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Nilpotent Rings and Nilpotent Polynomials

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Dedications

To my great parents, who never stop giving of themselves in countless ways.

To my beloved brothers and sisters.

To my dearest fiance who has been a constant source of support and hope, and to his family - my second family -.

To my sincere friends.

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Contents

Abstract

In this research, we will study the nilpotence property for a ring R , the polynomial ring ,which is denoted by $R[x]$, and many types of rings as Armendariz rings, Noetherian and Artinian rings , principal projective ($p.p.$) rings, and J-Reduced Rings.
In addition, a generalization of many well known facts, concerning nilpotent polynomials is presented.

Furthermore we will present many concepts ,which are related to the nilpotency property, such as Nilradical, Prime radical, Jacobson radical.

We also classify which of these standard nilpotence properties on ideals passes to polynomial rings, or from ideals in polynomial rings to ideals of coefficients in the base rings, which will be our main goal in this research.

Introduction

In 1870, the American mathematician, Benjamin Pierce first introduced the term nilpotent in the context of his work on the classification of Algebras. In Algebra, an element x of a ring R is said to be nilpotent if there exists some positive integer n such that $x^n = 0$.

Over the years, the mathematicians had talked about this concept in many papers, they also had studied its relation with many properties like Nilradical, Prime radical, and Jacobson radical.

Furthermore, they had studied how it affects many types of rings, as Polynomial Rings, Noetherian rings, Artinian rings, Armendariz Rings, Principal Projective Rings, and other types of rings which we will review them in our research.

This thesis is divided into four chapters:

Chapter one: In this Chapter we introduce basic definitions, fundamental notions, and several examples. In addition we present two important types of rings which are Noetherian rings, and Artinian rings. Furhermore we give many notions related to nilpotence which is the main notion in our thesis.

Chapter two: We present the nilpotency properties in details, and apply them on rings and polynomial rings, In addition we study these properties do in ideals and elements of these rings. We also apply them on specific rings, namely, Principal Projective Rings and Laurent Polynomial Rings, which we shall study extensively.

Chapter three: In this Chapter we study Armendariz Rings and present their types, namely, weak Armendariz and nil Armendariz rings and discuss the relation between them and Armendariz Rings. Furthermore we study the effects of Armendariz,as a property of a ring, on nilpotency property. In addition for any ring with an automorphism we study the skew inverse Laurent-serieswise Armendariz rings, and introduce the notions of a strongly Armendariz ring of inverse skew power series type, and an α-compatible ring.

Chapter four: Includes The Jacobson Radical of a ring, and the relation between it and nilpotence. We present many important definitions like quasi-regular left (or right) ideal and idempotent element which we need them in order to talk about The Jacobson Radical. We also study the Prime Radical of a ring. Furthermore we present the J-Reduced Rings and J-clean Rings that are related to The Jacobson Radical.

Chapter 1

Preliminaries

The object of the present Chapter is to present basic definitions, preliminary notions and some key results which we shall require for the development of the subject matter in this thesis.

1.1 Definitions and Examples

The purpose of this section is to review some basic definitions and examples that will be needed throughout the thesis.

Definition 1.1.1. [5] *A ring R is called* ***semicommutative ring*** *if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$.*

Definition 1.1.2. *An element x of a ring R is called* ***nilpotent*** *if there exists some positive integer n such that $x^n = 0$.*
And a ring is called ***nil*** *if every element of the ring is nilpotent.*

Example 1.1.1. *The nilpotent elements of the ring $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$ of integers modulo 8 are $\bar{0}, \bar{2}, \bar{4}, \bar{6}$,since $\bar{2}^3 = \bar{0}$, $\bar{4}^2 = \bar{0}$, $\bar{6}^3 = \bar{0}$.*

We denote the set of all nilpotent elements of R by $nil(R)$.

Definition 1.1.3. *A **subring** S of a ring R is a nonempty subset of R which is a ring under the same operations as R.*

Equivalently, a non-empty subset S of R is a subring if for $a, b \in S$, then $a - b$ and $ab \in S$. So S is closed under subtraction and multiplication. It is denoted by $S \trianglelefteq R$.

Example 1.1.2. *If n is any integer, then the set of all multiples of $n\mathbb{Z}$ is a subring of $\mathbb{Z}$.*

Definition 1.1.4. *A nonempty subset I of a ring R is called a **two-sided ideal(or simply an ideal)** if*

- $(I, +)$ *is a subgroup of* $(R, +)$, *i.e. for* $a, b \in I, a - b \in I$.
- $\forall a \in I$ *,and* $\forall r \in R :\quad a \cdot r \in I$, *i.e.* $IR \subseteq I$.
- $\forall r \in R$ *,and* $\forall a \in I :\quad r \cdot a \in I$, *i.e.* $RI \subseteq I$.

Definition 1.1.5. *A subset I of R is called a **right ideal** of R if*

- $(I, +)$ *is a subgroup of* $(R, +)$, *i.e. for* $a, b \in I, a - b \in I$.
- $\forall a \in I$ *,and* $\forall r \in R :\quad a \cdot r \in I$, *i.e.* $IR \subseteq I$.

*Similarly, a subset I of R is called a **left ideal** of R if*

- $(I, +)$ *is a subgroup of* $(R, +)$, *i.e. for* $a, b \in I, a - b \in I$.
- $\forall r \in R$ *,and* $\forall a \in I :\quad r \cdot a \in I$, *i.e.* $RI \subseteq I$.

*Note that **One-sided ideal** is either right or left ideal.*

Definition 1.1.6. *An ideal, I, of a ring is said to be **a nilpotent ideal**, if there exists a natural number k such that $I^k = 0$, i.e. I is nilpotent if and only if there is a natural number k such that the product of any k elements of I is 0.*

Note that an ideal is called **nil ideal** if each of its elements is nilpotent.

Example 1.1.3. *The ideal* $I = 2\mathbb{Z}_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ *is a nil ideal, since every element in* I *is nilpotent:*

$\bar{2}^3 = \bar{0}$, $\bar{4}^2 = \bar{0}$, $\bar{6}^3 = \bar{0}$.

Definition 1.1.7. *The sum of the nilpotent ideals of* R*, is called the* ***Wedderburn radical****, and is denoted by* $N_0(R)$.

Definition 1.1.8. *An ideal* I *of* R *is* ***finitely generated*** *if there is a finite subset* X *of* I *such that* $I =< X >$ *, i.e. such that* I *can be generated by a finite set of elements.*
i.e., if there is a finite set $X = \{x_1, x_2, \ldots, x_n\}$ *such that* $I = x_1R + x_2R + \ldots + x_nR$.
In the special case where $X = \{a\}$*, we write* $< \{a\} >=< a >= RaR$ *and such an ideal is called a principal ideal.*

Definition 1.1.9. *Let* R *be a commutative ring,and* I *an ideal of* R*,* ***The Quotient Ring*** *is* $(R/I, +, *)$ *, where* $\forall a, b \in R$ *, the addition and multiplication of cosets of* I *is defined as :*
$(+)$*:* $(I + a) + (I + b) = I + (a + b)$ *, where* $I + a = \{r + a : r \in I\}$
$(*)$*:* $(I + a) * (I + b) = I + ab$.

Definition 1.1.10. *Let* S *be a nonempty set and* $\bullet$ *is some binary operation* $S \times S \to S$*, then* $(S, \bullet)$ *is a* ***monoid*** *if it satisfies the following two axioms:*

(1) ***Associativity****;*

For all a*,* b *and* c *in* S*, the equation* $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ *holds.*

(2) ***Identity element****;*

There exists an element e *in* S *such that for every element* $a \in S$*, the equations* $e \bullet a = a \bullet e = a$ *hold.*

*Note that a **submonoid** is a subset of the elements of a monoid that are themselves a monoid under the same monoid operation.*

Example 1.1.4. *consider the monoid formed by the nonnegative integers under the operation $min(x+y,10)$. Then restricting x and y from all the integers to the set of elements $S=\{0,3,5,6,8,9,10\}$ forms a submonoid of the monoid $\{0,1,2,3,4,5,6,7,8,9,10\}$ under the operation $min(x+y,10)$.*

Definition 1.1.11. *Let $(R,+,.)$ and $(R',+',.')$ be rings, and let f be a function from R into R', then f is called **homomorphism** of R into R' if for all $a,b\in R$, we have the following:*
(1) $f(a+b)=f(a)+'f(b)$.
(2) $f(a.b)=f(a).'f(b)$.

Definition 1.1.12. *A homomorphism of a ring R into a ring R' is called :*

1. ***monomorphism** if f is a one to one function.*

2. ***epimorphism** if f is an onto function.*

3. ***isomorphism** if f is a one to one and onto function.*

Definition 1.1.13. ***An automorphism** α of a ring R is an isomorphism from R onto R.*

Polynomials's Multiplication Formula.

consider the polynomials;

$$f(x)=a_0+a_1x+\ldots+a_{n-1}x^{n-1}+a_nx^n \tag{1.1}$$

and

$$g(x) = b_0 + b_1x + \ldots + b_{m-1}x^{m-1} + b_mx^m \tag{1.2}$$

then ,

$$f(x)g(x) = r_0 + r_1x + \ldots + r_{m+n}x^{m+n} \tag{1.3}$$

where $r_i = \sum_{j=0}^{i} a_jb_{i-j} \qquad , i = 0, 1, 2, \ldots, m+n.$

Definition 1.1.14. *The* ***polynomial ring****,* $R[x]$*, in* x *over a ring* R *is defined as the set of polynomials in* x *, of the form :*

$$p(x) = p_0 + p_1x + \ldots + p_{n-1}x^{n-1} + p_nx^n \tag{1.4}$$

where p_0 *,* p_1 *, ... ,* p_n *, are the coefficients of* $p(x)$*, which are elements in a ring* R*. Addition and multiplication are defined as the definitions of addition and multiplication in* R *for the coefficients, the distributive and associative laws, and the exponent rule for* x *, that is,* $x^i.x^j = x^{i+j}$ *hold in* $R[x]$*.*

Definition 1.1.15. ***The ring of formal power series*** *in* x *with coefficients in* R *is denoted by* $R[[x]]$*, and is defined as follows. The elements of* $R[[x]]$ *are infinite expressions of the form* $f(x) = \sum_{i=0}^{\infty} a_ix^i$ *in which* $a_n \in R$ *for all* $n \in \mathbb{N}$

Definition 1.1.16. *Let* R *be a ring.* ***A left (right) annihilator*** *of a subset* U *of* R *is denoted by* $lAnn_R(U) = \{a \in R : aU = 0\}$ *,* $(rAnn_R(U) = \{a \in R : Ua = 0\})$*.*

1.2 Noetherian and Artinian Rings

In this section, we describe special types of rings which are the Noetherian and Artinian Rings.

Definition 1.2.1. *A ring R is* ***Noetherian Ring*** *if it satisfies the ascending chain condition $[ACC]$ on ideals; that is, given any chain of ideals : $I_1 \subseteq I_2 \subseteq \cdots$,there exists an $N \in \mathbb{N}$ such that : $I_n = I_N$ for $n \geqslant N$.*

Example 1.2.1. [13] $(\mathbb{Z}, +, *)$ *is a Noetherian Ring. This is because any ascending chain of ideals must terminate at the ideal $p\mathbb{Z}$ for some prime number p.*

Definition 1.2.2. *An ideal I in a ring R is said to be* ***maximal*** *if $I \neq R$ and for every ideal N such that $I \subseteq N \subseteq R$, either $N = I$ or $N = R$.*

Example 1.2.2. *For the ring of integers $\mathbb{Z}$, $n\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$ iff n is prime.*

Theorem 1.2.1. *Let R be a ring. The following are equivalent:*

1. *R is left Noetherian.*
2. *Every nonempty set S of left ideals of R has a maximal element I, i.e., there exists $I \in S$ such that $J \in S, I \subseteq J$ implies $I = J$.*
3. *Every left ideal of R is finitely generated.*

Similar theorems hold for right and two-sided Noetherian rings.

proof:

(1) $\Rightarrow$ (2) : *Suppose there exists a nonempty set of left ideals S of R without a maximal element. Note for any $I_k \in S$. Since I_k is not maximal in S, we may find $I_{k+1} \in S$ such that $I_k \subset I_{k+1}$. Thus since S is not empty, we may construct an infinite chain $I_1 \subset I_2 \subset I_3 \subset I_4 \subset \ldots$ which does not stabilize, contradicting the fact that R is left Noetherian. Thus every nonempty set of left ideals of R must have a maximal element.*

(2) $\Rightarrow$ (1) : *Let $S = I_1 \subset I_2 \subset I_3 \subset \ldots$ be an ascending chain of left ideals. By assumption S has a maximal element say I_N . Since I_N is maximal and $I_N \subseteq I_n$ for all $n \geq N$ we see that $I_N = I_n$ for all $n \geq N$ and so the chain stabilizes. Thus every ascending chain of left*

ideals stabilizes and so R is left Noetherian.

$(3) \Rightarrow (1)$: *Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ be an ascending chain of left ideals in R. Then $I = \cup I_n$ is a left ideal of R. By assumption $I = (x_1, \ldots, x_k)_L$. For each $1 \leq j \leq k, x_j \in I$ so $x_j \in I_{n_j}$ for some positive integer n_j. Thus $\{x_1, \ldots, x_k\} \subseteq I_{max(n_1,n_2,\ldots,n_k)}$ from which it follows that $I \subseteq I_{max(n_1,n_2,\ldots,n_k)}$ and hence $I = I_{max(n_1,n_2,\ldots,n_k)}$. Setting $N = max(n_1, n_2, \ldots, n_k)$ it then follows easily that $I_N = I_{N+s}$ for $s \in N$ and so the chain stabilizes. Thus any ascending chain of left ideals in R stabilizes and so R is left Noetherian.*

$(1) \Rightarrow (3)$: *Let I be a left ideal of R. Suppose that I is not finitely generated as a left ideal of R. Then in particular $I \neq (0)$. Hence we may choose nonzero $a_1 \in I$. Suppose we have chosen $a_1, \ldots, a_k \in I$. Then since I is not finitely generated as a left ideal, there exists $a_{k+1} \in I - (a_1, \ldots, a_k)_L$. Thus using the axiom of countable choice we may choose $a_1, a_2, \ldots$ such that the left ideals $(a_1, \ldots, a_k)_L = I_k$ form an infinite ascending chain of left ideals which does not stabilize. This contradicts the fact that R is left Noetherian and so we conclude that every left ideal is finitely generated.* □

Theorem 1.2.2. *A ring is Noetherian if every ideal is finitely generated.*

proof:

Suppose that R is Noetherian. Let I be an ideal of R, and let

$$S = \{K : K \trianglelefteq R, K \subseteq I \ , K \ \ is \ \ finitely \ \ generated\}.$$

Then $\{0_R\} \in S$, so S is not empty, so S contains a maximal element M. Then M is finitely generated, so there is a finite subset A of R with $M =< A >$. Let $x \in I$. Put $B = A \cup \{x\}$ and $J =< B >$. Then B is finite so J is finitely generated, and $J \subseteq I$ because $B \subseteq I$. Therefore $J \in S$. But $M \subseteq J$, and M is maximal, so $M = J$, so $x \in M$. This is true for all x in I, so $I \subseteq M$. But $M \subseteq I$, because $M \in S$. Therefore $I = M$, so I is finitely generated. □

Theorem 1.2.3. ***(Hilbert's Basis Theorem)***

Let R be a commutative ring with identity. If R is a Noetherian ring then so is the polynomial ring $R[x]$.

proof:

Let $I \subseteq R[x]$ be an ideal of $R[x]$, we will show that I is finitely generated.

