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Ideals In Near Rings

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This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Mathematics, Faculty of Graduate Studies, Hebron University, Palestine.

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Declaration

I declare that the master thesis entitled "Ideals In near rings" is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research ,except where due acknowledgment is made in the text and has not been submitted for the award of any other degree at any institution.

Hadeel Suliman

Signature:________________ Date:_____________

Dedications

To my mother who give me the help
To my husband who spent nights and days doing his best
To my sons
To my brothers
To my sincere friends
To my beloved country Palestine
And to all knowledge seekers.

Hadeel

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Abstract

This thesis aims to develop a better understanding of near ring and its ideals. We present the defenistion of near ring . Also we study some basic properties of near ring and give some examples .

We introduce the concept of ideal in near ring and various properties of it. In addition we present many concepts such as prime ideal, weekly prime ideal, semi-symmetric ideal, 3-prime ideal and IFP ideal. Also we present some results related to these concepts.

Contents

Introduction

In mathematics, a near-ring is an algebraic structure similar to a ring but satisfying fewer axioms. Near-rings arise naturally from functions on groups. The study and research of near-rings is very systematic and continuous. Near-ring newsletters containing complete and updated bibliography on the subject are published periodically by a team of mathematicians.

Historically the first step toward near rings was the axiomatic research done by Dikson in 1905. He showed that there do exist fields with only one distributive law (called near fields).

Boolean near rings was introduced by James R, Clay and Donald A. Lawver in 1968. The theory of near ring and its application was introduced by Gunter Pliz in 1983. In addition,Sarwer Jehan Abbasi presented the matrix near rings and generalized distributivity in his PhD thesis in 1989.

In 2002, Vasantha Kandasamy published the book about Smarandache Near-rings. Taylor and Francis Group published the book about Near Rings, Ideals, and Graph Theory in 2013.

Some years later, these near fields showed up again and proved to be useful in coordinating certain important classes of geometric planes. Many parts of the well-established theory of rings were transferred to near rings, and new near ring precise features were discovered, building up a theory of near rings step-by-step.

Every ring is a near ring. A natural example of a near ring (which is not a ring) is given by the set $M(G)$ of all mappings of an additive group G (not necessarily Abelian) into itself, with addition defined by $(f+g)(a) = f(a) + g(a)$ and multiplication by $(fg)(a) = f(g(a))$ (here, fg is the composition of the mappings $f$and g) .

This thesis aims to develop a better understanding of near ring and its ideals. The material of this thesis lies in three chapters.

Chapter 1: In this Chapter we introduce basic definitions, fundamental notations, and several examples. Furthermore we give theorems necessary for the rest of this thesis.

Chapter 2: Includes definition of near ring, distributively generated near ring,Boolean near ring and matrix near ring. In addition we present the N-group and its properties,furthermore we introduce the concept of derevation in near ring.

Chapter 3: We introduce the concept of ideal in near ring and various properties of it. In addition we present many concepts such as prime ideal, weekly prime ideal, semi-symmetric ideal, 3-prime ideal and IFP ideal. Also we will introduce some results related to these concepts.

Chapter 1

Preliminaries

The aim of this Chapter is to introduce basic definitions, fundamental notations, theorems,and several examples needed during the course of this thesis.

1.1 Semigroup

Definition 1.1.1. *[5] let G be a set. A binary operation $*$ on G is a function from $G \times G$ into G .*

A binary operation on a set G is simply a method by which the members of an ordered pair from G combine to yield a new member of G.
Examples of binary operations are ordinary addition, subtraction, and multiplication of integers. Division of integers is not a binary operation on the integers (because an integer divided by an integer need not be an integer).

Definition 1.1.2. *[5] A binary operation $*$ on a set $G$is associative if for all*

$a, b, c \in G$ we have $(a * b) * c = a * (b * c)$.

Definition 1.1.3. *[5] A binary operation $*$ on a set G is commutative if for all $a, b \in G$ we have $a * b = b * a$.*

Example 1.1.1. *Let $\mathbb{N}$ be the set of all natural numbers, the usual addition "+" on $\mathbb{N}$ is a commutative binary operation.*

Definition 1.1.4. *[5] Let $*$ be a binary operation on G, if there exist an element $e \in G$ such that $a * e = e * a, \forall a \in G$, then we say that e is an identity element with respect to $*$.*

Definition 1.1.5. *[19]*

1. *Let S be a set. S is said to be a semigroup if a binary operation $(.)$ is defined on S such that $(a.b).c = a.(b.c)$ for all $a, b, c \in S$*

2. *A semigroup S is said to be commutative if $a.b = b.a$ for all $a, b \in S$. A semigroup in which $a.b \neq b.a$ for at least a pair a, b is said to be a non commutative semigroup .*

3. *A semigroup which has an identity element $e \in S$ is called a monoid.*

Definition 1.1.6. *[19] Let (S,.) be a semigroup . A non empty proper subset P of S is said to be a subsemigroup if (P,.) is a semigroup .*

Definition 1.1.7. *Let $(S, .)$ be a semigroup. A non empty subset I of S is said to be a right ideal (left ideal) of S if $s \in S$ and $a \in I$ then $as \in I (sa \in I)$. An ideal is thus simultaneously a left and right ideal of S.*

Example 1.1.2. *Let* $\mathbb{Z}_9 = \{0, 1, 2, \cdots, 8\}$ *be the integers modulo* $9.\mathbb{Z}_9$ *is a semi-group under multiplication modulo 9 and* $P = \{0, 3, 6\}$ *is an ideal of* $\mathbb{Z}_9$.

