τ.
With help of the exact solution, which is known for a simple model problem, it is possible to determine the stability parameter τ, and below we present the derivation of τ that applies to the SUPG method.

Definition 3.1. *The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series*

$$f(a) + \frac{f'(a)}{1!}.(x-a) + \frac{f''(a)}{2!}.(x-a)^2 + \cdots ,$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}.(x-a)^n.$$

Let us use a Taylor series expansion to approximate the first and second derivatives of a function f about a certain point say x_0

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \cdots .$$

Now, take $x = x_0 + h$, to obtain

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \cdots . \tag{3.14}$$

Also, assuming $x = x_0 - h$, yields

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \cdots . \tag{3.15}$$

Subtract (3.15) from (3.14), to get

$$f(x_0+h) - f(x_0-h) = 2f'(x_0)h + 0 + \cdots ,$$

$$\Rightarrow f(x_0+h) - f(x_0-h) \simeq 2f'(x_0)h,$$

$$\Rightarrow f'(x_0) \simeq \frac{f(x_0+h) - f(x_0-h)}{2h}, \tag{3.16}$$

where h is a partition length.
Now add (3.15) to (3.14), to get

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + h^2 f''(x_0) + \cdots ,$$

$$\Rightarrow f(x_0+h) + f(x_0-h) \simeq 2f(x_0) + h^2 f''(x_0),$$

$$\Rightarrow f''(x_0) \simeq \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}. \tag{3.17}$$

Now, consider the following general homogenous model problem:

$$-\epsilon u'' + \beta u' = 0, \qquad \text{on} \quad \Omega, \tag{3.18}$$

$$u = 0, \qquad\qquad \text{on} \quad \Gamma.$$

In SUPG method, we multiply this equation by a test function $v + \tau v'$ and integrate over Ω, to get

$$\langle -\epsilon u'' + \beta u', v + \tau v' \rangle = 0,$$

$$\Rightarrow -\epsilon \langle u'', v\rangle + \beta\langle u', v\rangle - \epsilon\tau\langle u'', v'\rangle + \beta\tau\langle u', v'\rangle = 0,$$

$$\Rightarrow -\epsilon \langle u'', v\rangle + \beta\langle u', v\rangle + \tau\Big[- \epsilon\langle u'', v'\rangle + \beta\langle u', v'\rangle\Big] = 0.$$

Integrating by parts and using $u = 0$ on Γ provides

$$-\epsilon \langle u'', v\rangle + \beta\langle u', v\rangle + \tau\big[- \epsilon\langle u'', v'\rangle - \beta\langle u'', v\rangle\big] = 0.$$

If v is linear, then $\langle u'', v'\rangle = \langle u', v''\rangle = 0$, thus,

$$-\epsilon \langle u'', v\rangle + \beta\langle u', v\rangle + \tau\big[- \beta\langle u'', v\rangle\big] = 0,$$

$$\Rightarrow \langle -\epsilon u'' + \beta u' - \beta\tau u'' \ , \ v\rangle = 0,$$

$$\Rightarrow -\epsilon u'' + \beta u' - \beta\tau u'' = 0. \tag{3.19}$$

Now, integrate equation (3.19) over the subinterval $[x_{i-1}, x_i]$, to get

$$\langle -\epsilon u'' + \beta u' - \beta\tau u'' \ , \ 1\rangle = 0.$$

Fix u at x_i, we have

$$\langle -\epsilon u''(x_i) + \beta u'(x_i) - \beta\tau u''(x_i) \ , \ 1\rangle = 0.$$

Using the notation $u_i = u(x_i)$, then above equation becomes:

$$\Rightarrow \langle -\epsilon u''_i + \beta u'_i - \beta\tau u''_i \ , \ 1\rangle = 0,$$

$$\Rightarrow \langle (-\epsilon - \beta\tau) u''_i + \beta u'_i \ , \ 1\rangle = 0. \tag{3.20}$$

We can use the first and second derivatives approximations (3.16) and (3.17) for any index x_i in general as follows:

$$u'_i = \frac{u_{i+1} - u_{i-1}}{2h}.$$

$$u''_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

Thus, equation (3.20) can be simplified as follows

$$\langle (-\epsilon - \beta\tau)\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) + \beta\left(\frac{u_{i+1} - u_{i-1}}{2h}\right) \ , \ 1\rangle = 0,$$

$$\Rightarrow \left[(-\epsilon - \beta\tau)\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) + \beta\left(\frac{u_{i+1} - u_{i-1}}{2h}\right)\right]\langle 1 \ , \ 1\rangle = 0,$$

$$\Rightarrow \left[(-\epsilon - \beta\tau)\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) + \beta\left(\frac{u_{i+1} - u_{i-1}}{2h}\right)\right](x_i - x_{i-1}) = 0,$$

$$\Rightarrow \left[(-\epsilon - \beta\tau)\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) + \beta\left(\frac{u_{i+1} - u_{i-1}}{2h}\right)\right]h = 0,$$

$$\Rightarrow \left[(\frac{(-\epsilon - \beta\tau)}{h^2} - \frac{\beta}{2h})u_{i-1} + (\frac{2(\epsilon + \beta\tau)}{h^2})u_i + (\frac{(-\epsilon - \beta\tau)}{h^2} + \frac{\beta}{2h})u_{i+1}\right]h = 0,$$

$$\Rightarrow (\frac{-\epsilon}{h} - \frac{\beta\tau}{h} - \frac{\beta}{2})u_{i-1} + (\frac{2\epsilon}{h} + \frac{2\beta\tau}{h})u_i + (\frac{-\epsilon}{h} - \frac{\beta\tau}{h} + \frac{\beta}{2})u_{i+1} = 0. \tag{3.21}$$