Let f_1 be an element of least degree in I, and let $(g_1, \ldots, g_r)$ denote the ideal generated by the polynomials $g_1, \ldots, g_r$. For $i \geq 1$, if $(f_1, \ldots, f_i) \neq I$, then choose f_{i+1} to be an element of minimal degree in $I/(f_1, \ldots, f_i)$.

If $(f_1, \ldots, f_{i+1}) = I$, then we are done.

Let a_j be the leading coefficient of f_j. Since R is Noetherian, the ideal $(a_1, a_2, \ldots) \subseteq R$ is generated by $a_1, a_2, \ldots, a_m$ for some $m \in \mathbb{N}$.

We claim that $f_1, f_2, \ldots, f_m$ generates I.

Suppose not. Then our process chose an element f_{m+1}, and $a_{m+1} = \sum_{j=1}^{m} u_j a_j$ for some $u_j \in R$.

Since the degree of f_{m+1} is greater than or equal to the degree of f_j for $j = 1, \ldots, m$,then the polynomial:

$$g = \sum_{j=1}^{m} u_j f_j x^{degf_{m+1} - degf_j} \in (f_1, \ldots, f_m)$$

has the same leading coefficient and degree as f_{m+1}. The difference $f_{m+1} - g$ is not in $(f_1, f_2, \ldots, f_m)$ and has degree strictly less than f_{m+1}, a contradiction of our choice of f_{m+1}.

Thus $I = (f_1, f_2, \ldots, f_m)$ is finitely generated, and we are done. □

In the following example we see a ring which is noetherian, but it's ring of polynomials is not.

Example 1.2.3. [6] *Let R be the even integers $2\mathbb{Z}$, it is a commutative ring that has no multiplicative identity. So we shall show that $2\mathbb{Z}$ is Noetherian but $2\mathbb{Z}[x]$ is not.*

proof:

Let I be a non-zero ideal of $2\mathbb{Z}$. If $a \in I$ then $-a \in I$ because I is a subgroup of $(2\mathbb{Z}, +)$. Let a be the smallest positive element of I. Suppose that $b \in I$ with $b > 0$.
In $\mathbb{Z}$, there are integers q and r such that $b = qa + r$ and $0 \leq r < a$. Now,

$$qa = \underbrace{a + a + \ldots + a}_{\boldsymbol{q\ times}},$$

which is in I. Thus $b - qa \in I$ and so $r \in I$. Because $0 \leq r < a$, we must have $r = 0$. Therefore $I = \{qa : q \in \mathbb{Z}\}$ (of course, a is even). So every ideal of $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$. We know that $\mathbb{Z}$ satisfies ACC, so $2\mathbb{Z}$ must also satify ACC. If $2\mathbb{Z}[x]$ is Noetherian then it is finitely generated. Suppose that the generators are $f_1(x), \ldots, f_n(x)$, where $f_i(x)$ has degree d_i. Put $N = max\{d_1, \ldots, d_n\}$. Then if $g(x) \in \langle\{f_1(x), \ldots, f_n(x)\}\rangle$ then either the degree of $g(x)$ is at most N or every coefficient in $g(x)$ is divisible by 4.
Therefore $2x^{N+1} \notin \langle\{f_1(x), \ldots, f_n(x)\}\rangle$ but $2x^{N+1} \in 2\mathbb{Z}[x]$. This contradiction shows that $2\mathbb{Z}[x]$ is not Noetherian.

Theorem 1.2.4. *Let $M_n(R)$ be the ring of all $n \times n$ matrices with entries from R, if the ring R is not Noetherian then $M_n(R)$ is not Noetherian, and if the ring R has an identity and R is Noetherian then $M_n(R)$ is Noetherian.*

proof:
(1) : *Let $I_1 \subset I_2 \subset \ldots \subset I_m \subset I_{m+1} \subset \ldots$ be an infinite ascending chain of ideals of R. Then $M_n(I_1) \subset M_n(I_2) \subset \ldots \subset M_n(I_m) \subset M_n(I_{m+1}) \subset \ldots$ is an infinite ascending chain of ideals of $M_n(R)$. Therefore it is not noetherian.*
(2) : *Let $J_1 \subseteq J_2 \subseteq \ldots$ be an ascending chain of ideals of $M_n(R)$. Since R has an identity, there are ideals I_m of $M_n(R)$ such that $J_m = M_n(I_m)$ for $m = 1, 2, \ldots$. Then $I_1 \subseteq I_2 \subseteq \ldots$. Since R is Noetherian, there is some N such that $I_m = I_N$ whenever $m \geq N$. Then*

$J_m = M_n(I_m) = M_n(I_N) = J_N$ when $m \geq N$. So it is noetherian. $\square$

Definition 1.2.3. *A ring R is* ***Artinian Ring*** *if it satisfies the descending chain condition $[DCC]$ on ideals, that is, given any chain of ideals : $I_1 \supset I_2 \supset \cdots$, there exists an $N \in \mathbb{N}$ such that $I_n = I_N$ for $n \geqslant N$.*

Example 1.2.4. *$\mathbb{Z}/n\mathbb{Z}$ or any finite ring is Artinian.*

- *A ring is* ***left-Noetherian (Artinian)*** *if it satisfies the ascending (descending) chain condition on left ideals.*
- *A ring is* ***right-Noetherian (Artinian)*** *if it satisfies the ascending (descending) chain condition on right ideals.*
- *A ring is* ***Noetherian (Artinian)*** *if it is both left- and right-Noetherian (Artinian).*

We now give an example of a Noetherian rings that is not Artinian.

Consider the ring of integers $\mathbb{Z}$ which is a Noetherian ring , but not Arinian ,since for an ideal $I_1 = a\mathbb{Z}$ to contain an ideal $I_2 = b\mathbb{Z}$, a must divide b. For I_2 to contain a third ideal, $I_3 = c\mathbb{Z}$, b must divide c. If we continue this pattern, each successive ideal must have a generator that is a multiple of the generator of the ideal immediately previous in the chain. Since the integers are infinite, we can keep constructing multiples forever. Our chain will never bottom out, so it fails to satisfy the descending chain condition. Thus, $\mathbb{Z}$ is not Artinian.

Chapter 2

Nilpotent Rings

In this Chapter, the nilpotency properties are presented in details, and they are applied on rings and polynomial rings; In addition, we study which of these properties are true in ideals and elements of these rings. Also these propositions are checked on specific rings, namely, Principal Projective Rings and Laurent Polynomial Rings.

2.1 Nilpotency Properties

In this section we present many of nilpotency properties, and introduce theorems on rings and polynomial rings which have these properties.

Definition 2.1.1. *A ring R is called* ***reduced*** *if it has no nonzero nilpotent elements.*

Example 2.1.1. *The ring of integers $\mathbb{Z}$ is a reduced ring.*

Note that in a commutative ring R, $N_0(R)$ equals the set of all nilpotent elements of R. But this is not true for non-commutative rings.

Example 2.1.2. *Let R be the ring of 2×2 matrices over $\mathbb{Q}$, then R has only two ideals 0 and R(since $\mathbb{Q}$ is a field). So $N_0(R) = 0$ but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = 0$.*

Lemma 2.1.1. *In any ring R, the sum of two nilpotent ideals is a nilpotent ideal.*

proof:

Let I, J be ideals in R such that $I^n = 0$ and $J^m = 0$ for $n, m \in \mathbb{N}$. We claim that $(I + J)^{n+m-1} = 0$. That is, the product of $n+m-1$ elements of the form $u+v$,$u \in I$, $v \in J$, is 0. *Such a product can be written as a sum of products $w = w_1 w_2 \dots w_{n+m-1}$ where each $w_i \in I \cup J$. If at least n of these w_i 's are in I, then $w = 0$ as $I^n = 0$. If the number of the w_i 's belonging to I is smaller than n, then at least m of them lie in J, and hence $w = 0$ since $J^m =$* 0. ☐

Corollary 2.1.1. *We can generalize the previous lemma to any sum of finitely many nilpotent ideals.*

Lemma 2.1.2. [10] *An arbitrary sum of nil ideals is a nil ideal.*

proof:

It suffices to show that the sum of two nil ideals is again a nil ideal , so let I, J be nil ideals and $a+b = z \in I+J$, we have $a^k = 0$ for some $k \in \mathbb{N}$, hence $z^k = a^k + c$ for some suitable $c \in J$.

Since J is a nil ideal , we get $c^t = 0$ for some $t \in \mathbb{N}$ and $z^{tk} = 0$. ☐

Theorem 2.1.1. [14] *Let R be a finitely generated commutative ring without unity. Then the following are equivalent:*

(1) R is nil.

(2) R is nilpotent.

(3) every proper ideal of R is nilpotent.

(4) every proper ideal of R is nil.

proof:

(1) $\Rightarrow$ (2)*: Since every element of R is nilpotent, every generator of R is nilpotent. Let $x_1, \ldots, x_n$ be a set of generators for R with corresponding indices of nilpotency $k_1, \ldots, k_n$. So, the product of an arbitrary $k = k_1 + \ldots + k_n$ factors is*

$(m_{1,1}x_1 + r_{1,1}x_1 + \ldots + m_{n,1}x_n + r_{n,1}x_n) \ldots (m_{1,k}x_1 + r_{1,k}x_1 + \ldots + m_{n,k}x_n + r_{n,k}x_n)$

where each $m_{i,j} \in \mathbb{Z}$ and each $r_{i,j} \in R$. Observe that this product is a sum where each term is of the form $cx_{i_1} \ldots x_{i_k}$, where c is an integer or an element of R and the x_{i_j} 's may be repeated. So, each term has at least k generator factors. Consider an arbitrary term of the sum. For each x_i, let α_i be the number of generator factors equal to x_i. So , $\alpha_1 + \ldots + \alpha_k \geqslant k$. If $\alpha_i < k_i$ for each i, then it would be the case that $\alpha_1 + \ldots + \alpha_k < k_1 + \ldots + k_k = k$, a contradiction. Thus, there is some j with $\alpha_j \geqslant k_j$. So this term has a factor of ${x_j}^{k_j}$, which is equal to zero, since x_j has index of nilpotency k_j . Thus, each term of the sum is equal to zero, and so the entire sum is equal to zero. Therefore, the product of any k terms is equal to zero, hence $R^k = 0$.

(2) $\Rightarrow$ (3)*:By the definition of nil ideals.*

(3) $\Rightarrow$ (4)*:Since every every proper ideal of R is nilpotent, so they are also nil.*

(4) $\Rightarrow$ (1)*:We proceed by contradiction. Assume that every proper ideal of R is nil. Suppose that R contains an element a which is not nilpotent. Then a^2 is also not nilpotent. Note that $a^2 \in aR = \{ar \mid r \in R\}$. Thus, $aR = R$. Hence there is some $e \in R$ such that $a = ae$. Let $r \in R$. Then there is some $x \in R$ such that $r = ax$. So*

$$re = (ax)e = (ae)x = ax = r$$

Thus e is a multiplicative identity of R , a contradiction. Therefore, R is nil. □

Remark 2.1.1. *In a commutative ring with 1 , a finitely generated nil ideal is nilpotent.*

proof:

Let K be a finitely generated nil ideal of R , $K = k_1R+\ldots+k_mR$, with $k_i \in K$. since each k_i is nilpotent , then so is the ideal. $\square$

Proposition 2.1.1. [14] *If R is a nil ring, then $xy \neq y$ for all nonzero $x, y \in R$. If R is a finite ring, then R is nil if and only if for every nonzero $x, y \in R$, it is the case that $xy \neq y$.*

proof:

Let x and y be nonzero elements of the nil ring R. So $x^n = 0$ for some positive integer n. If $xy = y$ then $0 = (x^n)y = (x^{n-1})(xy) = (x^{n-1})y = (x^{n-2})(xy) = (x^{n-2})(y) = \ldots = xy = y$,a contradiction. Thus, $xy \neq y$.

Now, assume that R is finite and suppose that for every nonzero $x, y \in R, xy \neq y$. Let $x \in R$. Let n, m be positive integers with $n < m$. So, if $x^n \neq 0$ and $x^m \neq 0$, then $x^n \neq x^n x^{m-n} = x^m$. So, each positive power of x which is not zero must be distinct. But, R is finite, so it must be the case that $x^k = 0$ for some k. Thus, every element of R is nilpotent. $\square$

Definition 2.1.2. *Let R be a ring and let I be an ideal of R. We say that I is **a prime ideal** if whenever $ab \in I$ then either $a \in I$ or $b \in I$.*

Lemma 2.1.3. $N = \bigcap P_\alpha$.

Proof

Let $x \in N$, so x is a nilpotent element in R. Now $x^n = 0 \in P$, so $x.x^{n-1} = x^n \in P$, but P is a prime ideal, so $x \in P$ or $x^{n-1} \in P$.

If we assume that $x^{n-1} \in P$ and $x \notin P$, then $x^{n-1} \in P \Rightarrow x.x^{n-2} = x^{n-1} \in P \Rightarrow x \in P$ or $x^{n-2} \in P$, if we assume that $x^{n-2} \in P$ and $x \notin P$ and continue as a previous steps $n-1$ times, we will get that $x.x^{n-(n-1)} \in P$, so if we take the choice $x \in P$ and $x^{n-(n-1)} \notin P$, *this will give us that $x \in P$ and $x \notin P$ which is a contradiction.*

So from the first step, we have that $x \in P$ for every prime ideal. Therefore $x \in \bigcap P_\alpha$, hence $N \subseteq \bigcap P_\alpha$.

For the converse, we will show that for any non-nilpotent element a, there is some prime ideal that doesn't contain a.