Definition 1.1.8. *Let* $*$ *be a binary operation on* G *and* $a \in G$*, if there exist an element* $b \in G$ *such that* $a * b = b * a = e$*, then we say that* b *is an inverse of* a *and* b *will be denoted by* a^{-1}.

Definition 1.1.9. *[19] Let* G *be a non empty set together with a binary operation (usually called multiplication) that assign to each ordered pair* (a, b) *of elements of* G *an element in* G *denoted by* ab*. We say* G *is a group under this operation if the following three properties are satisfied*

1. *Associativity. The operation is associative .*
2. *Identity. There is an element* e *(called the identity)in* G *, such that* $ae = ea = a \ \forall a \in G$.
3. *Inverse . For each element* a *in* G*, there is an element* b *in* G *(called an inverse of* a*) such that* $ab = ba = e$ *.*

Example 1.1.3. *the set of integers* $\mathbb{Z}$*, the set of rational numbers* $\mathbb{Q}$*, and the set of real numbers* $\mathbb{R}$ *are all groups under ordinary addition .*
In each case the identity is zero and the inverse of a *is* $-a$.

Example 1.1.4. *The even integers under multiplication form a commutative semigroup that is not a group (since it contains no multiplicative identity)*

Definition 1.1.10. *if a nonempty subset H of a group G is itself a group under the operation of G, it is called a subgroup of G.*

We use the notation $H \leq G$ to mean H is a subgroup of G.

Definition 1.1.11. *A subgroup H of a group G is called a normal subgroup of G if $aH = Ha$ for all a in G. We denote this by $H \lhd G$.*

Remark 1.1.1. *A subgroup H of a group G is normal in G if and only if $xHx^{-1} \subseteq H$ for all x in G.*

Example 1.1.5. *Every subgroup of an Abelian group is normal (In this case, $ah = ha$ for a in the group and h in the subgroup.*

Definition 1.1.12. *The order of a semigroup / monoid / group is the cardinality of the set G if G has finite cardinality and ∞ otherwise, we denote the order of G by $|G|$. If $|G| < \infty$, the semigroup / monoid / group is said to be finite.*

If we define a groupoid as a set with binary operation on it, then we have the following schematic set inclusion:

$$\text{groupid} \supseteq \text{Semigroup} \supseteq \text{monoid} \supseteq \text{groups}$$

The following remark gives a condition by which a semigroup is a group.

Remark 1.1.2. *Let G be a semigroup . Then G is a group if and only if for all $a, b \in G$ the equations $ax = b$ and $ya = b$ have a unique solution in G.*

1.2 Rings

Definition 1.2.1. *[19] A ring $(R, +, .)$ is a non empty set R together with binary operations $+$ and $.$ defined on R such that the following axioms are satisfied :*

i: $(R, +)$ *is an abelian group*

ii: Multiplication is associative

iii: for all $a, b, c \in R$, the left distributive law $a(b + c) = (ab) + (ac)$, and the right distributive law $(a + b)c = (ac) + (bc)$, hold

Example 1.2.1. *The set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, and the set of real numbers $\mathbb{R}$ are all rings under usual addition and multiplication.*

Definition 1.2.2. *A ring in which multiplication is commutative is a commutative ring. A ring R with an element 1 such that $1.x = x.1 = x$ for all $x \in R$ is a ring with unity.*

[19] If $R_1, R_2, \cdots, R_n$ are rings, we can form the set $R_1 \times R_2 \times \cdots \times R_n$ of all ordered $n-$tuples $(r_1, r_2, \cdots, r_n)$ where $r_i \in R_i$. Define addition and multiplication of $n-$tuples component wise, then $R_1 \times R_2 \times \cdots \times R_n$ is a ring . We shall usually use additive terminology and notation, so that

$$R_1 + R_2 + \cdots + R_n$$

is the direct sum of the rings R_i.

Definition 1.2.3. *Let R be ring with unity. An element u in R is a unit of R if it has a multiplicative inverse in R. If every non zero element of R is a unit, then R is a division ring. A field is commutative division ring.*

Example 1.2.2. $\mathbb{Z}$ *is not a field since* 5, *for example, has no multiplicative inverse , so 5 is not a unit in* $\mathbb{Z}$.
clearly $\mathbb{Q}$ *and* $\mathbb{R}$ *are fields.*

Definition 1.2.4. *A subring of a ring is a non empty subset of the ring which is a ring under the induced operations from the whole ring.*

Definition 1.2.5. *A Subring N of a ring R satisfying $rN \subseteq N$ and $Nr \subseteq N$ for all $r \in R$ is an ideal of R . A subring N of R satisfying $rN \subseteq N$ for all $r \in R$ is a left ideal of R ,and the one satisfying $Nr \subseteq N$ for all $r \in R$ is a right ideal of R.*

Remark 1.2.1. *An ideal is to a ring as a normal subgroup is to a group.*

Example 1.2.3. *Let R be any commutative ring with* 1 *and $a \in R$.*
Let $(a) = \{xa | x \in R\}$, then (a) is an ideal of R.
Note that if R is not commutative, then (a) need not be an ideal, but it is certianly a left ideal of R.