From now on, we will call equation (3.21) by the "difference equation", and we need to use it to determine the stability parameter τ, see [23, 13]. Thus, rearrange this difference equation to get

$$\frac{\epsilon}{h}(-u_{i-1}+2u_i-u_{i+1})+\frac{\beta\tau}{h}(-u_{i-1}+2u_i-u_{i+1})+\frac{\beta}{2}(-u_{i-1}+u_{i+1})=0,$$

$$\Rightarrow \frac{\epsilon+\beta\tau}{h}(-u_{i-1}+2u_i-u_{i+1})+\frac{\beta}{2}(-u_{i-1}+u_{i+1})=0. \tag{3.22}$$

The following Taylor series expansions for u_{i-1} and u_{i+1} will be deployed in the sequel,

$$u_{i-1}=\sum_{n=0}^{\infty}\frac{u_i^{(n)}(x_{i-1}-x_i)^n}{n!},$$

$$=\sum_{n=0}^{\infty}\frac{u_i^{(n)}(-h)^n}{n!}=u_i-hu_i'+\frac{h^2}{2!}u_i''-\cdots,$$

and

$$u_{i+1}=\sum_{n=0}^{\infty}\frac{u_i^{(n)}(x_{i+1}-x_i)^n}{n!},$$

$$=\sum_{n=0}^{\infty}\frac{u_i^{(n)}(h)^n}{n!}=u_i+hu_i'+\frac{h^2}{2!}u_i''+\cdots.$$

Using these two expansions, equation (3.22) can simplified as:

$$\frac{\epsilon+\beta\tau}{h}\Big(-(u_i-hu_i'+\frac{h^2}{2!}u_i''-\cdots)+2u_i-(u_i+hu_i'+\frac{h^2}{2!}u_i''+\cdots)\Big)+$$
$$+\frac{\beta}{2}\Big(-(u_i-hu_i'+\frac{h^2}{2!}u_i''-\cdots)+(u_i+hu_i'+\frac{h^2}{2!}u_i''+\cdots)\Big)=0,$$

$$\frac{\epsilon+\beta\tau}{h}\Big(-h^2u_i''-\frac{1}{12}h^4u_i^{(4)}-\cdots\Big)+\frac{\beta}{2}\Big(2hu_i'+\frac{1}{3}h^3u_i^{(3)}+\cdots\Big)=0. \tag{3.23}$$

Now, the exact solution of (3.18) is $u(x)=c_1e^{\frac{\beta}{\epsilon}x}+c_2$, where the constants c_1 and c_2 can be determined with help of the boundary condition and

$$u'=c_1\frac{\beta}{\epsilon}e^{\frac{\beta}{\epsilon}x}.$$

$$u''=c_1(\frac{\beta}{\epsilon})^2e^{\frac{\beta}{\epsilon}x}.$$

$$\vdots$$

$$u^{(n)} = c_1(\frac{\beta}{\epsilon})^n e^{\frac{\beta}{\epsilon}x}. \tag{3.24}$$

Substitute (3.24) in (3.23) to get

$$(\frac{\epsilon+\beta\tau}{h})[-h^2c_1(\frac{\beta}{\epsilon})^2e^{\frac{\beta}{\epsilon}x}-\frac{1}{12}h^4c_1(\frac{\beta}{\epsilon})^4e^{\frac{\beta}{\epsilon}x}-\cdots]+\frac{\beta}{2}[2hc_1(\frac{\beta}{\epsilon})e^{\frac{\beta}{\epsilon}x}+\frac{1}{3}h^3c_1(\frac{\beta}{\epsilon})^3e^{\frac{\beta}{\epsilon}x}+\cdots]=0,$$

$$\Rightarrow c_1e^{\frac{\beta}{\epsilon}x}\left[(\frac{\epsilon+\beta\tau}{h})\big(-h^2(\frac{\beta}{\epsilon})^2-\frac{1}{12}h^4(\frac{\beta}{\epsilon})^4-\cdots\big)+\frac{\beta}{2}\big(2h\frac{\beta}{\epsilon}+\frac{1}{3}h^3(\frac{\beta}{\epsilon})^3+\cdots\big)\right]=0.$$