Let a be a non-nilpotent element in R, and S be the set of all ideals of R that don't contain any element of the form a^n. Since $0 \in S$, so $S \neq \phi$, and by Zorn's lemma, S has a maximal element M.

it suffices to show that M is a prime ideal, so suppose otherwise, $x, y \in M$ but $xy \in M$, then the set of elements z for which $xz \in M$ is an ideal of R that properly contains M, this implies that it contains $a^m, m \in \mathbb{Z}$, so $a^{n+m} \in M$, by assumption of M, $a^n = 0 \notin M$ which is a contradiction. Hence M is a prime ideal which doesn't contain a. so we get $\bigcap P_\alpha \subseteq N$. Therefore $N = \bigcap P_\alpha$ as we need. □

Theorem 2.1.2. *if R is a commutative Noetherian ring, then a power series $f(x)$ over a commutative ring R is nilpotent if and only if each coefficient of $f(x)$ is nilpotent.*

proof.

$\Rightarrow$: Let P_α be the collection of prime ideals of R and let N be the ideal of nilpotent elements of R and consider Lemma (2.1.3), now for each α, $P_\alpha[[x]]$ is a prime ideal of $R[[x]]$. Since $f(x)$ is nilpotent,$f(x) \in P_\alpha[[X]]$ for each α. Hence $f(x) \in \bigcap P_\alpha[[X]] = (\bigcap P_\alpha)[[X]] = N[[x]]$; that is, each coefficient of $f(x)$ is nilpotent. Suppose f is nilpotent , since R is Noetherian ring , the set of coefficients of $f(x)$ (A_f) is finitely generated. Since f is nilpotent , each a_i is nilpotent ,i.e. each coefficient of $f(x)$ is nilpotent.

$\Leftarrow$: let $f(x) = \sum\limits_{i=0}^{\infty} a_i x_i$, suppose A_f is nilpotent ,and $(A_f)^m = 0$, then $a_i^m = 0$ for all i. On the other hand if $a_i^m = 0$ for $i \geq 0$,then there is a positive integer k such that $(A_f)^k = 0$

, *so* $f^k = 0$ *,hence* f *is nilpotent .* $\square$

Lemma 2.1.4. *Let* R *be a right noetherian ring, then the sum of nilpotent ideals in* R *is nilpotent ideal.*

proof:

Let $B = \sum_{i \in I} A_i$ *be the sum of nilpotent ideals in* R*. Since* R *is noetherian (i.e. right noetherian),* B *is finitely generated as right ideal. Suppose* $B = (x_1, \ldots, x_n)$*, then each* x_i *lies in finitely many* $A_{i's}$ *, hence* B *is contained in the sum of a finite number of* $A_{i's}$*, say (after reindexing if necessary)* $A_1, \ldots, A_n$*. Thus* $B = A_1 + \ldots + A_n$*, then by corollary* (2.1.1) *,* B *is nilpotent.* $\square$

Remark 2.1.2. *In a right artinian ring, every nil right ideal is nilpotent.*

Definition 2.1.3. *By* ***a maximal nilpotent ideal****, we mean a nilpotent ideal that is not properly contained in a larger nilpotent ideal.*

Lemma 2.1.5. *If a ring* R *has a maximal nilpotent ideal* N*, then* N *contains all nilpotent ideals of* R*.*

proof:

If I *is another nilpotent ideal, then* $I+N$ *is again a nilpotent ideal by Lemma* (2.1.1)*.Because of the maximality of* N *we must have* $I+N = N$*, and thus* $I \subseteq N$*.* $\square$

Thus, a maximal nilpotent ideal , if it exists , is unique and is equal to the sum of all nilpotent ideals $N_0(R)$*. However, not every ring has such an ideal. That is to say, the sum of all nilpotent ideals of a ring is not always nilpotent(although it is nil for each of its* *elements " Lemma* (2.1.1) *").*

An example of this, the maximal nilpotent ideal $2\mathbb{Z}_8$ *of the ring* $\mathbb{Z}_8$.

The ***nilradical of a commutative ring*** *is the set of all nilpotent elements in the ring , which was denoted previously by* $nil(R)$.

But in the case of ***a noncommutative ring*** *, there are several analogues of the nilradical , like lower nilradical and upper nilradical.*

Definition 2.1.4. *The* ***upper nilradical*** *of a ring* R *is the sum of all nil ideals , and it is denoted by* $Nil^*(R)$.

Definition 2.1.5. *The* ***lower nilradical*** *of a ring* R *is the intersection of all prime ideals , we denote it by* $Nil_*(R)$.

Note that the lower nilradical is also called the ***prime radical****.*

Recall that for an element $a \in R$, $(a) = RaR$, *is the ideal generated by* a *in* R.

Definition 2.1.6. *An ideal* I *in a ring* R *is said to be a* ***semiprime ideal*** *if, for any ideal* J *of* R, $J^2 \subseteq I$ *implies that* $J \subseteq I$. *(For instance, a prime ideal is always semiprime.)*

Definition 2.1.7. *A ring* R *is called a* ***prime (resp., semiprime) ring*** *if* (0) *is a prime (resp., semiprime) ideal.*

Proposition 2.1.2. [8] *For any ring* R*, the following are equivalent:*

(1) R *is a semiprime ring.*

(2) $Nil_*(R) = 0$

(3) R *has no nonzero nilpotent ideals.*

(4) R *has no nonzero nilpotent left ideals.*

proof:

(1) $\iff$ (2) *is clear from Definition* (2.1.7)*. Next we shall prove*

$$(4) \Rightarrow (3) \Rightarrow (1) \Rightarrow (4).$$

The first two implications are also clear.

For (1) $\Rightarrow$ (4)*, let* R *be a semiprime ring and let* I *be a nilpotent left ideal. Choose* $n(\geq 1)$ *minimal such that* $I^n = 0$*. If* $n > 1$*, then* $(I^{n-1})^2 = I^{2n-2} \subseteq I^n = 0$*, implies that* $I^{n-1} = 0$*, which contradicts the minimality of* n*. Thus* $n = 1$*, and* $I = 0$*. Hence* R *has no nonzero nilpotent left ideals.* □

Kothe's Conjecture *states that, If* $Nil_*(R) = 0$*, then* R *has no nonzero nil one-sided ideals.*

Proposition 2.1.3. [7] *Let* R *be a semiprime ring with the ascending chain condition* (*ACC*) *for right annihilators. Then* R *has no nonzero nil one-sided ideals.*

proof:

Let I *be a nonzero one-sided ideal of* R *and let* $0 \neq a \in I$ *with* $rAnn(a)$ *as large as possible. Since* R *is semiprime, there is an element* $x \in R$ *such that* $axa \neq 0$*. Thus* axa *is a nonzero element of* I *such that* $rAnn(a) \subseteq rAnn(axa)$*. So* $rAnn(a) = rAnn(axa)$ *. We have* $ax \neq 0$*, i.e.,* $x \notin rAnn(a)$ *Thus* $x \notin rAnn(axa)$*. So,* $(ax)^2 \neq 0$*. Hence* $xax \notin rAnn(a)$ *implying that* $(ax)^3 \neq 0$*. Therefore,* ax *and hence, also* xa *is not nilpotent and* $ax \in I$ *or* $xa \in I$*.* □

Lemma 2.1.6. *Assume that* R *satisfies the ACC for right annihilators* $ann_r(a) = \{x \in R : ax = 0\}$*, where* $a \in R$*. Then:*

1. *Any nil one-sided ideal I is contained in Nil_*R.*

2. *Any nonzero nil right (resp., left) ideal J contains a nonzero nilpotent right (resp., left) ideal.*

In particular, if R is also semiprime, then every nil one-sided ideal is zero.

proof:

(1) : *Assume I is a nil right ideal $\subsetneq Nil_*(R)$. Among the elements in $I/Nil_*(R)$, choose a such that $ann_r(a)$ is maximal. For any $x \in R$, we claim that $axa \in Nil_*(R)$. For this, we may assume that $axa \neq 0$. Since $ax \in I$ is nilpotent, there exists an integer $k > 1$ such that $(ax)^k = 0 \neq (ax)^{k-1}$. Then*

$$ann_r(axa) \supsetneq ann_r(a)$$

since $x(ax)^{k-2}$ belongs to $ann_r(axa)$ but not to $ann_r(a)$. The maximality of $ann_r(a)$ now implies that $axa \in Nil_*(R)$, as claimed. Since $R/Nil_*(R)$ is semiprime, this implies that $a \in Nil_*(R)$, a contradiction.

If I is a nil left ideal instead, then for any $a' \in I$, $a'R$ is a nil right ideal, so $a'R \subseteq Nil_*(R)$. Therefore, we also have $I \subseteq Nil_*(R)$.

(2) : *Among the nonzero elements of J, choose b such that $ann_r(b)$ is maximal. It suffices to show here that $bxb = 0$ for all $x \in R$, for then we'll have $(bR)^2 = (Rb)^2 = 0$. If J is a right ideal, we can repeat the argument in (1) to get $bxb = 0$. Now assume J is a left ideal and $bxb \neq 0$. Then $xb \in J$ is nilpotent, so there exists an integer $k > 1$ such that $(xb)^k = 0 \neq (xb)^{k-1}$. But then $xb \in ann_r((xb)^{k-1})$ and $xb \notin ann_r(b)$, so we have*

$$ann_r(b) \subsetneq ann_r((xb)^{k-1}),$$

contradiction. $\square$

The last part of the lemma (2.1.6) *above leads to the following theorem of J.Levitzki.*

Theorem 2.1.3. [10] ***Levitzki's Theorem.*** *Let R be a right noetherian ring. Then every nil one-sided ideal I of R is nilpotent. We have $Nil_*R = Nil^*R$, and this is the largest nilpotent right (resp., left) ideal of R.*

proof:

In view of (1) *in the Lemma* (2.1.6) *above, it suffices to show that Nil_*R is nilpotent. Since R is right noetherian, there exists a maximal nilpotent ideal N in R. Then R/N has no nonzero nilpotent ideals, so R/N is semiprime. This shows that $N \supseteq Nil_*R$, and hence $Nil_*R = N$ is nilpotent.* □

Definition 2.1.8. *The* ***Jacobson radical*** *of a ring R is the intersection of all maximal ideals of R,we denote it by $J(R)$.*

Example 2.1.3. *The Jacobson radical of the ring $\mathbb{Z}/12\mathbb{Z}$ is $6\mathbb{Z}/12\mathbb{Z}$, which is the intersection of the maximal ideals $2\mathbb{Z}/12\mathbb{Z}$ and $3\mathbb{Z}/12\mathbb{Z}$.*

Definition 2.1.9. *A set $S \subseteq R$ is said to be* ***locally nilpotent*** *if every finite subset of S is nilpotent.*

Definition 2.1.10. ***Levitsky radical*** *is the sum of all locally nilpotent ideals,we denote it by L-rad(R).*

Definition 2.1.11. *A subset A of a ring R is called* ***left(resp. right)T-nilpotent*** *if, for any*

sequence of elements $\{a_1, a_2, \ldots\} \subseteq A$, there exists an integer $n \geq 1$ such that ;

$a_1a_2 \ldots a_n = 0$ (resp. $a_n \ldots a_2a_1 = 0$).

The set is called ***T-nilpotent*** *if it is both left and right T-nilpotent.*

Definition 2.1.12. *The* ***index of nilpotency of a nilpotent element*** x *in a ring* R *is the least positive integer* n *such that* $x^n = 0$.

The ***index of nilpotency of a subset*** I *of a ring* R *is the supremum of the indices of nilpotency of all nilpotent elements in* I. *If such a supremum is finite , then* I *is said to be* ***of bounded index of nilpotency.***

Example 2.1.4. *In the ring* $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$ *of integers modulo* 8*, the index of the nilpotent element* $\bar{2}$ *is equal to* 3 *(since* $\bar{2}^3 = \bar{0}$*).*

And if $I = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$ *is a subset of* $\mathbb{Z}_8$ *, then the index of nilpotency of* I *is equal to* 3*. (Since for the nilpotent elements in* I*,* $\bar{2}^3 = \bar{0}$ *,* $\bar{4}^2 = \bar{0}$ *,* $\bar{6}^3 = \bar{0}$*).*

Remark 2.1.3. *A nil subset* N *of a ring* R *has bounded index* $\leq t$ *if* $a^t = 0$ *for all* a *in* N*. In fact* $N(R) = \{a \in R : Ra \text{ is nil of bounded index}\}$.

Definition 2.1.13. *Let* R *be a ring. We say* R *has property* $(**)$*, whenever* $f(x) = a_0 + a_1x + \ldots + a_mx^m$*,* $g(x) = b_0 + b_1x + \ldots + b_nx^n$ *are elements of* $(R[x], +, \circ)$ *and* $f \circ g \in nil(R)[x]$*, then* $a_ib_j \in nil(R)$ *for* $i = 1, \ldots, m$ *,* $j = 0, 1, \ldots, n$.

Proposition 2.1.4. *Let* I *be a nil ideal of a ring* R*. Then* $\bar{R} = R/I$ *has property* $(**)$ *if and only if* R *has property* $(**)$.

proof:

Claim: *If* I *is a nil ideal, then* $nil(\overline{\mathrm{R}}) = \overline{\mathrm{nil(R)}}$.

proof of the claim: *We denote* $\overline{\mathrm{R}} = R/I$*, and* $\overline{\mathrm{nil(R)}} = nil(R)/I$.

Firstly,

$$
\begin{aligned}
nil(\overline{\mathrm{R}}) &= \{m + I \in R/I : m + I \ \ is \ \ nilpotent\} \\
&= \{m + I : (m + I)^k = I\} \\
&= \{m + I : m^k + I = I\} \\
&= \{m + I : m^k \in I\}.
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\mathrm{nil(R)}} &= nil(R)/I \\
&= \{m + I : m \in nil(R)\} \\
&= \{m + I : m^k = 0\}
\end{aligned}
$$

but $(m + I)^k = m^k + I = 0 + I = I.$

Therefore $nil(\overline{\mathrm{R}}) = \overline{\mathrm{nil(R)}}.$

Now let $f(x) = \sum_{i=0}^{m} a_i x_i$ *and* $g(x) = \sum_{j=0}^{n} b_j x_j$ *be elements of* $R[x]$, *so* $\overline{\mathrm{f}} = \sum_{i=0}^{m} \overline{\mathrm{a_i}} x_i$, *and* $\overline{\mathrm{g}} = \sum_{j=0}^{n} \overline{\mathrm{b_j}} x_j$, *where* $\overline{\mathrm{a_i}}$ *and* $\overline{\mathrm{b_j}} \in \overline{\mathrm{R}}$. *Now* $f \circ g \in nil(R)[x]$, *if and only if*

$$
\begin{aligned}
\overline{\mathrm{f} \circ \mathrm{g}} &= \overline{\mathrm{f}} \circ \overline{\mathrm{g}} \\
&= (\sum_{i=0}^{m} \overline{\mathrm{a_i}} x_i) \circ (\sum_{j=0}^{n} \overline{\mathrm{b_j}} x_j) \in nil(\overline{\mathrm{R}})[x].
\end{aligned}
$$

Also, $\overline{\mathrm{a_i}}\overline{\mathrm{b_j}} = (a_i + I)(b_j + I) = a_i b_j + I.$

Since $a_i b_j \in nil(R)$, *so* $\overline{\mathrm{a_i}}\overline{\mathrm{b_j}} \in nil(R)/I = \overline{\mathrm{nil(R)}} = nil(\overline{\mathrm{R}})$, *for* $i = 1, \ldots, m$ *and* $j = 0, 1, \ldots, n.$

Hence R *has a property* $(**)$ *if and only if* R/I *has a property* $(**)$. $\square$

2.2 Nilpotency of polynomial rings

In this section we further the study of nilpotent elements in $R[x]$.