Remark 1.2.2. *Let R be aring and N be an ideal of R. For any $r \in R$ let $r + N = \{r + n : n \in N\}$,define (+)and (.)on $r + N$ as $(r_1 + N) + (r_2 + N) = (r_1 + r_2) + N$ and $(r_1 + N).(r_2 + N) = (r_1.r_2) + N$.Then it follows that $(r + N, +, .)$ is a ring.*

Definition 1.2.6. *If N is an ideal in a ring R and $r \in R$, then the ring of cosets $r + N = \{r + n : n \in N\}$ under the induced operations is the quotient ring, or factor ring , and is denoted by R/N*

Definition 1.2.7. *A map ϕ of a ring R into a ring R' is a homomorphism if*

$$\phi(a + b) = \phi(a) + \phi(b)$$

and

$$\phi(a.b) = \phi(a).\phi(b)$$

for all elements a and b in R,if $R = R'$then a homomorphism ϕ is called endomorphism of R. .

Remark 1.2.3. *If $N$is an ideal of a ring R, then the canonical map $\phi : R \to R/N$ given by $\phi(a) = a + N$ for $a \in R$ is a homomorphism.*

Definition 1.2.8. *The kernal of a homomorphism ϕ of a ring R into a ring R' is the set of all elements of R mapped onto the additive identity $0'$ of R' by ϕ*

Definition 1.2.9. *[5]*

- *i: A maximal ideal of ring R is an ideal M different from R such that there is no proper ideal N of $\mathbb{R}$ properly containing M.*
- *ii: A proper ideal P of R is called prime if for any two ideals A and B, $AB \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$.*
- *iii: A ring R is called simple if it has no proper ideals other than $\{0\}$.*

The proof of the following theories can be found in any book from under graduate.

Theorem 1.2.1. *Let $\phi : R \to R'$ be a homomorphism, then* $\ker \phi = \{0\}$ *if and only if ϕ is one -one*

Theorem 1.2.2. *[5] Let R be a commutative ring with unity then M is a maximal ideal iff R/M is a field.*

Theorem 1.2.3. *[5] Let R be a commutative ring with unity then every maximal ideal of R is a prime ideal.*

Notation:

Let $X \subseteq R$. The intersection of all ideals of R containing X is called the ideal generated by X. It is denoted by $< X >$.

If $X = \{a\}$, then we write $< a >$ for $< X >$. For an element $a \in R$, the ideal $< a >$ is called the principal ideal generated by a.

Definition 1.2.10. *i: an additive map or additive function is a function that preserves the addition operation.*

ii: An additive mapping $d : R \to R$, where R is a ring, is called a derivation if

$$d(xy) = d(x)y + xd(y) \quad \textit{hold for all} \quad x, y \in R$$

iii: An additive mapping $F : R \to R$ is called generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$

1.3 Boolean rings

Definition 1.3.1. *Let R be a ring, an element $a \in R$ is said to be idempotent if $a = a^2$.*

Definition 1.3.2. *A ring R is said to be a Boolean ring if all its elements are idempotent.*

Lemma 1.3.1. *Let R be a Boolean ring, then $xy = -yx$ for all $x, y \in R$*

Proof. Consider $(x+y)^2$,then we have $(x+y)^2 = x^2 + xy + yx + y^2$.
Since R is Boolean, we also have

$$(x+y)^2 = x + y$$

so we have $x + y = (x+y)^2 = x + xy + yx + y$. Adding $-x - y$ to each side, we get $0 = xy + yx$ so, $xy = -yx$ $\square$

Lemma 1.3.2. *Let R be a Boolaen ring, then $y = -y$, for all $y \in R$.*

Proof. Since R is a Boolean ring, we have

$$-y = (-y)^2 = y^2 = y \text{ as desired}$$

$\square$

Definition 1.3.3. *The characteristic of a ring with unity R (written Char(R)) is the smallest positive integer n, such that $n.1_R = 0_R$. If there is no such n, then Char(R)=0.*

Theorem 1.3.3. *If R is a Boolean ring, then Char$(R) = 2$, and R is commutative.*

Proof. by Lemma (1.3.1), we have $xy = -yx$. so $xy + yx = 0$. If we set $x = y = 1$,,so $1+1 = 0$, hence Char$(R) = 2$. by Lemma 2, we have $(xy) = -(xy)$, $\forall x, y \in R$ and we know $xy + yx = 0$, then $xy = -yx = yx$, So $xy = yx$. Therefore R is commutative. $\square$

Definition 1.3.4. *An integral domain D is a commutative ring with unity containing no divisors of zero.*

Theorem 1.3.4. *Any Boolean ring which is also integral domain is isomorphic to $\mathbb{Z}_2$.*

Proof. Suppose a Boolean ring R is also an integral domain Then R contains unity 1. Also, $a^2 = a$ for all $a \in R$ implies $a^2 - a = 0$. Then $a(a-1) = 0$, giving as zero divisors a and $a - 1$ in R unless $a = 0$ or $a = 1$. So if R is to have no zero divisors it must only contain two elements : 0 and 1. Thus R must be isopmorphic to $\mathbb{Z}_2$ $\square$

1.4 Modules

Definition 1.4.1. *[5] Let A be a ring with 1. A left module M (or a left A-module denoted by ${}_AM$) over A consists of an abelian group M and a low of composition $A \times M \to M$ (denoted $(a, x) \longmapsto ax$) such that*

i. $a(bx) = (ab)x$ *for* $a, b \in A$ *and* $x \in M$

ii. $1x = x$ *for* $x \in M$

iii. $(a + b)x = ax + bx$ *for* $a, b \in A$ *and* $x \in M$

iv. $a(x + y) = ax + ay$ *for* $a \in A$ *and* $x, y \in M$

The last statement asserts that $P(a) : M \to M$ defied by $P(a)(x) = ax$ is an endomorphism of the underlying abelian group of the module, while the first three statements assert that $P : A \to \mathrm{End}(M)$ is a ring homomorphism.