But $c_1e^{\frac{\beta}{\epsilon}x}$ can not be zero, hence

$$\begin{aligned}
&(\frac{\epsilon+\beta\tau}{h})\left[-h^2(\frac{\beta}{\epsilon})^2-\frac{1}{12}h^4(\frac{\beta}{\epsilon})^4-\cdots\right]+\frac{\beta}{2}\left[2h\frac{\beta}{\epsilon}+\frac{1}{3}h^3(\frac{\beta}{\epsilon})^3+\cdots\right]=0,\\
&\Rightarrow -(\frac{\epsilon+\beta\tau}{h})\left[(\frac{\beta h}{\epsilon})^2+\frac{1}{12}(\frac{\beta h}{\epsilon})^4+\cdots\right]+\beta\left[\frac{\beta h}{\epsilon}+\frac{1}{3!}(\frac{\beta h}{\epsilon})^3+\cdots\right]=0,\\
&\Rightarrow -2(\frac{\epsilon+\beta\tau}{h})\left[\frac{1}{2!}(\frac{\beta h}{\epsilon})^2+\frac{1}{4!}(\frac{\beta h}{\epsilon})^4+\cdots\right]+\beta\left[\frac{\beta h}{\epsilon}+\frac{1}{3!}(\frac{\beta h}{\epsilon})^3+\cdots\right]=0,\\
&\Rightarrow -2(\frac{\epsilon+\beta\tau}{h})\left(\cosh(\frac{\beta h}{\epsilon})\ -\ 1\right)+\beta\sinh(\frac{\beta h}{\epsilon})=0,\\
&\Rightarrow (-2\frac{\epsilon}{h})\left(\cosh(\frac{\beta h}{\epsilon})-1\right)-(\frac{2\beta\tau}{h})\left(\cosh(\frac{\beta h}{\epsilon})-1\right)+\beta\sinh(\frac{\beta h}{\epsilon})=0,\\
&\Rightarrow (\frac{2\beta\tau}{h})\left(\cosh(\frac{\beta h}{\epsilon})\ -\ 1\right)=\big(-2\frac{\epsilon}{h}\big)\left(\cosh(\frac{\beta h}{\epsilon})\ -\ 1\right)+\beta\sinh(\frac{\beta h}{\epsilon}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tau &= -\frac{\epsilon}{\beta}+\frac{h}{2}\frac{\sinh(\frac{\beta h}{\epsilon})}{\big(\cosh(\frac{\beta h}{\epsilon})\ -\ 1\big)},\\
&= \frac{h}{2}\left(\frac{\sinh(\frac{\beta h}{\epsilon})}{\big(\cosh(\frac{\beta h}{\epsilon})\ -\ 1\big)}\ -\ \frac{2\epsilon}{\beta h}\right),\\
&= \frac{h}{2}\left(\frac{\sinh(\frac{\beta h}{\epsilon})}{\big(\cosh(\frac{\beta h}{\epsilon})\ -\ 1\big)}\ -\ \frac{1}{pe}\right), \qquad (3.25)
\end{aligned}$$

where, $pe = \frac{\beta h}{2\epsilon}$.

Let $\quad m = \dfrac{\beta h}{\epsilon},$ and consider

$$\frac{\sinh(m)}{\cosh(m)\ -\ 1} = \frac{\frac{1}{2}(e^m - e^{-m})}{\frac{1}{2}(e^m + e^{-m})\ -\ 1} = \frac{e^m - \frac{1}{e^m}}{e^m + \frac{1}{e^m}\ -\ 2}, \tag{3.26}$$

$$= \frac{\frac{e^{2m}}{e^m} - \frac{1}{e^m}}{\frac{e^{2m}}{e^m} + \frac{1}{e^m} - \frac{2e^m}{e^m}},$$

$$= \frac{e^{2m}\ -\ 1}{e^{2m} - 2e^m\ +\ 1} = \frac{(e^m - 1)(e^m + 1)}{(e^m - 1)(e^m - 1)},$$

$$= \frac{e^m\ +\ 1}{e^m\ -\ 1} = \frac{e^{\frac{m}{2}}(e^{\frac{m}{2}} + e^{\frac{-m}{2}})}{e^{\frac{m}{2}}(e^{\frac{m}{2}} - e^{\frac{-m}{2}})},$$

$$= \frac{\frac{1}{2}(e^{\frac{m}{2}} + e^{\frac{-m}{2}})}{\frac{1}{2}(e^{\frac{m}{2}} - e^{\frac{-m}{2}})},$$

$$= \frac{\cosh(\frac{m}{2})}{\sinh(\frac{m}{2})},$$

$$= \coth(\frac{m}{2}),\ \text{see [13].} \tag{3.27}$$

Substitute (3.26) in (3.27) to obtain

$$\frac{\sinh\left(\frac{\beta h}{\epsilon}\right)}{\left(\cosh\left(\frac{\beta h}{\epsilon}\right)\ -\ 1\right)} = \coth\left(\frac{\beta h}{2\epsilon}\right), \tag{3.28}$$

also equation (3.28) in (3.25) yields

$$\tau\ =\ \frac{h}{2}\left(\coth\left(\frac{\beta h}{2\epsilon}\right)\ -\ \frac{1}{pe}\right),$$

$$\Rightarrow \tau\ =\ \frac{h}{2}\left(\coth\left(pe\right)\ -\ \frac{1}{pe}\right),\ \text{see [8, 21].} \tag{3.29}$$

Equation (3.29) is called the "coth-formula".

Chapter 4

Numerical results

In this chapter we discuss the numerical solution of a one dimensional convection-diffusion problem of the form :

$$-ku'' + pu' + qu = x, \qquad u \in (0,1), \tag{4.1}$$
$$u(0) = u(1) = 0,$$

where k, p and q are arbitrary nonzero constants, and u is the unknown function. We apply the usual FEM, the SUPG method, and the ADM, and we compare their numerical solutions by the exact one. Throughout this chapter the MATLAB software is used to obtain the numerical approximations.