Lemma 2.2.1. [4] *Let $f(x)$ be a polynomial over a commutative ring R , $f(x)$ is nilpotent if and only if each coefficient of f(x) is nilpotent.*

proof:

Let $p(x) = \sum_{i=0}^{n} a_i x_i$ be nilpotent. We proceed by induction on n, the degree of $p(x)$. If $p(x) = a_0$ has degree 0, then $(p(x))^m = a_0^m = 0$ for some m. Thus a_0 is nilpotent, and all of the coefficients of $p(x)$ are nilpotent.

Now suppose now that for some $d \geq 0$, if $r(x)$ is a nilpotent polynomial of degree at most d, then all of the coefficients of $r(x)$ are nilpotent. Now suppose $p(x)$ has degree $n = d+1$ and that $(p(x))^m = 0$ for some $m \geq 1$. We prove by induction that the nm coefficient of $(p(x))^m$ is a_n^m. For the base case $m = 1$, the n coefficient of $p(x)$ is indeed a_n. Suppose now that for some $m \geq 1$, the nm coefficient of $p(x)$ is a_n^m.

Now the $n(m+1)$ coefficient of $(p(x))^{m+1}$ is by definition $\sum_{i+j=n(m+1)} a_i b_j$, where b_j is the j^{th} coefficient of $(p(x))^m$. If $i > n$, then a_i does not exist. (Equivalently, is zero.) If $i < n$, then $j > nm$, so that b_j does not exist. (Equivalently, is zero.) Thus the only remaining term is the $(i, j) = (n, nm)$ term, and we have $\sum_{i+j=n(m+1)} a_i b_j = a_n b_{nm} = a_n^{m+1}$. Thus the result holds.

Now the nm coefficient of $(p(x))^m$ is a_n^m on one hand and zero on the other. Thus $a_n^m = 0$, so that a_n is nilpotent. Now $p(x) - a_n x^n$ is a unit (in a commutative ring ,the sum of a unit u and a nilpotent element x is a unit ,since $u^{-1}x$ is nilpotent, so $1+u^{-1}x$ is a unit, and thus $u(1+u^{-1}x) = u+x$ is a unit), and has degree at most d. Thus by the induction hypothesis all coefficients of $p(x) - a_n x^n$ are nilpotent, and so all coefficients of $p(x)$ are nilpo-

tent. □

But the result in the previous Lemma is not true in general for noncommutative rings. For example, let $R = M_n(k)$, the $n \times n$ full matrix ring over some ring $k \neq 0$. Consider the polynomial $f(x) = e_{12} + (e_{11} - e_{22})x - e_{21}x^2$, where the e_{ij} 's are the matrix units. In this case $f(x)^2 = 0$, but $e_{11} - e_{22}$ is not nilpotent. (In fact, when $n = 2$ this is a unit.)

The trouble seems to arise from the fact that if $f(x)$ is nilpotent, then so is the ideal $< f >$ over a commutative ring, but not necessarily over a noncommutative ring. Indeed, for a general ring R, $f(x)R[x]$ is a nilpotent right ideal of $R[x]$ if and only if A_fR is a nilpotent right ideal of R, where A_f is the set of coefficients of $f(x)$.

Theorem 2.2.1. [4] *Let R be a ring, and let $f(x) = \sum_{i=0}^{m} a_i x_i$ and $g(x) = \sum_{j=0}^{n} a_j x_j$ be polynomials in $R[x]$. If $f(x)Rg(x)R = 0$, then a_iRb_jR is a nilpotent right ideal, for each pair (i, j).*

proof:

We work by induction on $k = i + j$. If $k = 0$ then $i = j = 0$. We know $a_0Rb_0R = 0$ by looking at the degree zero term in $f(x)Rg(x)R = 0$. Thus a_0Rb_0R is trivially nilpotent. So, we may assume $k \geq 1$ and that the result is true for all values smaller than k as the inductive assumption. From the degree k coefficient in $f(x)Rg(x)R = 0$ we obtain

$$\sum_{t=0}^{k} a_t r b_{k-t} s = 0 \tag{2.1}$$

for arbitrary r,$s \in R$. In particular, specializing r in Equation (2.1) to ub_0v (with u,$v \in R$) we have $\sum_{t=0}^{k} a_t u b_0 v b_{k-t} s = 0$. But then $a_k u b_0 v b_0 s = -\sum_{t=0}^{k-1} a_t u b_0 v b_{k-t} s \in \sum_{t<k} a_t R b_0 R$. The right ideal on the right hand side is nilpotent (since it is a finite sum of such, by

inductive hypothesis). Thus, $(a_kRb_0R)^2 \subset a_kRb_0Rb_0R$ *is a nilpotent right ideal, hence so is* a_kRb_0R*. Repeat the argument in the previous paragraph by specializing* r *to* ub_jv *instead of* ub_0v*, and induct on* j*. Note that the previous paragraph consists of checking the base case. In this more general case, Equation* (2.1) *then yields*

$$a_{k-j}ub_jvb_js = -\left(\sum_{t=0}^{k-j-1} a_tub_jvb_{k-t}s\right) - \left(\sum_{t=k-j+1}^{k} a_tub_jvb_{k-t}s\right)$$

In particular, $a_{k-1}Rb_jRb_jR \subseteq \left(\sum_{t<k-j} a_tRb_jR\right) + \left(\sum_{i'=j'=k,j'<j} a'_iRb'_jR\right)$*, which is nilpotent by the inductive assumptions on* k *and* j*. By the same reasoning as in the previous paragraph, we obtain that* $a_{k-j}Rb_jR$ *is nilpotent, finishing both inductions.* □

Proposition 2.2.1. [4] *Let* R *be a ring, and let* $f(x) = \sum_{i=0}^{m} a_ix_i$ *,* $g(x) = \sum_{j=0}^{n} a_jx_j \in R[x]$*. The set* $f(x)Rg(x)R$ *is nilpotent iff* a_iRb_jR *is a nilpotent right ideal for each pair* (i, j)*.*

proof:

$\Rightarrow$ *: by the previous Theorem.*

$\Leftarrow$ *: is a consequence of the fact that there are only finitely many pairs* (i, j)*.* □

Proposition 2.2.2. [4] *Let* $n \in \mathbb{Z} > 0$*, let* R *be a ring, and for each* $1 \leq l \leq n$*, let* $f_l(x) = \sum_{i=0}^{n_l} a_{i,l}x_i \in R[x]$*. The set* $\prod_{l=1}^{n} f_l(x)R$ *is nilpotent if and only if* $\prod_{l=1}^{n}(a_{i_l,l}R)$ *is a nilpotent right ideal, for all choices* $0 \leq i_l \leq n_l$ *(for each* l*).*

proof:

$\Rightarrow$ *: we simplify to the case* $\prod_{l=1}^{n} f_l(x)R$ *,so let* $S = (i_1, i_2, \ldots, i_n)|0 \leq i_l \leq n_l$ *be the set of n-tuples whose entries consist of indices for the coefficients of the polynomials. We can order* S *by saying*

$$(i_1, i_2, \ldots, i_n) < (j_1, j_2, \ldots, j_n)$$

if and only if there exists an integer m in the range $1 \leq m \leq n$ so that $\sum\limits_{l=m'}^{n} i_l = \sum\limits_{l=m'}^{n} j_l$ *when* $1 \leq m' < m$ *but* $\sum\limits_{l=m}^{n} i_l < \sum\limits_{l=m}^{n} j_l$. *Given an n-tuple $s = (i_1, i_2, \ldots, i_n) \in S$, let $B_s = \prod_{l=1}^{n}(a_{i_l,l}R)$. We work by induction on the well-ordered set S to prove that B_s is nilpotent for each $s \in S$. The claim is clearly true for the n-tuple $s_0 = (0, 0, \ldots, 0)$. Fix $s = (i_1, i_2, \ldots, i_n) \in S$, and suppose by induction that the claim is true for all n-tuples $t < s$. Set $k = \sum i_l$. Looking at the degree k coefficient in the equation $\prod_{l=1}^{n} f_l(x)R = 0$ yields*

$$\sum_{j_1+j_2+\ldots+j_n=k} \left(\prod_{l=1}^{n} (a_{jl,l} r_l) \right) = 0 \tag{2.2}$$

with $r_l \in R$ arbitrary. Specializing each r_l to $s_{l,l} \prod_{l'=l_1}^{n}(a_{i_{l'},l'} s_{l,l'})$ (for arbitrary $s_{l,l'} \in R$, for all $l' > l$ then, Equation (2.2) implies that $C_s = \prod_{l=1}^{n}(\prod_{l'=l}^{n}(a_{i_{l'}}, l'R))$ belongs to $\sum\limits_{t<s} B_t$, which is nilpotent. On the other hand ${B_s}^n \subseteq C_s$, and so B_s is similarly nilpotent.This completes the induction.

$\Leftarrow$ *: this implication is easy.* $\square$

Corollary 2.2.1. *Let R be a ring and $f(x) \in R[[x]]$. Suppose $A_f R$ is finitely generated. The right ideal $f(x)R[[x]]$ is nilpotent if and only if $A_f R$ is nilpotent.*

2.3 Principal Projective Rings (p.p. rings)

In this section we talk about the Principal Projective Rings (briefly p.p. rings), and study its nilpotency.

Definition 2.3.1. *A ring R is called **a right(left) p.p. ring** if each principal right(left) ideal of R is projective, or equivalently, if the right(left) annihilator of each element of R is generated by an idempotent. A ring is called a **p.p. ring** if it is both right and left p.p. ring.*

Definition 2.3.2. [19] *Let R be a ring. For a subset X of a ring R, we define $N_R(X) = \{a \in R : xa \in nil(R)$, for all $x \in X\}$, which is called **the nilpotent annihilator** of X in R.*

If X is a singleton, say $X = r$, we use $N_R(r)$ in place of $N_R(\{r\})$.

Clearly, for any nonempty subset X of R, we have ;

$$N_R(X) = \{a \in R : xa \in nil(R), for\ \ all\ \ x \in X\}$$

$$N_L(X) = \{b \in R : bx \in nil(R), for\ \ all\ \ x \in X\}.$$

Example 2.3.1. [19] *Let $\mathbb{Z}$ be the ring of integers and $T_2(\mathbb{Z})$ the 2×2 upper triangular matrix ring over $\mathbb{Z}$. We consider the subset $X = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$, so ;*

$$N_{T_2(\mathbb{Z})}(X) = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, m \in \mathbb{Z} \right\}.$$

Proposition 2.3.1. [19] *Let X, Y be subsets of R. Then, we have the following:*

(1) *$X \subseteq Y$ implies $N_R(X) \supseteq N_R(Y)$.*

(2) *$X \subseteq N_R(N_R(X))$.*

(3) *$N_R(X) = N_R(N_R(N_R(X)))$.*

proof:

(1): by definition.

(2): it is clear , since $\forall x' \in X$, we have $x'a' \in nil(R)$, $\forall a' \in N_R(X)$.

(3): Applying **(2)** to $N_R(X)$, we obtain $N_R(X) \subseteq N_R(N_R(N_R(X)))$.

Since $X \subseteq N_R(N_R(X))$, we have $N_R(X) \supseteq N_R(N_R(N_R(X)))$, by **(1)**. Therefore, $N_R(X) = N_R(N_R(N_R(X)))$. □

Lemma 2.3.1. *Let R be a subring of S. Then, for any subset X of R, we have $N_R(X) = N_S(X) \cap R$.*

proof.

Let $r \in N_R(X)$. Then, $r \in R$ and $xr \in nil(R)$, for each $x \in X$, and so $xr \in nil(S)$, for each $x \in X$. Hence, $r \in N_S(X) \cap R$ and so $N_R(X) \subseteq N_S(X) \cap R$.

Assume that $a \in N_S(X) \cap R$. Then, $a \in R$ and $xa \in nil(S)$, for each $x \in X$. As $X \subseteq R$, we have $xa \in nil(R)$, for each $x \in X$. Thus $a \in N_R(X)$ and so $N_R(X) \supseteq N_S(X) \cap R$. Therefore, $N_R(X) = N_S(X) \cap R$. $\square$

Definition 2.3.3. [21] *A ring R is said to be* ***a nilpotent p.p. ring*** *if for any element $p \in R$ with $N_R(p) \neq R$, $N_R(p)$ is generated as a right ideal by a nilpotent element.*

Lemma 2.3.2. [15] *Let R be a semicommutative ring. If $ab \in nil(R)$, for $a, b \in R$, then $aRbR \subseteq nil(R)$.*

proof:

Suppose $ab \in nil(R)$. Then, $abs \in nil(R)$ for any $s \in R$, since $nil(R)$ is an ideal of R. Thus, there exists a positive integer n such that $(abs)^n = absabs \ldots abs = 0$, and so $arbsarbs \ldots arbs = 0$, for any $r \in R$, because R is a semicommutative ring. Hence, $arbs \in nil(R)$, for each $r \in R$ and $s \in R$. Therefore $aRbR \subseteq nil(R)$. $\square$

Proposition 2.3.2. [5] *Let R be a semicommutative ring. Then, R is a nilpotent p.p. ring if and only if $R[x]$ is a nilpotent p.p. ring.*

proof:

Suppose that R is a nilpotent p.p. ring. Let $f(x) = a_0 + a_1x + \ldots + a_mx^m \in R[x]$, with

$N_R[x](f(x)) \neq R[x]$. We show that $N_R[x](f(x))$ is generated by a nilpotent element. If $g(x) = b_0 + b_1x + \ldots + b_nx^n \in N_R[x](f(x))$, then we have

$$f(x)g(x) = \Big(\sum_{i=0}^{m} a_ix^i\Big)\Big(\sum_{j=0}^{n} b_jx^j\Big) = \sum_{s=0}^{m+n}\Big(\sum_{i+j=s} a_ib_j\Big)x^s \in nil(R[x]).$$

We have the following system of equations by Lemma (2.3.3):

$$\triangle_s = \sum_{i+j=s} a_ib_j \in nil(R[x]), s = 0, 1, \ldots, m+n.$$

We will show that $a_ib_j \in nil(R)$ by induction on $i+j$. If $i+j = 0$, then $a_0b_0 \in nil(R), b_0a_0 \in nil(R)$.