The right module can also be defined in a similar way

Definition 1.4.2. *M is said to be a unitary $A-$module if A is a division ring.*

Remark 1.4.1. *If the ring A is commutative, we need not realy distinguish between right and left modules since $ab = ba$, however, in the non commutative case the distinction is important.*

Various elementary facts (like $0x = 0$ for all $x \in M$) follow easily from the definition, and we shall assume these facts without further discussion.

Example 1.4.1. *Let $A = \mathbb{Z}$. Then if M is any abelian group.*

We may define a $\mathbb{Z}-$module structure on M by

$(n, x) \to nx =$ the sum of n copies of x if n is positive, $-n$ copies of the negative of x if n is negative, and 0 if $n = 0$

Example 1.4.2. *If R is any ring and I is a left ideal in R, then I is a left module over R .*

Analogously of course, right ideals are R right modules.

Definition 1.4.3. *Let M be a left $R-$module and N is a subgroup of M . Then N is a submodule if for any $n \in N$ and $r \in R$, the product rn is in N (or $nr \in N$ for right modules).*

Example 1.4.3.

1. *If $A = \mathbb{Z}$ then a submodule is just a subgroup .*
2. *If A is viewed as a left module over itself, then a submodule is just a left ideal.*

   Similarly, if it is viewed as a right module over itself then a submodule is just a right ideal .

Let A and B be a rings and M a left $A-$module . If A_* is an additive subgroup of A, B_* is an additive subgroup of B and N is an additive subgroup of M, then we define aN to be the additive subgroup of M generated by all products ax with $a \in A$ and $x \in N$. The basic formulas for products of the additive subgroups in a ring are true in this more general situation:

$$(A_*B_*)N = A_*(B_*N), \qquad (A_* + B_*)N = A_*N + B_*N.$$

$$\text{and} \quad A_*(N + L) = A_*N + A_*L.$$

$$\text{where} \quad A_*B_* = \{ab : a \in A_* \text{ and } \quad b \in B_*\}$$

Definition 1.4.4. *Let M and M' be left $A-$modules. A function $f : M \to M'$ is called module homomorphism if it is consistent with the actions:*

$$f(x + y) = f(x) + f(y) \ \ \textit{for} \ \ x, y \in M$$

$$f(ax) = af(x) \quad \textit{for} \ \ a \in A, \quad x \in M.$$

This may be simplified to

$$f(ax + y) = af(x) + f(y) \quad \textit{for} \;\; a \in A, \quad x, y \in M.$$

Remark 1.4.2. *Let* $f : M \to M'$ *be a homomorphism of left modules. Then* $\ker f$ *is a submodule of* M *and* Imf *is a submodule of* M'

Chapter 2

Near ring

A near ring is a non empty set with two binary operations addition and multiplication satisfying all the ring axioms except possibly, one of the distributive laws and commutativity of addition.

2.1 Definition and basic properties

In this section we present the definition of near ring and illustrate it with several examples .

Definition 2.1.1. *[7] A near ring is a non empty set N together with two binary operations "$+$" and "$.$" such that:*

i. $(N,+)$ *is a group (not necessarily abelian)*

ii. $(N,.)$ *is a semigroup.*

iii. *for all* $n_1, n_2, n_3 \in N,\ (n_1+n_2)n_3 = n_1n_3+n_2n_3$ *(right distributive law)*

This near ring will be termed as right near ring. If $n_1(n_2+n_3) = n_1n_2+n_1n_3$ instead of condition (iii) , then we call N a left near ring.

In this thesis by a near ring we will mean only a right near ring unless otherwise is specified.

Example 2.1.1. *[5] Let $(G,+)$ be a group with its identity element 0, then the set $M(G) = \{f : G \to G | f$ is a function$\}$ is near ring under the usual addition and composition of mappings; the additive identity of $M(G)$, that is , the zero mapping, is also denoted by 0*

In this near ring the left distributive law fails to hold.

To verify this, take $a, b, c \in G, \quad a \neq 0$

$$\text{Define } f_a : G \to G \text{ by } f_a(g) = a, \ \forall\ g \in G$$

$$f_b : G \to G \text{ by } f_b(g) = b, \ \forall\ g \in G$$

$$f_c : G \to G \text{ by } f_c(g) = c, \ \forall g \in G$$

Let $g \in G$. Now $\Big[f_a \circ (f_b + f_c)\Big](g) = f_a\Big[(f_b + f_c)(g)\Big] = f_a\Big[f_b(g) + f_c(g)\Big] = f_a(b+c) = a$

also, $\Big[(f_a \circ f_b) + (f_a \circ f_c)\Big](g) = (f_a \circ f_b)(g) + (f_a \circ f_c)(g) = f_a(f(g)) + f_a(f_c(g)) = f_a(b)+f_a(c) = a+a \neq a$ (since $a \neq 0$). Therefore $f_a \circ (f_b+f_c) \neq (f_a \circ f_b)+(f_a \circ f_c)$. This shows that N fails to satisfy the left distributive law. This provided an example of a right near ring that is not a left near ring and hence, it is not a ring.

Example 2.1.2. *[19] Let $\mathbb{Z}_{12} = \{0, 1, 2, \cdots, 11\}$. $(\mathbb{Z}_{12},+)$ is a group under "$+$"*

modulo 12.