4.1 Streamline Upwind Petrov-Galerkin Method

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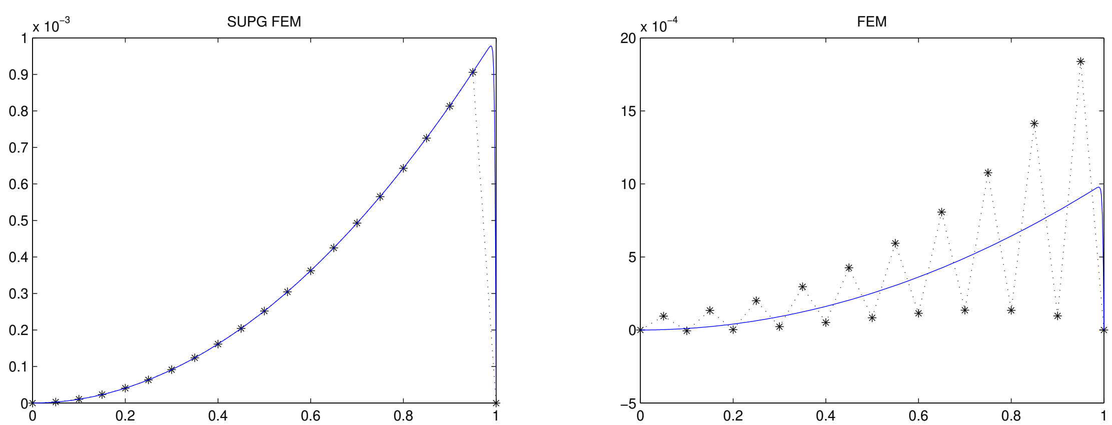

Figure 4.1: To the left: the SUPG solution. To the right: the FEM solution

Figure 4.1 shows the solution of (4.1)with $k = 1, p = 500$ and $q = 1$. Here, zero is nearly a solution of the auxiliary equation, hence the particular solution is of the form $y_p = x(Ax + B)$. The stability parameter τ obtained by the coth-formula is equal to 0.0230000000, and the numerical solution is obtained with 20 nodal elements.
Note that the numerical solution by the SUPG method is more close to the exact solution than the solution obtained by the usual FEM.
Increasing the number of nodes enhances the numerical solution of the two methods, but in the usual FEM the problem of oscillations remains unsolved as it is clear from figure 4.2.

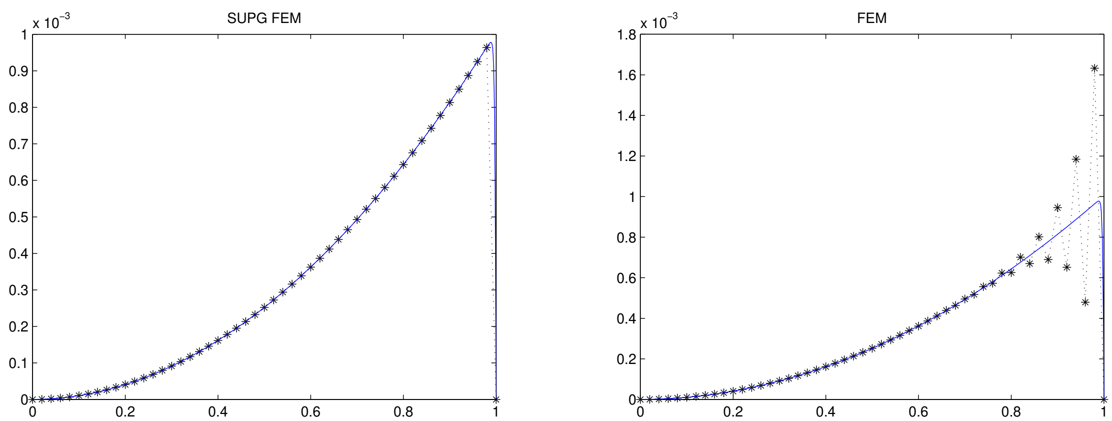

Figure 4.2: To the left: the SUPG solution. To the right: the FEM solution

Figure 4.2 is the SUPG and the FEM solutions of (4.1) with the same conditions and parameters as of figure 4.1, but $n = 50$ and thus $\tau = 0.0080009080$. Note that the numerical solution by the SUPG method is more close to the exact solution than the solution obtained by the usual FEM and we notice that the difference between the exact solution and numerical solution becomes less when the number of nodal elements becomes larger.
With mesh refinement, Figure 4.3, with $n = 100$ and thus $\tau = 0.003067836$, the FEM solution becomes better but still the spurious oscillations exist.