Now, suppose that s is a positive integer such that $a_ib_j \in nil(R)$, when $i + j < s$. We will show that $a_ib_j \in nil(R)$, when $i + j = s$.

Consider the following equation:

$$(*) : \triangle_s = a_0b_s + a_1b_{s-1} + \ldots + a_sb_0 \in nil(R).$$

Multiplying Eq.$(*)$ by b_0 from left, we have

$$b_0a_sb_0 = b_0 \triangle_s -(b_0a_0)b_s - (b_0a_1)b_{s-1} - \ldots - (b_0a_{s-1})b_1.$$

By the induction hypothesis, $a_ib_0 \in nil(R)$, for each $i, 0 \leq i < s$, and so $b_0a_i \in nil(R)$, for each $i, 0 \leq i < s$. Thus, $b_0a_sb_0 \in nil(R)$ and so $b_0a_s \in nil(R), a_sb_0 \in nil(R)$.

Multiplying Eq.$(*)$ by $b_1, b_2, \ldots, b_{s-1}$ from the left side, respectively, yields $a_{s-1}b_1 \in nil(R)$, $a_{s-2}b_2 \in nil(R), \ldots, a_0b_s \in nil(R)$, in turn. This means that $a_ib_j \in nil(R)$, when $i+j = s$. Therefore, by induction we obtain $a_ib_j \in nil(R)$, for each i, j, and so $b_j \in N_R(a_i)$, for each $i, 0 \leq i \leq m$ and $j, 0 \leq j \leq n$. If $N_R(a_i) = R$, for each $i, 0 \leq i \leq m$, then $a_ir \in nil(R)$ for each $i, 0 \le i \le m$ and each $r \in R$. So, for any $u(x) = u_0 + u_1x + \ldots + u_tx^t \in R[x]$, we have $a_iu_j \in nil(R)$ for each $i, 0 \le i \le m$ and each $j, 0 \le j \le t$. Thus,

$$f(x)u(x) = \sum_{s=0}^{m+t}\Big(\sum_{i+j=s} a_iu_j\Big)x^s \in nil(R[x]).$$

by Lemma (2.3.3), and so $u(x) \in N_R[x](f(x))$. Thus, we obtain $N_R[x](f(x)) = R[x]$. This is contrary to the fact that $N_R[x](f(x)) \ne R[x]$. Thus, there exists an $i, 0 \le i \le m$, such that $N_R(a_i) \ne R$. Since R is a nilpotent p.p. ring, there exists some $c \in nil(R)$ with $N_R(a_i) = cR$. Now, we show that $N_R[x](f(x)) = cR[x]$. Since $b_j \in N_R(a_i) = cR$ for each $j, 0 \le j \le n$, there exists $r_j \in R$ such that $b_j = cr_j$, and so $g(x) = c(r_0+r_1x+\ldots+r_nx^n) \in cR[x]$. Hence, $N_R[x](f(x)) \subseteq cR[x]$. On the other hand, for $h(x) = h_0 + h_1x + \ldots + h_px^p \in R[x]$, we have

$$f(x).ch(x) = \Big(\sum_{i=0}^{m} a_ix^i\Big)\Big(\sum_{j=1}^{p} ch_jx^j\Big) = \sum_{s=0}^{m+p}\Big(\sum_{i+j=s} a_ich_j\Big)x^s \in nil(R[x]).$$

Since $nil(R)$ is an ideal of R and $c \in nil(R)$, we obtain $a_ich_j \in nil(R)$ and so $f(x).ch(x) \in nil(R[x])$, by Lemma (2.3.3). Hence, $N_R[x](f(x)) \supseteq c.R[x]$, and so $N_R[x](f(x)) = c.R[x]$, where $c \in nil(R[x])$. Therefore, $R[x]$ is a nilpotent p.p. ring.

Conversely, assume that $R[x]$ is a nilpotent p.p. ring. Let $p \in R$, with $N_R(p) \ne R$. If $N_R[x](p) = R[x]$, then we have $N_R(p) = N_R[x](p) \cap R = R$, by Lemma (2.3.1), which is a contradiction. Thus, we obtain $N_R[x](p) \ne R[x]$. Since $R[x]$ is a nilpotent p.p. ring, there exists $u(x) = u_0 + u_1x + \ldots + u_sx^s \in nil(R[x])$ such that $N_R[x](p) = u(x).R[x]$. Since $u(x) = u_0 + u_1x + \ldots + u_sx^s \in nil(R[x])$, we obtain $u_i \in nil(R)$ for each $i, 0 \le i \le s$, by Lemma (2.3.3). Now, we show that $N_R(p) = u_0.R$. Since $u_0 \in nil(R)$ and $nil(R)$ is an ideal of R, we have $pu_0r \in nil(R)$ for each $r \in R$. Thus, $u_0r \in N_R(p)$, for each $r \in R$, and so $N_R(p) \supseteq u_0.R$. Suppose that $m \in N_R(p)$. Then, $m \in N_R[x](p)$, and so there exists

$p(x) = p_0 + p_1x + \ldots + p_qx^q \in R[x]$ such that $m = u(x)p(x)$. Hence, $m = u_0p_0 \in u_0.R$, and so $N_R(p) \subseteq u_0.R$. Therefore, $N_R(p) = u_0.R$, and so R is a nilpotent p.p. ring. □

Proposition 2.3.3. [19] *Let R be a semicommutative ring. Then, $f(x) = a_0 + a_1x + \ldots + a_nx^n \in R[x]$ is a nilpotent element of $R[x]$ if and only if $a_i \in nil(R)$, for all $0 \leq i \leq n$.*

proof:

$\Rightarrow$: Assume $f(x)$ is nilpotent, for $n = 0$ (constant polynomial), we have $f(x) = a_0$, and $(f(x))^m = 0$, which implies that $(a_0)^m = 0$, thus all coefficient of $f(x)$ are nilpotent. Now by induction on n, we get that each coefficient of $f(x)$ is nilpotent (i.e. $\in nil(R)$).

$\Leftarrow$: Suppose that $a_i \in nil(R)$, for all $0 \leq i \leq n$, we need to show that $f(x)$ is nilpotent. For $m =$ index of nilpotency of $\{a_0, a_1, \ldots a_n\}$, we have:

$$\begin{aligned}
(f(x))^m &= (a_0 + a_1x + \ldots + a_nx^n)^m \\
&= \sum_{j=1}^{m} \binom{m}{j} (a_0 + a_1x + \ldots + a_{n-1}x^{n-1})^j (a_nx^n)^{m-j} \\
&= \sum_{j=1}^{m} \binom{m}{j} (a_0 + a_1x + \ldots + a_{n-1}x^{n-1})^j ((a_n)^m x^{nm})^{-j} \\
&= (\sum_{j=1}^{m} \binom{m}{j} (a_0 + a_1x + \ldots + a_{n-1}x^{n-1})^j) * (0). \quad (Since (a_n)^m = 0) \\
&= 0.
\end{aligned}$$

Therefore, $(f(x)^m) = 0$, so $f(x)$ is nilpotent in $R[x]$. □

2.4 Laurent Polynomial Ring

In this section we introduce the Laurent Polynomial Rings, and conclude how some of the nilpotency properties are passing from the ring to this type of rings.

Definition 2.4.1. [19] ***The ring of Laurent polynomial*** *in x, with coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers.*
We denote this ring by $R[x; x^{-1}]$.

If $f(x)$ is a nilpotent element of $R[x; x^{-1}]$, then we say that $f(x) \in nil(R[x; x^{-1}])$.

Lemma 2.4.1. *Let R be a semicommutative ring. Then, $f(x) = \sum_{i=k}^{n} a_i x^i \in R[x; x^{-1}]$ is a nilpotent element of $R[x; x^{-1}]$ if and only if $a_i \in nil(R)$, for each $i, k \leq i \leq n$.*

proof:

There exists a positive integer t such that $f(x).x^t \in R[x]$. Note that $(f(x))^k = 0$ if and only if $(f(x).x^t)^k = 0$, where k is a positive integer. Then, the proof is complete by Proposition (2.3.3). $\square$

Lemma 2.4.2. [22] *Let R be a semicommutative ring, $f(x) = \sum_{i=k}^{m} a_i x^i \in R[x; x^{-1}]$ and $g(x) = \sum_{j=l}^{n} a_j x^j \in R[x; x^{-1}]$. Then, we have $f(x)g(x) \in nil(R[x; x^{-1}])$ if and only if $a_i b_j \in nil(R)$, for each $i, k \leq i \leq m$ and for each $j, l \leq j \leq n$.*

proof.

Suppose that $a_i b_j \in nil(R))$, for each $i, k \leq i \leq m$ and for each $j, l \leq j \leq n$. Then,

$$f(x)g(x) = \sum_{s=k+l}^{m+n} \left(\sum_{i+j=s} a_i b_j \right) x^s \in nil(R[x; x^{-1}]),$$

by Lemma (2.4.1). So it suffices to show that $a_ib_j \in nil(R)$ for each i, j, when $f(x)g(x) \in nil(R[x; x^{-1}]$. There exist positive integers u and v such that $f(x)x^u \in R[x]$ and $g(x)x^v \in R[x]$. Since $(f(x)g(x))^k = 0$ if and only if $(f(x)x^ug(x)x^v)^k = 0$, where k is a positive integer.

Same as the proof of Proposition (2.3.2), we obtain that $a_ib_j \in nil(R)$, for each i, j. □

Proposition 2.4.1. [19] *Let R be a semicommutative ring. If R is a nilpotent p.p. ring, then so is $R[x; x^{-1}]$.*

proof:

Let $f(x) = \sum_{i=k}^{m} a_ix_i \in R[x; x^{-1}]$, with $N_R[x; x^{-1}](f(x)) \neq R[x; x^{-1}]$. We show that $N_R[x; x^{-1}](f(x))$ is generated by a nilpotent element. If $g(x) = \sum_{j=l}^{n} b_jx_j \in N_R[x; x^{-1}](f(x))$, then $f(x)g(x) \in nil(R[x; x^{-1}])$. Then, we obtain $a_ib_j \in nil(R)$, for each i, j, by lemma (2.4.2), and so $b_j \in N_R(a_i)$ for each $j, l \leq j \leq n$ and each $i, k \leq i \leq m$. If $N_R(a_i) = R$, for each $i, k \leq i \leq m$, then for each $h(x) = \sum_{j=s}^{t} h_jx_j \in R[x; x^{-1}]$, we have $a_ih_j \in nil(R)$, for each $i, k \leq i \leq m$ and $s \leq j \leq t$. Thus, $f(x)h(x) \in nil(R[x; x^{-1}])$, by lemma (2.4.2), and so $h(x) \in N_R[x; x^{-1}](f(x))$. Hence, we obtain $N_R[x; x^{-1}](f(x)) = R[x; x^{-1}]$, which is a contradiction. Thus, there exists an $i, k \leq i \leq m$, such that $N_R(a_i) \neq R$. Since R is a nilpotent p.p. ring, there exists some $c \in nil(R)$, with $N_R(a_i) = cR$. Now, we show that $N_R[x; x^{-1}](f(x)) = c.R[x; x^{-1}]$. Since $b_j \in N_R(a_i)$, for each $j, l \leq j \leq n$, there exists $r_j \in R$ such that $b_j = c.r_j$. Thus, $g(x) = \sum_{j=l}^{n} b_jx_j = c(\sum_{j=l}^{n} r_jx_j) \in c.R[x; x^{-1}]$. Hence, $N_R[x; x^{-1}](f(x)) \subseteq c.R[x; x^{-1}]$. Let $q(x) = \sum_{j=v}^{t} q_jx_j \in R[x; x^{-1}]$. Since $c \in nil(R)$ and $nil(R)$ is an ideal of R, we obtain $a_icq_j \in nil(R)$, for each i, j, and so $f(x).cq(x) \in nil(R[x; x^{-1}])$, by Lemma (2.4.2). Thus, $N_R[x; x^{-1}](f(x)) \supseteq c.R[x; x^{-1}]$.

Hence, $N_R[x; x^{-1}](f(x)) = c.R[x; x^{-1}]$, where $c \in nil(R[x; x^{-1}])$. Therefore, $R[x; x^{-1}]$ is a nilpotent p.p. ring. □

Chapter 3

Armendariz Rings

In this Chapter we study Armendariz Rings and present its types, namely, weak Armendariz and nil Armendariz rings and discuss the relation between these rings and Armendariz Rings. Furthermore we study the effects of Armendariz property of a ring on nilpotency property. In addition we study the skew inverse Laurent-serieswise Armendariz rings for any ring with an automorphism, and introduce the notions of a strongly Armendariz ring of inverse skew power series type, and an α-compatible ring.

3.1 Armendariz rings

We study in this secion the Armendariz Rings, and see that Armendariz property pass from the ring into its polynomial ring.

Definition 3.1.1. [7] *A ring R is called* ***Armendariz*** *if for any $f(x) = \sum\limits_{i=0}^{n} a_i x_i$,and $g(x) = \sum\limits_{j=0}^{n} b_j x_j \in R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for all i and j.*

if R is an Armendariz ring, then nil(R) is a subring of R and nil(R)[x] $= nil(R[x])$. Hence nil(R) is a locally nilpotent subring of R, when R is an Armendariz ring.

The following example shows that there exist non Armendariz rings such that the set of its nilpotent elements is a locally nilpotent ideal.

Example 3.1.1. [7] *Let $\mathbb{Z}$ be the ring of integers and let*

$$R = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} ; a - b \equiv c \equiv 0(mod 2) \right\}$$

. Then R is not Armendariz. Since

$$nil(R) = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} ; c \equiv 0(mod 2) \right\}$$

, hence nil(R) is a locally nilpotent ideal of R.