Define "." on $\mathbb{Z}_{12}$ by $a.b = a$ for all $a \in \mathbb{Z}_{12}$. clearly $(\mathbb{Z}_{12}, +, .)$ is a near ring.

Theorem 2.1.1. *[5] Let N be a near ring, for $n, n_1 \in N$*

i. $0n = 0$ *and* $(-n)n_1 = -nn_1$

ii. $-(n + n_1) = -n_1 - n$

Proof. i: Let $n, n_1 \in N$, Now $0n = (0 + 0)n = 0n + 0n$, this implies $0n = 0n + 0n$, so $0n - 0n = 0n + 0n - 0n$ then $0 = 0n$, $\forall n \in N$. Next let $n_1 \in N$, then $(-n)n_1 + nn_1 = (-n + n)n_1 = 0n_1 = 0$. This means $(-n)n_1 = -nn_1$.

ii: Take $n, n_1 \in N$. Now $(n + n_1) + (-n_1 + (-n)) = (n + (n_1 - n_1) - n) = (n + (-n)) = 0$ therefore, $-(n + n_1) = -n_1 - n$

□

Definition 2.1.2. *[5] Let N be a near ring*

i. $N_0 = \{n \in N : n0 = 0\}$ *is called the zero symmetric part of N.*

ii. $N_c = \{n \in N : n0 = n\}$ *is called a constant part of N*

iii. A near ring N is called a zero symmetric near ring if $N = N_0$.

iv. A near ring N is called a constant near ring if $N = N_c$

Definition 2.1.3. *[5]*

i: An additive subgroup M of a near ring N with

$MM \subset M$ is called a subnear ring of N, it is denoted by $M \leq N$.

ii: A subnear ring M of N is called left invariant (right invariant, respectively) if $NM \subseteq M$ ($MN \subseteq M$, respectively).

If M is both left and right invariant then we say that it is invariant.

Example 2.1.3. *[5] N_0 and N_c are subnear rings of the near ring N.*

Solution:

First we show that N_0 is a subnear ring of N. we show that N_0 is a subgroup of N. Let $x, y \in N_0$, then $x0 = 0$ and $y0 = 0$. Now $(x - y)(0) = x0 - y0 = 0$. Therefore $x-y \in N_0$, then $(N_0, +)$ is a subgroup of $(N, +)$.Next, take $n_1, n_2 \in N_0$, now $(n_1n_2)0 = n_1(n_20) = n_10 = 0$. Therefore $n_1n_2 \in N_0$ and so $N_0N_0 \subset N_0$. Hence, N_0 is a subnear ring of N. We now show N_c is a subnear ring of N. Let $x, y \in N_c$, this implies $(x - y)(0) = x0 - y0 = x - y$ this means $x - y \in N_c$ So, $(N_c, +)$ is a subgroup of $(N, +)$.

Let $n_{c_1}, n_{c_2} \in N_c$,this implies$(n_{c_1}.n_{c_2})(0) = n_{c_1}(n_{c_2}0) = n_{c_1}.n_{c_2}$and so $n_{c_1}.n_{c_2} \in N_c$ Hence, $N_cN_c \subseteq N_c$. Therefore, N_c is a subnear ring of N.

Example 2.1.4. $(2\mathbb{Z}, +, .)$ *is a subnear ring of the near ring* $(\mathbb{Z}, +, .)$.

Definition 2.1.4. *i) $M_0(G)$ is the zero symmetric subnear ring of $M(G)$ consisting of all zero preserving maps from G to itself .*

ii) $M_c(G)$ is the constant subnear ring of $M(G)$ consisting of all constant maps from G to itself.

Example 2.1.5. *[5] Let $(G, +)$ be a group then:*

i. $\Big(M(G)\Big)_0 = M_0(G)$

ii. $\Big(M(G)\Big)_c = M_c(G)$

Solution:

i. Let $f \in \Big(M(G)\Big)_0$. This implies $f : G \to G$ such that $f0 = 0$.

We show that $f \in M_0(G)$. Now $f(0) = f(0(0)) = (f0)(0) = 0(0) = 0$. So, $f \in M_0(G)$.

Conversely, suppose that $f \in M_0(G) \Rightarrow f(0) = 0$. Now we show that $f \in \Big(M(G)\Big)_0$.

Let $g \in G$. Now $(f0)(g) = f(0(g)) = f(0) = 0 = 0(g)$. This shows that $(f0)(g) = 0(g)\ \forall g \in G$, which yields $f0 = 0$. Therefore, $f \in \Big(M(G)\Big)_0$. Hence $\Big(M(G)\Big)_0 = M_0(G)$.

ii. Let $f \in \Big(M(G)\Big)_c$. Then $f0 = f$. Now we show that $f \in M_c(G)$.

Let $g \in G$. Now $f(g) = f0(g) = f(0g) = f(0)$.and for $g \neq g_1 \in G$,we have$f(g_1) = f0(g_1) = f(0g_1) = f(0)$. This means that if is a constant mapping and so $f \in M_c(G)$. Therefore, the inclusion $\Big(M(G)\Big)_c \subseteq M_c(G)$ is clear Let $f \in M_c(G)$. Then f is a constant mapping for any $g_1 \in G$. Now $(f0)(g_1) = f(0(g_1)) = f(0) = f(g_1)$ This implies that $(f0)(g_1) = f(g_1)\ \forall g_1 \in G$. This means $f \in \Big(M(G)\Big)_c$. Then $M_c(G) \subseteq \Big(M(G)\Big)_c$. Hence $\Big(M(G)\Big)_c = M_c(G)$.