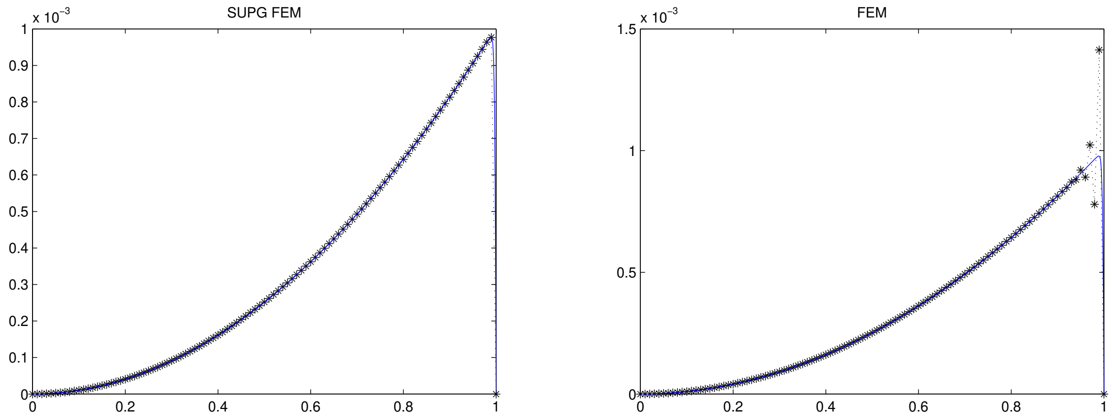

Figure 4.3: To the left: the SUPG solution. To the right: the FEM solution

Figure 4.3 shows the FEM and the SUPG solutions of(4.1) with the same conditions and parameters as of Figure 4.1.

Clearly, the numerical solution by the SUPG method is more accurate than the numerical solution by the usual FEM, also note that when the number of nodal elements n becomes larger, then the value of stability parameter τ getting less, i.e., the SUPG method is approaching the usual FEM.

Now, decreasing the value of p, means decreasing the size of the convection term, i.e., increasing the size of the diffusion term. The figure below shows the solution of the differential equation (4.1) with $k = 1, p = 250$ and $q = 1$.

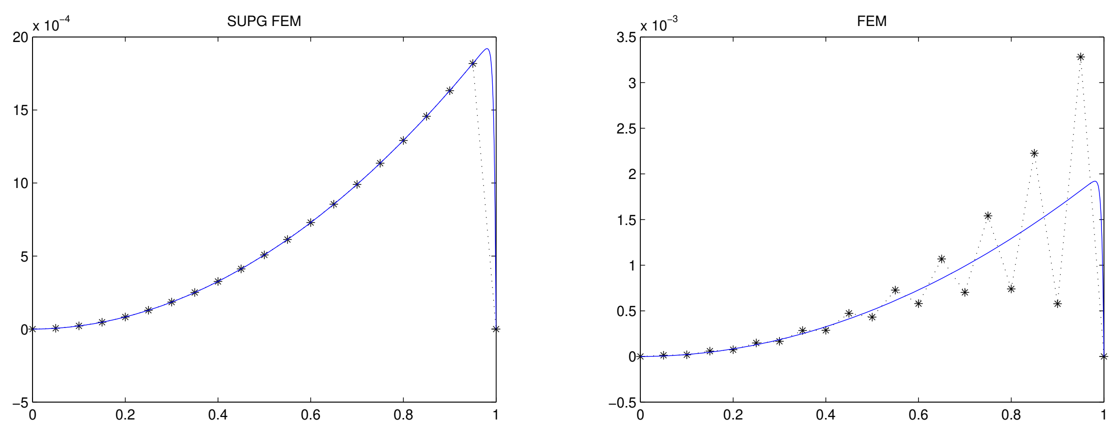

Figure 4.4: To the left: the SUPG solution. To the right: the FEM solution

Figure 4.4 shows the FEM and the SUPG solutions of equation (4.1), the numerical

solution is obtained with 20 nodes, thus $\tau = 0.0210001863$. Here, zero is not a solution of the auxiliary equation, i.e., $y_p = Ax + B$.
Obviously, the numerical solution obtained by the SUPG method is more close to the exact solution than the solution obtained by the usual FEM.
Now, it is easy to note that when we increase the number of nodes then the numerical solution of the two methods will improve, see Figure 4.5.

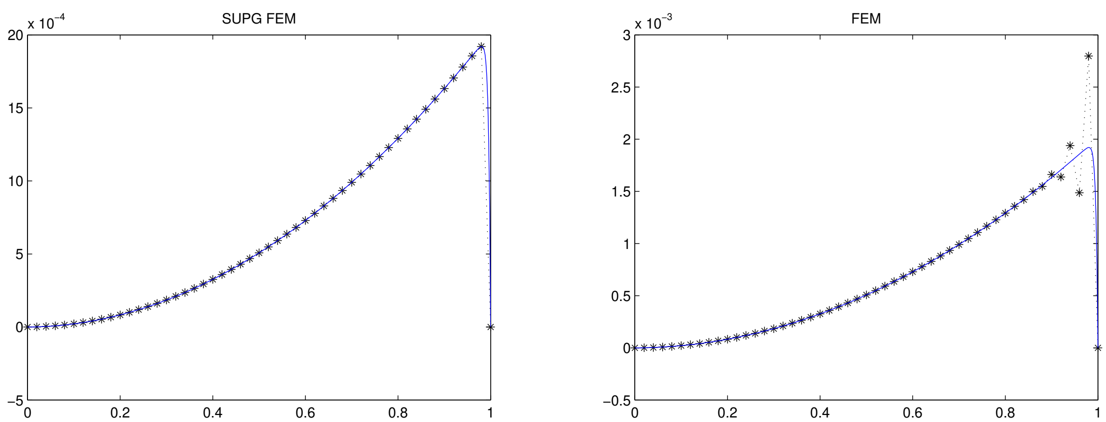

Figure 4.5: To the left: the SUPG solution. To the right: the FEM solution

Figure 4.5 shows the solution of (4.1) with the same conditions and parameters as of Figure 4.4 and the numerical solution is obtained with 50 nodal elements and thus, the stability parameter $\tau = 0.0061356730$.
Clearly, the SUPG method gives a finer numerical solution than the FEM, also note that the difference between the exact solution and the numerical solution becomes less when the number of nodes becomes larger.
Refining the mesh, Figure 4.6, with $n = 100$ and $\tau = 0.0018942548$, the FEM solution becomes better but still the spurious oscillations exist.