Definition 3.1.2. *If $f(x) \in R[x]$, $coef(f(x))$ denotes the subset of R of the coefficients of $f(x)$, if $A \subseteq R[x]$, then $coef(A)$ is the set of all the coefficients of all polynomials in A.*

Definition 3.1.3. [2] *A ring R is said to be **weak Armendariz** if whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $ab \in nil(R)$ for all $a \in coef(f(x))$ and $b \in coef(g(x))$.*

Proposition 3.1.1. [19] *Suppose R is an Armendariz ring, if $f_1, f_2, \ldots, f_n \in R[x]$ are such that $f_1 f_2 \ldots f_n = 0$, then $a_1 a_2 \ldots a_n = 0$, where a_i is a coefficient of f_i.*

proof:

Suppose $f_1 f_2 \ldots f_n = 0$, and let a_i be any coefficient of f_i. Now we have $f_1(f_2 \ldots f_n) = 0$,so $a_i b = 0$ for any coefficient b of $f_2 \ldots f_n$.Thus $a_1 f_2 \ldots f_n = 0$, thus $(a_1 f_2)(f_3 \ldots f_n) = 0$. Since $a_1 a_2$ is a coefficient of $a_1 f_2$, we have $(a_1 a_2)c = 0$ for each coefficient c of $f_3 \ldots f_n$. Hence $a_1 a_2 f_3 \ldots f_n = 0$. Continuing, we see that $a_1 a_2 \ldots a_n = 0$ $\square$

Theorem 3.1.1. [19] *A ring R is Armendariz iff $R[x]$ is Armendariz.*

proof:

$\Leftarrow$: Any subring of an Armendariz ring is again Armendariz.

$\Rightarrow$: Suppose that R is an Armendariz ring, and let $f(T), g(T) \in R[x][T]$ with $fg = 0$. write $f(T) = f_0+f_1T+\ldots+f_nT^n$, and $g(T) = g_0+g_1T+\ldots+g_mT^m$, where $f_i, g_j \in R[x]$. We need each $f_ig_j = 0$. Let $k = deg(f_0)+\ldots+deg(f_n)+deg(g_0)+\ldots+deg(g_m)$, where the degree is as polynomials in x , and the degree of the zero polynomial is taken to be zero. Then $f(x^k) = f_0 + f_1x^{k1} + \ldots + f_nx^{kn}$, $g(x^k) = g_0 + g_1x^{k1} + \ldots + g_mx^{km} \in R[x]$, and the set of coefficients of $f_i's(resp., g_i's)$ equals the set of coefficients of $f(x^k)(resp.g(x^k))$. Since $f(T)g(T) = 0$, and x commutes with elements of R, $f(x^k)g(x^k) = 0$. Since R is an Armendariz ring , each coefficient of f_i annihilates each coefficient of g_j. Thus $f_ig_j = 0$, therefore $R[x]$ is an Armendariz ring. $\square$

Theorem 3.1.2. [19] *Let R be a ring and $n \geq 2$, then $R[x]/(x^n)$ is Armendariz if and only if R is a reduced ring.*

proof:

$\Rightarrow$: suppose that $R[x]/(x^n)$ is Armendariz, let $r \in R$ with $r^n = 0$. Since r and $\bar{x}$ commutes, $0 = r^n - \bar{x}^nT^n = (r - \bar{x}T)(r^{n-1} + r^{n-2}\bar{x}T + \ldots + \bar{x}^{n-1}T^{n-1})$, where T is an indeterminate, and $\bar{x} \in R[x]/(x^n)$. Now $R[x]/(x^n)$ Armendariz gives $r\bar{x}^{n-1} = 0$, and hence $r = 0$, it easily follow that R is reduced.

$\Leftarrow$: suppose that R is reduced.Denote $\bar{x} \in R[x]/(x^n)$ by u, so $R[x]/(x^n) = R[u] = R+Ru+ \ldots + Ru^{n-1}$, where u commutes with elements of R and $u^n = 0$. Let $f, g \in R[u][T]$ with $fg = 0$. We can write $f = f_0+f_1u+\ldots+f_{n-1}u^{n-1}$, and $g = g_0+g_1u+\ldots+g_{n-1}u^{n-1}$, where $f_i, g_i \in R[T]$. Now for f_iu^i and g_ju^j, where $i+j \geq n$, the coefficients of f_iu^i annihilate the coefficients of g_ju^j since $u^{i+j} = 0$. We show that if $i + j < n$, then $f_ig_j = 0$ and hence since R is reduced and thus Armendarize, the coefficients of f_i annihilate the coefficients of g_j. Thus the coefficients of f annihilate the coefficients of g.

Now $0 = fg = (f_0 + f_1u + \ldots + f_{n-1}u^{n-1})(g_0 + g_1u + \ldots + g_{n-1}u^{n-1}) = f_0g_0 + (f_0g_1 + f_1g_0)u + (f_0g_2 + f_1g_1 + f_2g_0)u^2 + \ldots + (f_0g_{n-1} + f_1g_{n-2} + \ldots + f_{n-1}g_0)u^{n-1}$.

So,

$$0 = f_0g_0 = f_0g_1 + f_1g_0 = f_0g_2 + f_1g_1 + f_2g_0 = \ldots = f_0g_{n-1} + f_1g_{n-2} + \ldots + f_{n-1}g_0$$

Now if $i + j < n$, we see that $f_ig_j = 0$. □

3.2 Nil-Armendariz rings

In this section, we introduce a type of Armendariz rings which is the Nil-Armendariz rings, and see the relation between them.

Definition 3.2.1. [7] *A ring R is said to be* ***nil-Armendariz*** *if whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in nil(R)[x]$, then $ab \in nil(R)$ for all $a \in coef(f(x))$ and $b \in coef(g(x))$.*

Proposition 3.2.1. *Let R be a ring and $I \trianglelefteq R$ a nil ideal. Then R is nil-Armendariz if and only if R/I is nil-Armendariz.*

proof:

We denote $\overline{\mathrm{R}} = R/I$. Since I is nil, then $nil(\overline{\mathrm{R}}) = \overline{\mathrm{nil(R)}}$. Hence $f(x)g(x) \in nil(R)[x]$ if and only if $\bar{\mathrm{f}}(x)\bar{\mathrm{g}}(x) \in nil(\overline{\mathrm{R}})[x]$. And, if $a \in coef(f(x))$ and $b \in coef(g(x))$, then $ab \in nil(R)$ if and only if $\bar{\mathrm{a}}\bar{\mathrm{b}} \in nil(\overline{\mathrm{R}})$.

Therefore R is nil-Armendariz if and only if $\overline{\mathrm{R}}$ is nil-Armendariz. □

Lemma 3.2.1. *Let R be a nil-Armendariz ring and $n \geq 2$. If $f_1(x), f_2(x), \ldots, f_n(x) \in R[x]$ such that $f_1(x)f_2(x)\ldots f_n(x) \in nil(R)[x]$, then if $a_k \in coef(f_k(x))$ for $k = 1, \ldots, n$, we have $a_1a_2\ldots a_n \in nil(R)$.*

proof:

We use induction on n. The case $n = 2$ is clear by definition of nil-Armendariz ring. Suppose $n > 2$. Consider $h(x) = f_1(x)\ldots f_{n-1}(x)$. Then $h(x)f_n(x) \in nil(R)[x]$ and hence, since R is nil-Armendariz $a_ha_n \in nil(R)$, where $a_h \in coef(h(x))$ and $a_n \in coef(f_n)$. Therefore, for all $a_n \in coef(f_n(x))$,

$$f_1(x)\ldots f_{n-2}(x)(f_{n-1}(x)a_n) = h(x)a_n \in nil(R)[x],$$

and by induction, since the coefficients of $f_{n-1}(x)a_n$ are $a_{n-1}a_n$, where a_{n-1} is a coefficient of $f_{n-1}(x)$, we obtain $a_1a_2\ldots a_{n-1}a_n \in nil(R)$ for $a_k \in coef(f_k(x)), k = 1, \ldots, n$. $\square$

Lemma 3.2.2. [2] *Let R be a nil-Armendariz ring.*

(a) *If a, b are nilpotent, then ab is nilpotent.*

(b) *If a, b, c are nilpotent, then $(a + b)c$ and $c(a + b)$ are nilpotent.*

(c) *If a, b, c are nilpotent, then $a + bc$ is nilpotent.*

(d) *If a, b are nilpotent, then $a - b$ is nilpotent.*

proof:

(a): Suppose a, b are nilpotent and $b^m = 0$. Then $(a-abx)(1+bx+b^2x^2+\ldots+b^{m-1}x^{m-1}) = a \in nil(R)[x]$. Since R is nil-Armendariz, so $ab \in nil(R)$ [by Lemma (3.2.1)].

(b): Suppose a, b, c are nilpotent and $a^n = b^m = 0$. Then $(1 + \ldots + a^{n-1}x^{n-1})(1 - ax)(1 - bx)(1 + \ldots + b^{m-1}x^{m-1})c = c$. If we multiply the polynomials in the middle, we obtain $(1 + \ldots + a^{n-1}x^{n-1})(1 - (a + b)x + abx^2)(1 + \ldots + b^{m-1}x^{m-1})c = c$. Now, since R is nil-Armendariz, and $c \in nil(R)[x]$, by Lemma (3.2.1), we can choose the appropriate coefficients from each polynomial to obtain $(a+b)c \in nil(R)$. Similarly we see that $c(a+b) \in nil(R)$.

(c): Suppose a, b, c are nilpotent. By **(a)** , bc is nilpotent, and by **(b)**, $b(a+bc)$ is also nilpotent. Hence

$$(1-bx)(c+(a+bc)x) = c + ax - b(a+bc)x^2 \in nil(R)[x].$$

Now, since R is nil-Armendariz, $1.(a+bc) = a+bc$ is nilpotent.

(d): Suppose a, b are nilpotent. Now by applying **(c)** several times we can see that, since a^2, a and $-b$ are nilpotent, $a^2 - ab$ is nilpotent; hence $a^2 - ab - ba$ is nilpotent; hence $a^2 - ab - ba + b^2$ is nilpotent. Therefore $(a-b)^2$ is nilpotent, which means that $a-b$ is nilpotent. □

Theorem 3.2.1. [2] *If R is a nil-Armendariz ring, then $nil(R)$ is a subring of R.*

proof:

Let $a, b \in nil(R)$, we need to show that both $a-b$ and ab are nilpotent elements (i.e. $\in nil(R)$).Which is satisfied by Lemma (3.2.2). □

Lemma 3.2.3. *If R is a nil-Armendariz ring with no nonzero nil ideals, then R is Armendariz.*

proof:

Since R has no nonzero nil ideals, it does not contain any nonzero nil one-sided ideals. Suppose $f(x), g(x) \in R[x]$ such that $f(x)g(x) = 0$. Let $a \in coef(f(x))$ and $b \in coef(g(x))$. For all $r \in R$, since $rf(x)g(x) = 0$, R is nil-Armendariz, and $ra \in coef(rf(x))$, we have that rab is nilpotent. Hence Rab is a nil one-sided ideal. Then $Rab = 0$ and thus $ab = 0$. Therefore R is Armendariz. $\square$

Lemma 3.2.4. [7] *If R is nil-Armendariz, then $nil(R[x]) \subseteq nil(R)[x]$.*

proof:

Suppose $f(x) \in nil(R[x])$ and $f(x)^m = 0$. By Lemma (3.2.1), we have that $a_1 \dots a_m \in nil(R)$, where $a_i \in coef(f(x))$ for $i = 1, \dots, m$. In particular, for every $a \in coef(f)$, a^m is nilpotent.

Therefore $a \in nil(R)$ for all $a \in coef(f(x))$ and hence $f(x) \in nil(R)[x]$. $\square$

Theorem 3.2.2. [2] *Let R be a nil-Armendariz ring. Then, $R[x]$ is nil-Armendariz if and only if $nil(R[x]) = nil(R)[x]$.*

proof:

$(\Rightarrow)$: If $R[x]$ is nil-Armendariz, by Theorem (3.2.1), we have that $nil(R[x])$ is a subring of $R[x]$. Clearly $nil(R)x^k$ is nil for any $k \geq 0$, and thus $nil(R)[x] \subseteq nil(R[x])$. Now, since R is a subring of $R[x]$, it is nil-Armendariz and, by Lemma (3.2.4), we have the other inclusion. Hence $nil(R[x]) = nil(R)[x]$.

$(\Leftarrow)$: Now suppose that $nil(R[x]) = nil(R)[x]$. Let $f(y), g(y) \in (R[x])[y]$ such that $f(y)g(y) \in nil(R[x])[y]$. If

$$f(y) = f_0 + f_1 y + \dots + f_n y^n, \ \ where \ \ f_i = \sum_{k=1}^{s_i} f_{i_k} x^k,$$

$$g(y) = g_0 + g_1 y + \dots + g_m y^m, \ \ where \ \ g_j = \sum_{l=1}^{t_j} f_{j_l} x^l,$$

and $M > max(s_i, t_{j_{i,j}})$, then, by evaluating at x^M, we obtain polynomials $f'(x) = f(x^M)$ and $g'(x) = g(x^M)$ whose coefficients are all the $f'_{i_k}s$ and $g'_{j_l}s$. Also, since $nil(R[x]) =$

$nil(R)[x], f'(x)g'(x) \in nil(R)[x]$. Since R is nil-Armendariz, $f_{i_k}g_{j_l} \in nil(R)$. Now, since $nil(R)$ is a subring of R, we have that $f_ig_j \in nil(R)[x]$. Finally, since $nil(R)[x] = nil(R[x])$, f_ig_j is nilpotent, hence $R[x]$ is nil-Armendariz. □

Proposition 3.2.2. *If R is an Armendariz ring then R is nil-Armendariz.*

proof:

Suppose $f(x), g(x) \in R[x]$ such that $f(x)g(x) \in nil(R)[x]$. Since R is Armendariz, by Lemma (3.2.4), $f(x)g(x)$ is nilpotent and there exists $k \geq 1$ such that $(f(x)g(x))^k = 0$. Hence, since R is Armendariz, for all $a \in coef(f(x))$ and $b \in coef(g(x))$, by choosing the corresponding coefficient in each polynomial, we have $abab \dots ab = 0$ and thus, $ab \in nil(R)$. Therefore R is nil-Armendariz. □

Hence nil-Armendariz rings stand as a generalization of Armendariz rings and a particular case of weak Armendariz rings. For those rings, the property of being nil-Armendariz clearly passes to subrings, and in fact, most of the results can be proved for nil-Armendariz rings.

3.3 skew inverse Laurent-serieswise Armendariz

In this section, for any ring with an automorphism we study the skew inverse Laurent-serieswise Armendariz rings, and introduce the notions of a strongly Armendariz ring of inverse skew power series type, and an α-compatible ring.