Definition 2.1.5. *[6] Let N be a near ring, if $(N, +)$ is abelian, we call N an abelian near ring.*

Definition 2.1.6. *[19] Let N be an abelian near ring, if $N^\star = N/\{0\}$ is a* *multiplicative group then N is called a near field.*

Example 2.1.6. *[19] Let $\mathbb{Z}_2 = \{0, 1\}$. $(\mathbb{Z}_2, +)$ is a group under "$+$" modulo 2, and "." is defined as $1.1 = 1 = 1.0$ and $0.0 = 0.1 = 0$, then $(\mathbb{Z}_2, +, .)$is a near field.*

Definition 2.1.7. *[19] Let N be a near ring. $|N|$ or $O(N)$ denotes the order of the near ring N. i.e: the number of elements in N if N has a finite number of elements and ∞ otherwise. If $|N| < \infty$ we say the near ring is finite and if $|N| = \infty$ we call N an infinite near ring.*

Definition 2.1.8. *[5] Let N be a near ring*

i: The element $e \in N$is said to be an identity if $n.e = e.n = n$, for all $n \in N$.

ii: An element $n \in N$is said to be invertable if there exists an element $m \in N$ such that $n.m = m.n = e$, for all $n \in N$

Definition 2.1.9. *[5] Let N be a near ring. An element $n \in N$ is said to be left cancelable (right cancelable, respectivly) if $a, b \in N$ and $na = nb$, then $a = b$ $(an = bn \Rightarrow a = b$, respectivly)*

Definition 2.1.10. *[5] Let N be a near ring*

i: a nonzero element n is said to be a right zero divisor (left zero divisor, respectively) if there exists a nonzero element $a \in N$ such that $an = 0(na = 0$, respectively).

ii: A near ring N is said to be integral near ring if it has no zero divisors.

Theorem 2.1.2. *[5] Let N be a near ring. If $n \in N_0$ is left cancelable, then n is not a left zero divisor.*

Proof. Suppose $n \in N_0$ is left cancelable, let us also suppose that n is a left zero divisor, then $\exists 0 \neq n_1 \in N$ such that $nn_1 = 0$. Since $n \in N_0$, $nn_1 = 0 = n0$. Again, since n is left cancelable, we have $n_1 = 0$, which is a contradiction. $\square$

Theorem 2.1.3. *[5] Let N be a near ring. An element $n \in N$ is right cancelable if and only if n is not a right zero divisor.*

Proof. Suppose that $n \in N$ is right cancelable. Let us also suppose that n is a right zero divisor then $\exists n_1 \neq 0$ such that $n_1 n = 0$. This implies $n_1 n = 0 = 0n$. Now, as n is right cancelable, we get $n_1 = 0$, which is a contradiction.
Therefore, n is not a right zero divisor.
Conversely, suppose that n is not a right zero divisor, let $n, n_1, n_2 \in N$ such that $n_1 n = n_2 n$. This implies $n_1 n - n_2 n = 0$. This means that $(n_1 - n_2)n = 0$ since n is not a right zero divisor, we get that $n_1 - n_2 = 0$. This implies that $n_1 = n_2$.
Therefore, n is right cancelable. $\square$

Definition 2.1.11. *[19] Let $\{N_i\}$ be the collection of near rings ($i \in I$, I an indexing set). $N_1 \times N_2 \times \cdots \times N_r = \prod N_i$ with the component wise defined operations $(+)$ and $(.)$, N is called the direct product of the near rings N_i.*

Example 2.1.7. *[19] Consider the near field $\mathbb{Z}_2$. Let $\mathbb{Z}$ be the near ring $N = \mathbb{Z} \times \mathbb{Z}_2$. Here $N = \{(z_1, z_2) : z_1 \in \mathbb{Z}$ and $z_2 \in \mathbb{Z}_2\}$ is a near ring under component wise addition $(+)$ and multiplication $(.)$.*

Definition 2.1.12. *[1] Let N be a near ring. The center, $Z(N)$ is the subset of elements in N that commute with every element of N.*
In symbols

$$Z(N) = \left\{ a \in N \middle| ax = xa \quad \textit{for all } x \in N \right\}$$

2.2 Distributively generated near ring

In this section we introduce the concept of distributively generated near ring and some of its properties .

Lemma 2.2.1. *[9] Let N be a finite near ring with identity such that$-1x = x$ implies $x = 0$, then $(N, +)$ is abelian .*

Proof. Let $\alpha \in aut(N)$ such that$\alpha(x) = -x$, $\alpha^2 = I$ and $\alpha(x) = 0 \iff x = 0$. Want show that N is abelian, now $\alpha(x + y) = -(x + y) = (x + y)^{-1}$. Also $\alpha(x+y) = \alpha(x)+\alpha(y) = -x+-y = x^{-1}+y^{-1} = (y+x)^{-1}$, so $(x+y)^{-1} = (y+x)^{-1}$. Then $x + y = y + x \forall x, y \in N$.Therefore N is abelian. □

Example 2.2.1. *[5] $M(G)$ is abelian if and only if G is abelian*

Solution:

Suppose $M(G)$ is abelian, let $a, b \in G$. Define the constant mappings

$$f_a : G \to G \text{ by } f_a(x) = a, \ \forall x \in G$$

$$f_b : G \to G \text{ by } f_b(x) = b, \ \forall x \in G$$

clearly $f_a, f_b \in M(G)$. Since $M(G)$ is a belian, we have that $f_a + f_b = f_b + f_a$. Now $(f_a + f_b)(x) = (f_b + f_a)(x)$ implies that $f_a(x) + f_b(x) = f_b(x) + f_a(x)$. Therefore $a + b = b + a$. This shows that G is abelian. The converse is clear.