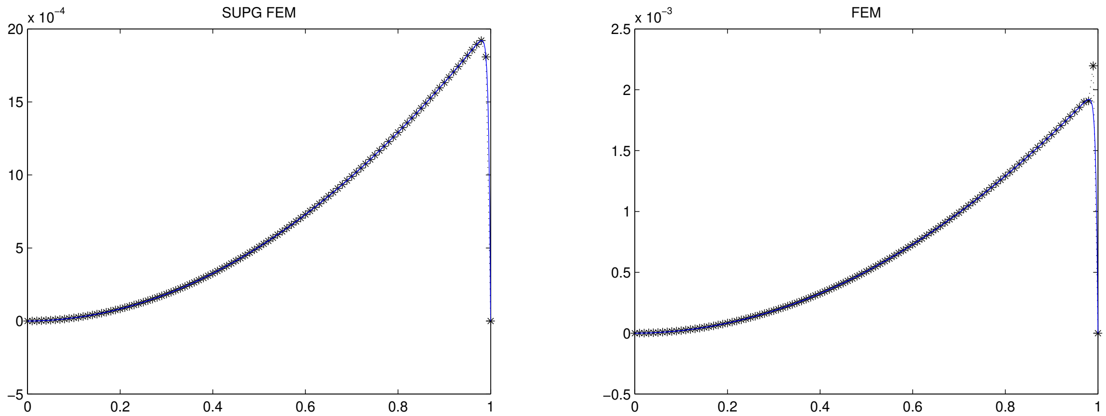

Figure 4.6: To the left: the SUPG solution. To the right: the FEM solution

Note that Figure 4.6 shows the solution of equation (4.1) with the same conditions and parameters as of Figure 4.4 but $n = 100$, and thus the stability parameter, by the coth-formula, is $\tau = 0.0018942548$. Obviously, the numerical solution obtained by the SUPG method is more close to the exact solution than the solution obtained by the usual FEM.
We remark that when the number of nodes n is getting larger, the value of the stability parameter τ is getting smaller, thus, the SUPG method approaching the usual FEM.

4.2 Artificial Diffusion Method

In this section we will discuss the numerical solution of the differential equation (4.1) by the ADM and we will compare its numerical solution by the exact one, and the solution obtained by the FEM.

Figure 4.7 shows the solution of (4.1) with $k = 1, p = 500$ and $q = 1$. Note that, since p is very large, then zero is nearly a solution of the auxiliary equation, thus $y_p = x(Ax + B)$.

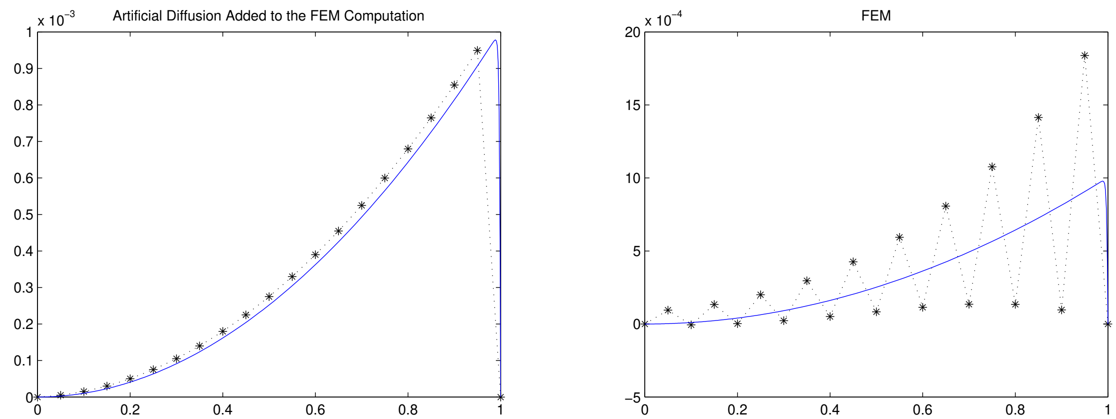

Figure 4.7: To the left: the ADM solution. To the right: the FEM solution

Figure 4.7 is the ADM and the FEM solutions of (4.1) with $n = 20$ and thus $\tau = 11.5000000000$. It is clear that the numerical solution obtained by the ADM is more accurate than the numerical solution obtained by the usual FEM.
Increasing the number of nodal elements improve the numerical solution of the two methods, see Figure 4.8.