Definition 3.3.1. [9] *Let R be a ring equipped with an automorphism α. We denote by $R((x^{-1}, \alpha))$ **the inverse skew Laurent series ring** over the coefficient ring R formed* *by formal series*

$$f(x) = \sum_{i=-\infty}^{n} a_i x^i,$$

where x is a variable, n is an integer and $a_i \in R$. In the ring $R((x^{-1}, \alpha))$, addition is defined as usual and multiplication is defined with respect to the relation:

$$\forall i, \qquad x^i a = \alpha^i(a) x^i.$$

Now we will study the ring $R((x^{-1}, \alpha))$, and introduce a skew inverse Laurent-serieswise Armendariz ring as a generalization of the standard Armendariz condition from polynomials to skew inverse Laurent series.

Definition 3.3.2. [9] *A ring R is called **power-serieswise Armendariz**, if $a_i b_j = 0$, for all i, j , whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x]]$ satisfy $f(x)g(x) = 0$.*

Definition 3.3.3. [9] *A ring R is called **a skew inverse Laurent-serieswise Armendariz** (or simply, SIL -Armendariz) ring, if for each elements $f(x) = \sum_{i=-\infty}^{n} a_i x^i$,and $g(x) = \sum_{j=-\infty}^{m} b_j x^j \in R((x^{-1}, \alpha))$, $f(x)g(x) = 0$ implies that $a_i \alpha^i(b_j) = 0$ for each $i \leq n$ and $j \leq m$.*

Theorem 3.3.1. [9] *Let R be a ring with an automorphism α and $A = R((x^{-1}, \alpha))$. Then the following statements are equivalent:*

1. *R is SIL-Armendariz.*
2. *For each $f(x) = \sum_{i=-\infty}^{0} a_i x^i$,and $g(x) = \sum_{j=-\infty}^{0} b_j x^j$ in A, if $f(x)g(x) = 0$, then $a_i \alpha^i(b_j) = 0$, for each $i, j \leq 0$.*
3. *For each $f(x) = \sum_{i=-\infty}^{n} a_i x^i$,and $g(x) = \sum_{j=-\infty}^{m} b_j x^j$ in A, if $f(x)g(x) = 0$, then $a_0 b_j = 0$, for each $j \leq m$.*

4. For each $f(x)=\sum_{i=-\infty}^{0} a_i x^i$ *,and* $g(x)=\sum_{j=-\infty}^{0} b_j x^j$ *in* A*, if* $f(x)g(x)=0$*, then* $a_0 b_j=0$*, for each* $j\leq 0$*.*

proof:

(1) $\Rightarrow$ (2) : From definition, take $n=m=0$.

(2) $\Rightarrow$ (3) : It is clear.

(3) $\Rightarrow$ (4) : Take $n=m=0$.

(4) $\Rightarrow$ (1): Let $f(x)=\sum_{i=-\infty}^{n} a_i x^i$,and $g(x)=\sum_{j=-\infty}^{m} b_j x^j$ be series of $R((x^{-1},\alpha))$ with $f(x)g(x)=0$. We have;

$$\begin{aligned} f(x)g(x) &= (\sum_{i=-\infty}^{n} a_i x^i)(\sum_{j=-\infty}^{m} b_j x^j) \\ &= (\sum_{i=-\infty}^{n} a_i x^{i-n})x^n(\sum_{j=-\infty}^{m} b_j x^j) \\ &= (\sum_{i=-\infty}^{n} a_i x^{i-n})(\sum_{j=-\infty}^{m} \alpha^n(b_j) x^{j+n}) = 0. \end{aligned} \tag{a.1}$$

By multiplying x^{-n-m} from the right-hand side of Eq. $(a.1)$, we obtain

$$(\sum_{i=-\infty}^{n} a_i x^{i-n})(\sum_{j=-\infty}^{m} \alpha^n(b_j) x^{j-m}) = 0.$$

So $a_m \alpha^m(b_j)=0$, for each $j\leq m$, by (4). This implies that

$$\begin{aligned} f(x)g(x) &= (\sum_{i=-\infty}^{n-1} a_i x^i)(\sum_{j=-\infty}^{m} b_j x^j) \\ &= (\sum_{i=-\infty}^{n-1} a_i x^{i-n+1})x^{n-1}(\sum_{j=-\infty}^{m} b_j x^j) \\ &= (\sum_{i=-\infty}^{n-1} a_i x^{i-n+1})(\sum_{j=-\infty}^{m} \alpha^{n-1}(b_j) x^{j+n-1}) = 0 \end{aligned} \tag{a.2}$$

By multiplying x^{-n-m+1} from the right-hand side of Eq. $(a.2)$, we obtain

$$(\sum_{i=-\infty}^{n-1} a_i x^{i-n+1})(\sum_{j=-\infty}^{m} \alpha^{n-1}(b_j)x^{j-m} = 0.$$

Hence $a_{m-1}\alpha^{m-1}(b_j) = 0$, for each $j \leq m$, by (4). By continuing in this way, we get $a_i\alpha^i(b_j) = 0$, for each $i \leq n$ and $j \leq m$ and the result follows. □

Definition 3.3.4. [9] *Let R be a ring with an automorphism α. We say that R is a **strongly Armendariz ring of inverse skew power series type** (or simply, strongly ISP -Armendariz ring), if R satisfies the following condition.*
$\forall\ f(x), g(x) \in R((x^{-1}, \alpha)),\ f(x)g(x) = 0 \iff ab = 0, \forall\ a \in C_f, b \in C_g$
where C_f and C_g are the sets of all coefficients of elements $f(x)$ and $g(x)$, respectively.

Definition 3.3.5. [7] *A ring R is α-**compatible** if for each $a, b \in R, ab = 0$ implies $a\alpha(b) = 0$.*

Lemma 3.3.1. [9] *Let α be an automorphism of a ring R. If R is strongly $ISP-$ $Armendariz$, then we have the following statements:*

1. *R is an α-compatible ring.*
2. *If $ab = 0$, then $a\alpha^k(b) = \alpha^k(a)b = 0$ for all integers k.*
3. *If $\alpha^k(a)b = 0$ for some integer k, then $ab = 0$.*

Recall that $N_0(R), N_*(R), L_rad(R), N^*(R), nil(R)$ and $rad(R)$ denote the Wedderburn radical, the lower nil radical, the Levitsky radical, the upper nil radical, the set of all nilpotent elements and the Jacobson radical, respectively.

Theorem 3.3.2. [7] *Let R be a ring with an automorphism α and A_1, A_2 be two subrings of $R((x^{-1}, \alpha))$, as follows:*

$$\begin{aligned} A_1 &= \{f(x) = \sum_{i=-m}^{n} a_i x^i : a_i \in R, m, n \in \mathbb{N}\} \\ A_2 &= \{f(x) = \sum_{i=0}^{n} a_i x^i : a_i \in R, n \in \mathbb{N}\} \end{aligned}$$

If R is a strongly ISP-Armendariz ring, then $(rad(A_1) \cap R)$ and $(rad(A_2) \cap R)$ are nil.

proof:

Let $a \in R$ be an element of $rad(A_1)$. Then $1 - ax$ is an invertible element in A_1, (since if $1 - ax$ is not invertible, then it is contained in a maximal ideal M of R. In particular, since x is from M, we see that $1 \in M$ which is a contradiction). So there exists an element $\sum_{i=-m}^{n} b_i x^i \in A_1$ such that

$$(1 - ax)(\sum_{i=-m}^{n} b_i x^i) = 1.$$

Therefore we have:

$$\begin{aligned} & b_{-m} = 0; \\ & b_{-m+1} - a\alpha(b_{-m}) = 0; \\ & \vdots \\ & b_{-1} - a\alpha(b_{-2}) = 0; \\ & b_0 - a\alpha(b_{-1}) = 1; \\ & b_1 - a\alpha(b_0) = 0; \\ & \vdots \\ & b_n - a\alpha(b_{n-1}) = 0; \\ & a\alpha(b_n) = 0. \end{aligned}$$

So by replacing $b_{-m} = 0$ in Equation $b_{-m+1} - a\alpha(b_{-m}) = 0$, we get $b_{-m+1} = 0$. By continuing in this way, we obtain $b_{-m} = \ldots = b_{-1} = 0$ and $b_0 = 1$. Now, by replacing $b_0 = 1$ in Equation $b_1 - a\alpha(b_0) = 0$, we have $b_1 = a$. So $b_2 - a\alpha(b_1) = 0$ implies that $b_2 = a\alpha(a)$.

Continuing in this way, we get $a\alpha(a)\alpha^2(a)\ldots\alpha^n(a) = 0$ and hence $a^{n+1} = 0$, since R is $\alpha-$*compatible* by Lemma (3.3.1). Thus $a \in nil(R)$ and so $(rad(A_1)\cap R)$ is nil. Similarly, we can show that $(rad(A_2)\cap R)$ is nil and the proof is complete. $\square$

Chapter 4

Nilpotence and The Jacobson Radical

This chapter studies the relation between the Jacobson Radical of a ring, and the relation between it and nilpotence. We present many important definitions like quasi-regular left (or right) ideal and idempotent element which we need in order to talk about the Jacobson Radical. We also study the Prime Radical of a ring. Furthermore we present the J-Reduced Rings and J-clean Rings that are related to the Jacobson Radical.

4.1 Nilpotence and The Jacobson Radical

In this section, we present The Jacobson Radical of a ring, some of its properties (Jacobson Semisimple), and its relation with nilpotence. We also present many important definitions like quasi-regular left (or right) ideal and idempotent element.

In a ring $(R, +, .)$, we define the circle operation $\circ$ on R, as $a \circ b = a + b - ab$, $\forall a, b \in R$.

Definition 4.1.1. *An element $a \in R$ of a ring R is said to be* ***left quasi-regular (resp. right)****, if there exists an element $x \in R$ such that $x \circ a = 0$ (resp. $a \circ x = 0$). In this case the element x is called* ***a left (resp. right) quasi-inverse*** *of the element a. And* *an element is called* ***quasi-regular element*** *if it is left and right quasi-regular.*

A ring R is called ***quasi-regular ring****, if each of its elements is quasi-regular.*

Definition 4.1.2. *Let A be an ideal (one- or two-sided) in a ring R. If each element of A is quasi-regular, then A is a* ***quasi-regular ideal****.*

Definition 4.1.3. *an* ***idempotent element****, or simply* ***an idempotent****, of a ring is an element e such that $e^2 = e$.*

Proposition 4.1.1. *An idempotent element e of a ring R is left quasi-regular if and only if $e = 0$.*

proof:

($\Rightarrow$) If $e = e^2$ and x is a left-quasi-inverse of e, then $x + e - xe = 0$, and therefore $xe + e^2 - xe^2 = 0$. Thus $e^2 = 0$. But $e^2 = e$ therefore $e = 0$.

($\Leftarrow$) if $e = 0$, then trivially an idempotent element e of a ring R is left quasi-regular. $\square$

Corollary 4.1.1. *The Jacobson radical of a ring does not contain nonzero idempotents.*

If a, b are elements of a ring R with unity element e, then

$$(b - e)(a - e) = ba - b - a + e = e - (b \circ a).$$

Thus $(b - e)(a - e) = e$ if and only if $b \circ a = 0$, hence it follows that a is quasi-regular if and only if $a - e$ is a unit.

Proposition 4.1.2. *Every nilpotent element of a ring R is left quasi-regular.*

proof:

If $a^n = 0$ with some exponent $n \geq 2$, then put $x = -a - a^2 - \ldots - a^{n-1}$. Now $x \circ a = x + a - xa = -a^2 - \ldots - a^{n-1} - (-a^2 - \ldots - a^n) = 0$, and therefore x is a left quasi-inverse of a. □

Note.

Quasi-regularity is important because it provides elementary characterizations of the Jacobson radical for rings without an identity element:

- The Jacobson radical of a ring is the sum of all quasi-regular left (or right) ideals.
- The Jacobson radical of a ring is the largest quasi-regular ideal of the ring.

Lemma 4.1.1. *Let x be a nilpotent element of a ring R, then $1 + x$ is a unit of R.*

proof:

x is nilpotent, so $x^n = 0$ for some $n > 0$. Then, by direct computation, we see that

$$(1 + x)(1 - x + x^2 + \ldots + (-1)^n x^n) = 1 + (-1)^n x^{n+1} = 1 + 0 = 1,$$

therefore,1+x is a unit in R. □

Proposition 4.1.3. [1] *For a ring R, the following are equivalent:*
(a) $x \in J(R)$.
(b) For all $r \in R$, $1 + rx$ has a left inverse (i.e. there is an $s \in R$ such that $s(1 + rx) = 1$).
(c) For all $r \in R$, $1 + rx$ is a unit.

proof:

$(a) \Rightarrow (b)$: Since $J(R)$ is a left ideal, it suffices to show $x \in J(R) \Rightarrow 1 + x$ has a left inverse.

If not, then $R(1+x)$ is a proper left ideal and hence there is a maximal left ideal M with $1+x \in M$. But $x \in M$, so $1 \in M$, a contradiction.

$(b) \Rightarrow (c)$: By assumption$1+rx$ has left inverse, say $s(1+rx) = 1$. Hence $s = 1 - srx$ also has a left inverse, again by assumption; say $ts = 1$. But if an element in a ring has both a left and right inverse, the two are equal. Here we conclude $t = 1 + rx$. Hence s is a $2-$sided inverse of $1 + rx$, *i.e.* $1 + rx$ is a unit.

$(c) \Rightarrow (a)$: Suppose $1+rx$ is a unit for all r, and let M be a maximal left ideal. If $x \notin M$, then $Rx + M = R$, so there is an $r \in R$ and $y \in M$ such that $rx + y = 1$. Then $y = 1 - rx$ is a unit by assumption, so $1 \in Ry$ and $Ry = R$, contradicting $y \in M$. $\square$

Proposition 4.1.4. *If I is a left ideal consisting of nilpotent elements, then $I \subset J(R)$.*

proof:

If $x \in I$ then rx is nilpotent for all $r \in R$, so $1+rx$ is a unit for all $r \in R$. Hence, by Proposition (4.1.3) $x \in J(R)$. $\square$

Corollary 4.1.2. *If R is commutative, then $nil(R) \subset J(R)$.*

Equality need not hold in this corollary. For example if $R = \mathbb{Z}_{(p)}$ then $nil(R) = 0$ and $J(R) = (p)$.

Lemma 4.1.2. *Let I be any ideal of R lying in $J(R)$. Then $J(R/I) = J(R)/I$.*

proof:

$J(R/I) = \bigcap(M + I)$, where M is a maximal ideal of the ring R, but since $I \subseteq J(R)$, then $\bigcap(M+I) = (\bigcap M)+I$, so $J(R/I) = J(R)/I$. $\square$

Definition 4.1.4. *A ring R is called **Jacobson semisimple** if $J(R) = 0$.*

Note that *Jacobson semisimple rings* are also called *semiprimitive rings*.