Definition 2.2.1. *[19] Let N be a near ring if $(N, .)$ is commutative . We call N self commutative near ring.*

Definition 2.2.2. *[8] Let N be a near ring:*

i: An element $d \in N$ is said to be distributive if for all $m, n \in N$, $d(m+n) = dm + dn$.

We denote $N_d = \{d \in N | d \text{ is distributive}\}$.

ii: If $N = N_d$ then we say that N is a distributive near ring.

iii: A near ring with the property that the set N_d generates $(N, +)$ is called a distributively generated near ring(denoted by d.g near ring)

Example 2.2.2. *Recall the near ring $M(G)$ (the set of all mappings of an additive group G into itself). If S is a multiplicative semigroup of endomomorphisms of G and N' is the subnear ring generated by S, then N' is a d.g near ring.*

Lemma 2.2.2. *[9] Let N be a distributively generated near ring such that $(N, +)$ is abelain then N is a ring.*

Proof. Let $a, b, c \in N$, since N is a distributively generated near ringwe can write a as a finite sum of distributive element. So $a = a_1 + a_2 + ... + a_n$ such that a_i is distributive element .

Now $a(b+c) = (a_1 + a_2 + ... + a_n)(b+c) = a_1(b+c) + a_2(b+c) + ... + a_n(b+c) = a_1b + a_1c + a_2b + a_2c + ... + a_nb + a_nc = a_1b + a_2b + ... + a_nb + a_1c + a_2c + ... + a_n = (a_1 + a_2 + ... + a_n)b + (a_1 + a_2 + ... + a_n)c = ab + ac$. So the left distributive law holds .Therefore N is a ring ☐

Definition 2.2.3. *[18] A near ring N that contains more than one element is said to be a division near ring if the set $N^{\star}$ of non zero elements is a multiplicative group.*

Remark 2.2.1. *[18] Every division ring is an example of a division near ring.*

Lemma 2.2.3. *[18] If N is a near ring with identity 1, then $(-1)(-1) = 1$. Furthermore if $(-1)n = n(-1)$, $\forall n \in N$, then $N^{\star}$ is commutative.*

Theorem 2.2.4. *[18] A necessary and sufficient condition for a d.g near ring D with more than one element to be a division near ring is that, for all non zero $a \in D$, $aD = D$*

Proof. Necessity. There is an element $e \in D$ s.t $ae = ea = a$, for $a \neq 0$ in D. Clearly $aD \subseteq D$. Suppose $a \neq 0$ is in D. then $\exists$ an element $b \in D$ such that $ab = e \in aD$. Thus $x = a(bx)$, $\forall x \in D$ and so $x \in aD$. Hence $aD = D$.

Sufficiency. If a and b are non zero elements of D, then $ab \neq 0$. For if not, $\exists a_e$ and b_e such that $aa_e = a$ and $bb_e = a_e$. Thus $0 = abb_e = aa_e = a$, Which is a contradiction.

Now let r be a non zero left distributive element of D, then $\exists e \in D, re = r$. But $r(r - er) = rr - rer = 0$, then we have $r = er$, so e is a two sided identity for r.

Since we know from the first part of the proof that the set of non zero elements is closed under multiplication and multiplication is associative, it only remains to prove that e is a right identity for the non zero elements of D and every non zero element of D has a right inverse.

Let $d \neq 0$ be an element in D. Then $(de - d)r = der - dr = dr - dr = 0$. Since $r \neq 0$ we have that $de = d$.

Also $dD = D$ implies that there is a $d' \in D$such that $dd' = e$.

Thus we have a shown that the distributively generated near ring D is a division near ring. $\square$

Theorem 2.2.5. *[9] Let N be a distributively generated near ring with identity 1 such that $(N, +)$ is a belian and $(xy)^2 = x^2y^2,\ \forall x, y \in N$, then N is commutative.*

Proof. By Lemma (2.2.2) N is a ring, suppose $x, y \in N$ then

$$\Big[x(y+1)\Big]^2 = x^2(y+1)^2 = x^2(y+1)(y+1) = x^2(y^2+y+y+1) = x^2(y^2+2y+1) = x^2y^2+2x^2y+x^2 \tag{2.1}$$

But also

$$\Big[x(y+1)\Big]^2 = (xy+x)(xy+x) = (xy)^2 + xyx + xxy + x^2 = x^2y^2 + xyx + x^2y + x^2 \tag{2.2}$$

By $(2.1), (2.2)$ we have $2x^2y = xyx + x^2y$ this becomes $yx = xy$ and the theorem is proved. $\square$

Theorem 2.2.6. *[9] Let N be a distributively generated near ring with 1, such that $(N, +)$ is a belian and $(xy)^2 = (xy^2)x,\ \forall x, y \in N$, then N is commutative.*

Proof. By Lemma (2.2.2) N is a ring. Suppose $x, y \in N$.

Now, repeating the argument for $y+1$ instead of y, we obtain

$$\Big[x(y+1)\Big]^2 = \Big[x(y+1)^2\Big]x,$$

hence $x(xy) = (xy)x$.