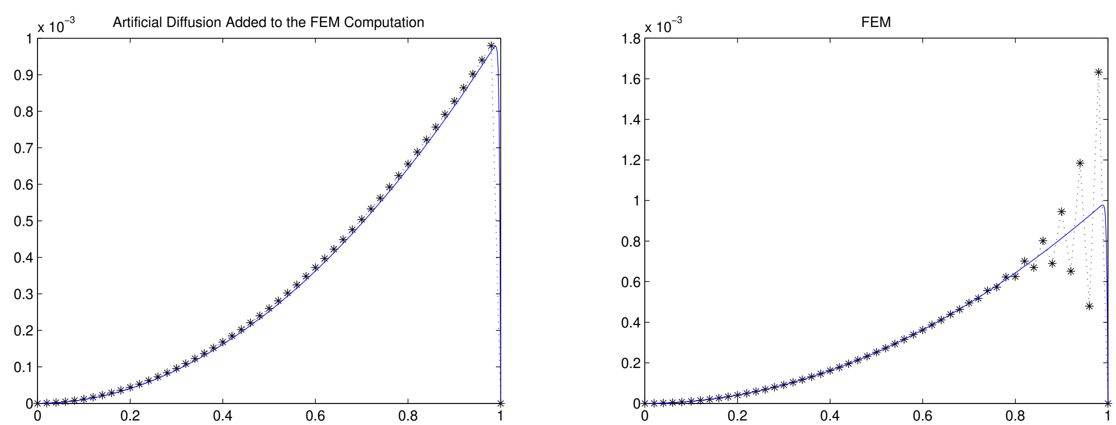

Figure 4.8: To the left: the ADM solution. To the right: the FEM solution

Figure 4.8 shows the solution of (4.1) with the same conditions and parameters as of Figure 4.7 but $n = 50$ and the stability parameter $\tau = 4.0000000000$.
Clearly that, the numerical solution obtained by the ADM is closer to the exact solution than the solution obtained by the usual FEM and the SUPG method.

With mesh refinement, Figure 4.9, with $n = 100$ and thus $\tau = 1.5000000000$, is obtained.

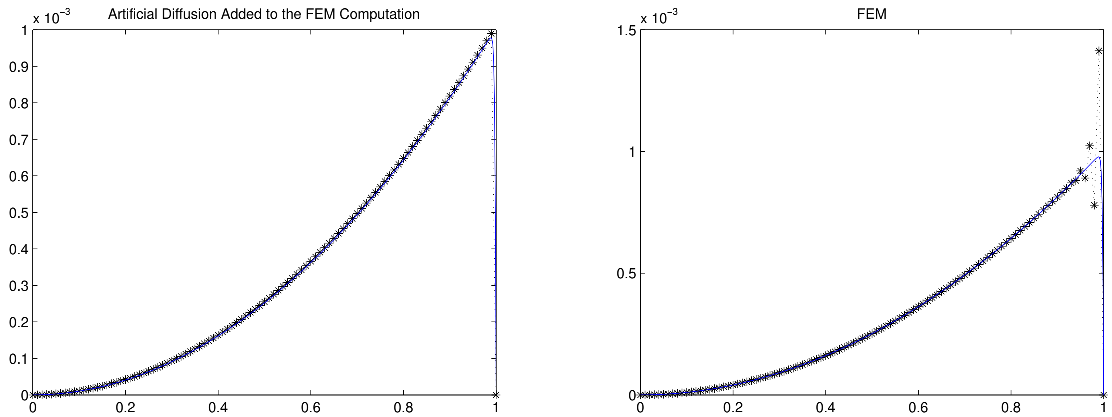

Figure 4.9: To the left: the ADM solution. To the right: the FEM solution

Figure 4.9 shows the FEM and the ADM solutions with the same conditions and parameters as of Figure 4.7.

Clearly, the numerical solution obtained by the ADM nearly matches the exact solution and is closer to the exact solution than the solution obtained by the usual FEM and the SUPG method.

Therefore, the numerical solution obtained by the ADM is more accurate than the numerical solutions obtained by other methods.

Now, let us decrease the value of p, the Figure below show the solution of the equation (4.1) with $k = 1, p = 250$ and $q = 1$.

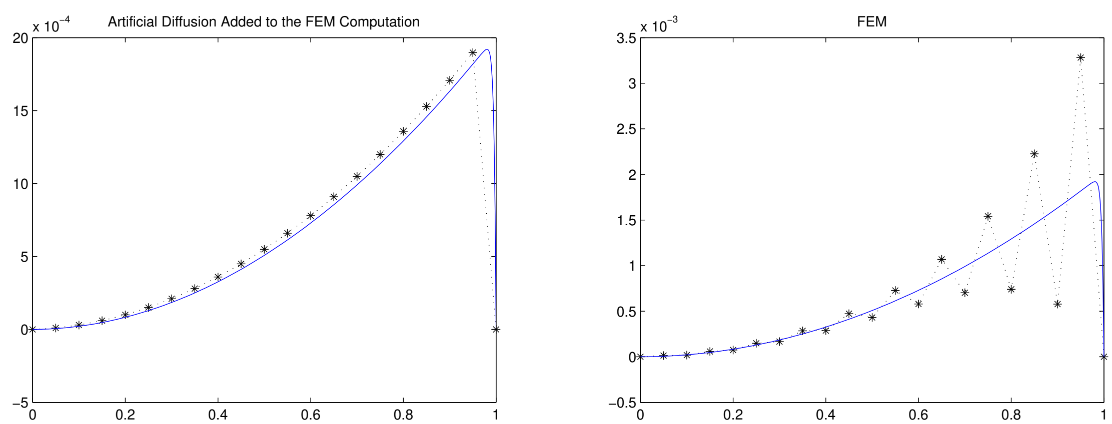

Figure 4.10: To the left: the ADM solution. To the right: the FEM solution

Figure 4.10 shows the FEM and the ADM solutions of the differential equation (4.1) with $n = 20$ and $\tau = 5.2500000000$. Here, zero is not nearly a solution of the auxiliary equation, thus, $y_p = Ax + B$.
Obviously, the numerical solution by the ADM is closer to the exact solution than the solution obtained by the usual FEM.
Notice that, increasing the number of nodal elements enhances the numerical solution of the two methods, see Figure 4.11.