Lemma 4.1.3. *If a left (resp., right) ideal $I \subseteq R$ is nil, then $I \subseteq J(R)$.*

proof:

Let $y \in I$. Then for any $x \in R, xy \in I$ is nilpotent. It follows that $1 - xy$ is a unit. Therefore, by Proposition (4.1.3), we have $y \in J(R)$. Therefore $I \subseteq J(R)$. $\square$

Lemma 4.1.4. *There are equivalent ways of expressing the definition of maximal ideals. Given a ring R and a proper ideal I of R (that is $I \neq R$), I is a maximal ideal of R if any of the following equivalent conditions hold:*

- *There exists no other proper ideal J of R so that $I \nsubseteq J$.*
- *For any ideal J with $I \subseteq J$, either $J = I$ or $J = R$.*
- *The quotient ring R/I is a simple ring (where a simple ring is a non-zero ring that has no two-sided ideal besides the zero ideal and itself).*

Theorem 4.1.1. *Let R be a ring , then $J(R/J) = 0$.*

proof:

The maximal ideals of the ring R/J are precisely the ideals of the form M/J , where M is a maximal ideal of R. So by lemma (4.1.4), we get that $J(R/J) = 0$. $\square$

Theorem 4.1.2. *Let R be a left artinian ring. Then $J(R)$ is the largest nilpotent left ideal, and it is also the largest nilpotent right ideal.*

proof:

In view of Lemma (4.1.3), we are done if we can show that $J = J(R)$ is nilpotent. Applying the left DCC (descending chain condition) to

$$J \supseteq J^2 \supseteq J^3 \supseteq \ldots,$$

there exists an integer k such that

$$J^k = J^{k+1} = \ldots = I \quad (say).$$

We claim that $I = 0$. Indeed, if $I \neq 0$, then, among all left ideals L such that $I.L \neq 0$, we can choose a minimal one, say L_0 (by the left DCC). Fix an element $a \in L_0$ such that $I.a \neq 0$. Then

$$I.(Ia) = I^2 a = Ia \neq 0$$

so by the minimality of L_0, we have $I.a = L_0$. Thus, $a = ya$ for some $y \in I \subseteq rad(R)$. But then $(1-y)a = 0$ implies that $a = 0$ since $1-y$ is a unit. This is a contradiction, so we must have $I = J^k = 0$. $\square$

The theorem we just proved and the lemma preceding it have the following pleasant consequence:

Corollary 4.1.3. *In a left artinian ring, any nil left ideal is nilpotent.*

4.2 The Prime Radical

In this section, we study the Prime Radical of a ring and review the strongly nilpotent concept which is related to the Prime Radical.

Recall that the prime radical is the intersection of all the prime ideals, and we will denote it by $P(R)$.

Lemma 4.2.1. *A ring R is semiprime iff $P(R) = 0$.*

proof:

$\Rightarrow$: Suppose that R is a semiprime ring, so (0) is a semiprime ideal (i.e. for any ideal J, $J^2 \subseteq 0 \Rightarrow J \subseteq 0$).

Recall that every prime ideal is a semiprime ideal, we have (0) a semiprime ideal and it is the smallest semiprime ideal, so $P(R) = \bigcap S$, (S : prime ideals) is a prime ideal and so it is a semiprime ideal, since (0) is the smallest one , we get that $P(R) = 0$.

$\Leftarrow$: suppose that $P(R) = 0$. Now $P(R)$ is a prime ideal, so in our case, (0) is a prime ideal and also a semiprime ideal, therefore R is a semiprime ring. $\square$

Definition 4.2.1. *An element a of a ring R is* ***strongly nilpotent*** *in case for every sequence $a_0, a_1, \ldots$ in R if*

$$a_0 = a, \quad and \quad a_{n+1} \in a_n R a_n \quad for\ all \quad n \geq 0,$$

there must exist some $n \in \mathbb{N}$ with $a_n = 0$.

That this really is a statement about the nilpotence of a maybe made a bit more clear with the following characterization.

Lemma 4.2.2. *An element $a \in R$ is strongly nilpotent iff for every sequence $x_1, x_2, \ldots$ in R, there exists some $n \in \mathbb{N}$ with*

$$a(x_1 a(x_2 a(\ldots a(x_{n-1} a(x_n) a x_{n-1}) a \ldots a) x_2) a x_1) a = 0.$$

proof:

$\Rightarrow$: In our case, we have

$$x_1 = a.$$

$$x_2 = a(x_1)a. \qquad (x_2 \in x_1Rx_1)$$

$$x_3 = a(x_1a(x_2)ax_1)a. \qquad (x_3 \in x_2Rx_2)$$

$$.$$

$$.$$

$$x_{n+1} = a(x_1a(x_2a(\dots a(x_{n-1}a(x_n)ax_{n-1})a\dots)ax_2)ax_1)a.$$

$$= a_n = 0.$$

Therefore $a(x_1a(x_2a(\dots a(x_{n-1}a(x_n)ax_{n-1})a\dots a)x_2)ax_1)a = 0.$

$\Leftarrow$: reversible steps. $\square$

Theorem 4.2.1. *The prime radical of a ring R is*

$$P(R) = \{a \in R : a \ \ is \ \ strongly \ \ nilpotent\}.$$

proof:

Suppose that $a \notin P(R)$, so there is some prime ideal I with $a \notin I$. Since I is prime, there is a function $\sigma : R/I \to R/I$ such that $\sigma(x) \in xRx/I$ for each $x \in R/I$.

Now define a sequence $a_0, a_1, \dots$ in R/I recursively by

$$a_0 = a, \ \ and \ \ a_{n+1} = \sigma(a_n), for \ \ all \ \ n \geq 0$$

Since $a_n \neq 0$ for all $n \in \mathbb{N}$, a is not strongly nilpotent.

On the other hand suppose that $a \in R$ is not strongly nilpotent. Then there is a sequence

$$a = a_0, a_1, a_2, \dots$$

in R with $0 \neq a_{n+1} \in a_n R a_n$ for all $n \geq 0$. Then there is an ideal I maximal w.r.t. $a_n \notin I$ for all $n \in P$. We claim that I is prime. Indeed, suppose that K, J are two ideals neither contained in I. Then by maximality, there must be some $n \in \mathbb{N}$ with $a_n \in K+I$ and $a_n \in J+I$. But then $a_{n+1} \in a_n R a_n \subseteq KJ + I$. So $KJ \nsubseteq I$, and I is prime. Of course, this means that $a \notin P(R)$. □

Corollary 4.2.1. *For every ring R, its prime radical $P(R)$ is a nil ideal.*

Corollary 4.2.2. *For a ring R the following are equivalent:*

1. *R is semiprime.*
2. *$P(R) = 0$.*
3. *$I^2 = 0$ implies $I = 0$ for every left (right/two-sided) ideal I of R.*
4. *$aRa = 0$ implies $a = 0$.*
5. *R has no non-zero nilpotent left (right/two-sided) ideals.*
6. *$IJ = 0$ implies $I \cap J = 0$ for every pair I, J of left (right/two-sided) ideals.*

proof:

(1) $\Leftrightarrow$ (2) : This is simply by lemma (4.2.1).

(6) $\Rightarrow$ (3) $\Rightarrow$ (4) , and (3) $\Leftrightarrow$ (5) are all trivial.

$(2) \Rightarrow (6)$: If $IJ = 0$, then $IJ \subseteq K$ for every prime ideal K, so $I \cap J \subseteq K$ for every prime ideal K. Thus, by $(2), I \cap J = 0$.

$(4) \Rightarrow (2)$: By (4) there is a map $\sigma : R/\{0\} \to R/\{0\}$ such that $\sigma(a) \in aRa$ for each $a \neq 0$ in R.

Then for each $a \neq 0$ in R the sequence $a, \sigma(a), \sigma^2(a), \ldots$ is never 0, so a is not strongly nilpotent. Thus, by Theorem (4.2.1), $P(R) = 0$. $\square$

Remark 4.2.1. *Any nilpotent element in R is in all prime ideals of R.*
i.e., if $nil(R) = \{a \in R; a^n = 0 \;\; for \;\; some \;\; n \in N\}$ *and* $P(R) = \cap P$, *where the intersection is taken over all prime ideals of R, then* $nil(R) \subseteq P(R)$.

We now have the following important Theorem which establishes a relationship between the set of nilpotent elements and the prime ideals.

Theorem 4.2.2. *The set of all nilpotent elements in a commutative ring R with* 1 *is the intersection of all prime ideals, i.e.,* $nil(R) = P(R)$.

4.3 J-Reduced Rings

In this section, we present the J-Reduced Rings and J-clean Rings that are related to The Jacobson Radical.

Definition 4.3.1. *The ring R is called* ***J-reduced*** *whenever nilpotent elements belong to the Jacobson radical* $J(R)$.

Lemma 4.3.1. *If* $R/J(R)$ *is a reduced ring, then R is* $J-reduced$.

proof:

[10] Assume that $a^n = 0$ for some $n \geq 2$. Then $\overline{a^n} = \overline{0} \in R/J(R)$. Since $R/J(R)$ is reduced,

then $\overline{a} = \overline{0}$ and so $a \in J(R)$, as asserted. $\square$

Clearly, every reduced ring is $J-reduced$ but the converse does not hold in general as the following example shows.

Example 4.3.1. [10] *Let $R = \mathbb{Z}_4$. Since $J(R) = 2R$, $R/J(R) \cong \mathbb{Z}_2$ and so $R/J(R)$ is reduced. By Lemma* (4.3.1)*, R is $J - reduced$ but it is not reduced.*

Let $J^*(R)$ denote the subset $\{x \in R : \exists n \in \mathbb{N} \textit{ such that } x^n \in J(R)\}$ of R. It is obvious that $J(R) \subseteq J^*(R)$, but the following example shows that the reverse inclusion does not hold.

Example 4.3.2. [10] *Let R denote the matrix ring $M_2(\mathbb{Z}_2)$. Then*

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = J^*(R) \right\} \tag{4.1}$$

while $J(R) = 0$.

Proposition 4.3.1. *Let R be a ring. If $J(R) = J^*(R)$, then R is $J - reduced$.*

proof:

Assume that $J(R) = J^*(R)$ and $a^n = 0$ for some $n \geq 2$. Then $a^n \in J(R)$ and so $a \in J^*(R) = J(R)$, therefore R is $J - reduced$. $\square$

Definition 4.3.2. *A ring R is* ***J-clean*** *if for any element $a \in R$, there exists an idempotent $e \in R$ such that $a - e \in J(R)$.*

Corollary 4.3.1. [10] *Every $J - clean$ ring is $J - reduced$.*

proof:

Assume that R is $J-clean$ and let $a^n = 0$ for some $n \geq 2$. Then there exists an idempotent $e \in R$ such that $a - e = j \in J(R)$. Since $a^n = 0$, $1 - a = 1 - e - j \in U(R)$ (where $U(R)$ is the set of all invertible elements), and so $1 - e \in U(R)$. Hence $e = 0$. That is, $a = j \in J(R)$, as asserted. $\square$

Definition 4.3.3. [21] *We say that **idempotents lift modulo I** if every idempotent in R/I can be lifted to R (i.e., for any idempotent $a + I$ of R/I, there exists an idempotent e of R such that $a + I = e + I$).*

Proposition 4.3.2. [10] *Let R be a ring in which idempotents lift modulo $J(R)$. If R is $J-reduced$, then $R/J(R)$ is abelian.*

proof:

Assume that R is $J-reduced$ and let $\overline{e}^2 = \overline{e} \in R$. Since idempotents lift modulo $J(R)$, we may assume that $e^2 = e \in R$. If $r \in R$, then $(er - ere)^2 = 0$ and $(re - ere)^2 = 0$ and so $er - ere \in J(R)$ and $re - ere \in J(R)$ by assumption. Thus $er - re \in J(R)$ and so $\overline{er} = \overline{re}$, as required. $\square$

Theorem 4.3.1. [2] *Let I be an ideal of a $J-reduced$ ring R, and let S be a subring of R containing I. If S/I is $J-reduced$, then so is S.*

proof:

Given $a^n = 0$ in S, then $a \in J(R)$ as R is J-reduced. For any $s \in S$, we can find some $r \in R$ such that $r(1 - as) = 1$. Furthermore, $\overline{a} \in J(S/I)$. Hence, we can find some $t \in S$ such that $1 - (1 - as)t \in I$. This implies that $r - r(1 - as)t \in I$, and so $r - t \in I$. We infer that $r \in S$. Therefore $1 - as \in S$ is left invertible. Likewise, $1 - as \in S$ is right invertible.

Hence, $1 - as$ is invertible , and then $a \in J(S)$. Therefore S is $J-reduced$. $\square$

Corollary 4.3.2. *Let I be an ideal of a $J-reduced$ ring R, and let S be a $J-reduced$ subring of R. Then $I+S$ is $J-reduced$.*

proof:

Obviously, $I \subseteq I+S \subseteq R$. It is easy to check that $(I+S)/I$ is $J-reduced$. Therefore $I+S$ is $J-reduced$, by Theorem (4.3.1). $\square$

Proposition 4.3.3. *Let I be a nil ideal of a ring R. Then the following are equivalent.*

(1) R is $J-reduced$.

(2) R/I is $J-reduced$.

proof:

(1) $\Rightarrow$ (2): Write $\overline{a}^n = \overline{0}$, then $a^n \in I$. As I is nil, there exists some $m \in N$ such that $a^{mn} = 0$. Hence, $a \in J(R)$, and so $\overline{a} \in J(R/I)$, as desired.

(2) $\Rightarrow$ (1): Assume that $a^n = 0$ for some $n \in \mathbb{N}$. Then $\overline{a}^n = \overline{0}$ and so $\overline{a} \in J(R/I)$. For any $r \in R$, $\overline{1} - \overline{ar}$ is invertible in (R/I), we have some $t \in R$ such that $1-(1-ar)t \in I$. Since I is nil, we get $(1-ar)t$ is invertible in R, and so $1-ar$ is invertible . This implies that $a \in J(R)$, as desired. $\square$

Proposition 4.3.4. *A ring R is $J-reduced$ if and only if eRe is $J-reduced$ for all idempotent elements $e \in R$.*

proof:

If $(eae)^n = 0$ in eRe, then $(eae)^n = 0$ in R. Since R is $J-reduced$, then $eae \in J(R)$, and so $eae \in eJ(R)e$. That is, $eae \in J(eRe)$. Therefore eRe is $J-reduced$.

The converse is trivial. $\square$

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