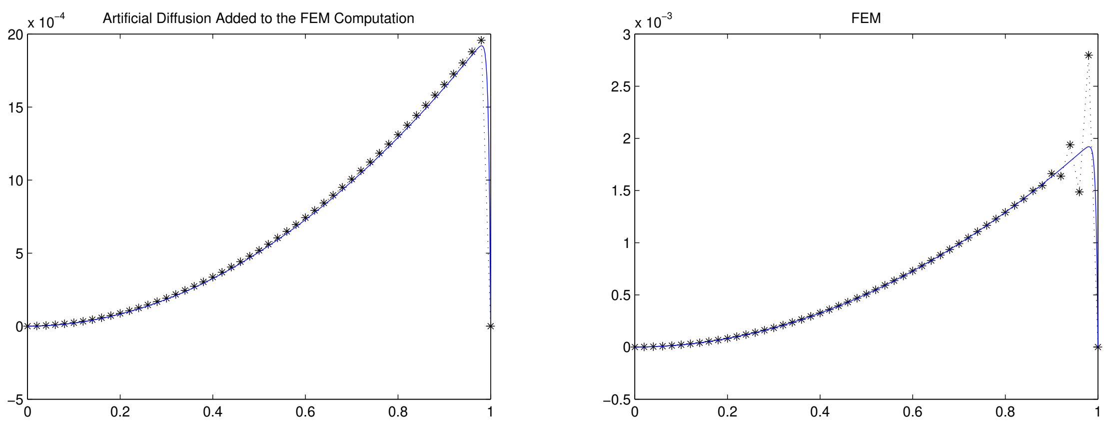

Figure 4.11: To the left: the ADM solution. To the right: the FEM solution

Figure 4.11 is the ADM and the FEM solutions of (4.1) with the same conditions and parameters as of Figure 4.10, but $n = 50$ and thus $\tau = 1.5000000000$. Note that the numerical solution obtained by the ADM is better than the solutions obtained by the usual FEM and the SUPG method.
Clearly, the difference between the exact solution and the numerical solution becomes smaller when the number of nodal elements becomes larger.
With mesh refinement, Figure 4.12, with $n = 100$ and thus $\tau = 0.2500000000$, is obtained.

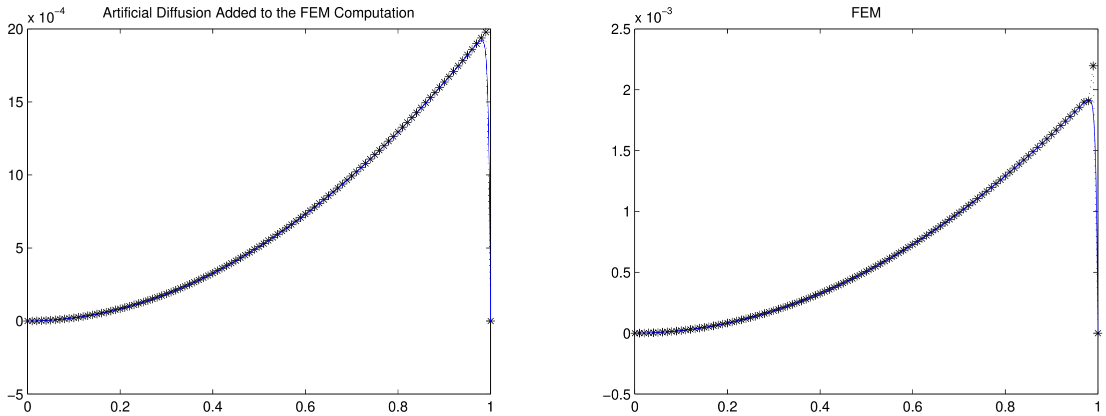

Figure 4.12: To the left: the ADM solution. To the right: the FEM solution

Figure 4.12 shows the FEM and the ADM solutions of equation (4.1) with the same conditions and parameters as of Figure 4.10.
Note that the numerical solution obtained by the ADM nearly matches the exact solution and this numerical solution is more accurate than the numerical solution obtained by other methods.

Conclusion.

In this thesis we conclude that the numerical solution obtained by the usual FEM for a diffusion-dominated problem is stable, whereas, for convection-dominated problems, the numerical solution using the FEM is not stable. Therefore, it is not recommended to use the usual FEM but the stabilized finite element methods such as SUPG and the ADM.
Using MATLAB software we conclude that the numerical solution obtained by the ADM is more accurate than the numerical solution obtained by SUPG. The error between the exact solution and the numerical solution becomes smaller when the number of nodal elements becomes larger, but the instability and the spurious oscillations still exist in the usual finite element solutions.

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