Now repeat in this argument $x+1$ instead of x in (2.1), hence $x(xy) + xy = (xy)x + yx$. By $(2.1), (2.2)$ we have $xy = yx$ and theorem is proved. $\square$

Theorem 2.2.7. *[9] Let N be a finite distributively generated near ring with identity 1 s.t. $(-1)x = x$ implies $x = 0$ and $(xy)^2 = (xy^2)x,\ \forall x, y \in N$ then N is commutative near ring.*

Proof. By lemma(2.2.1) $(N, +)$ is abelian and by lemma (2.2.2) N is a ring. Suppose $x, y \in N$. Repeating this argument for $y+1$ instead of y, we obtain $\Big(x(y+1)\Big)^2 = \Big[x(y+1)^2\Big]x$ implies $x(xy) = (xy)x$. Now, repeating the argument $x+1$ instead of x, we obtain $\Big[(x+1)y\Big]^2 = \Big[(x+1)y^2\Big](x+1)$ implies $x(xy)+xy = x(yx) + yx$ then by $(2.1), (2.2)$ we have $xy = yx$ and theorem is proved. $\square$

Corollary 2.2.7.1. *[9] Let N be a finite near ring with identity 1 such that $(-1)x = x$ implies $x = 0$ and $(xy)^2 = (yx^2)y,\ \forall x, y \in N$. Then N is commutative.*

2.3 Boolean near ring

In this section we introduce the concept of Boolean near rings. Using any Boolean ring with identity, we construct a class of Boolean near rings, called special

Boolean near rings.

Definition 2.3.1. *[5] Let N be a near ring. An element $n \in N$ is said to be*

i. idempotent if $n^2 = n$

ii. nilpotent element if $\exists$ a (least) positive integer k such that $n^k = 0$

Example 2.3.1. *[5]*

1. *Let $\mathbb{Z}_6 = \{0, 1, ...5\}$. $(\mathbb{Z}_6, +)$ is a group under $'+'$ module 6. Define $'.'$ on $\mathbb{Z}_6$ by $a.b = a$ for all $a \in \mathbb{Z}_6$, clearly every elements in $\mathbb{Z}_6$ is idempotent elements*

2. *In the near ring $(\mathbb{Z}_8, +, .)$ the element 2 is nilpotent because $2^3 = 8 = 0$.*

Notation:

If S and T are subsets of N then we write $ST = \{st | s \in S \text{ and } t \in T\}$.

For any natural number n, we write $S^n = S \times S \times \cdots \times S$ (n times).

Definition 2.3.2. *[5] Let N be a near ring*

i: A subset S of N is called nilpotent if $\exists$ a positive integer k such that $S^k = \{0\}$.

ii: A subset S of N is called nill if every element of S is a nilpotent element.

Theorem 2.3.1. *[5] Let S be a subset of N. If S is nilpotent, then S is nill.*

Proof. Suppose S is nilpotent. Then there exists a positive integer k such that $S^k = \{0\}$. This implies that $s^k = 0$ for all $s \in S$. Therefore, s is a nilpotent for all $s \in S$, and hence S is nill. □

Definition 2.3.3. *[10] A near ring N is said to be idempotent near ring if $x^2 = x, \ \forall x \in N$.*

If N is an idempotent ring, then $\forall a, b \in N \ \ a + a = 0$ and $a.b = b.a$

Example 2.3.2. *[10] Given a non trivial group $(N, +)$, define multiplication by $a.b = a, \ \forall a, b \in N$, then $(N, +, .)$ is an idempotent near ring which is not commutative.*

Definition 2.3.4. *[10] A near ring $(B, +, .)$ is Boolean near ring if there exists a Boolean ring $(A, +, \wedge, 1)$ with identity such that "." is defined in terms of $+, \wedge$ and 1, and for any $b \in B \ \ b.b = b$*

Remark 2.3.1. *[10] There exist Boolean near rings which are not Boolean rings.*

The following Theorem gives example of Boolean near ring that not Boolean ring. Firs recall that in a Boollean ring $(B, +, \wedge, 1)$ one can define complementation $\prime$, by $a' = a + 1$ and sup, $\vee$, by $a \vee b = (a' \wedge b')'$.

Theorem 2.3.2. *[10] Let $(B, +, \vee, 1)$ be a Boolean near ring with identity. Fix $x \in B$ and define multiplication on B by $a.b = b \wedge (a \vee x)$, then $(B, +, .)$ is a Boolean near ring which is a Boolean ring if and only if $x = 0$.*

Proof. For $a, b, c \in B$ we have $a.(b.c) = \Big[(a \vee x) \wedge (b \vee x)\Big] \wedge c$ and$(a.b).c = (\Big[(a \vee x) \wedge b\Big] \vee x) \wedge c = \Big[(a \vee x) \wedge (b \vee x)\Big] \wedge c$ so that $a.(b.c) = (a.b).c$.
Also $(a + b).c = c \wedge \Big[(a + b) \vee x\Big] = c \wedge (a \vee x) + c \wedge (b \vee x) = (a.c) + (b.c)$ if $x = 0, (B, +, .) = (B, +, \wedge, 1)$. Now $x.(x+x) = x0 = 0 \wedge (x \vee x)$ and $(x.x) + (x.x) = 0$,

so that $(B, +, .)$ is not a ring if $x \neq 0$.

Also $b.b = (b \vee x) \wedge b = b,\ \forall b \in B$, so that $(B, +, .)$ is a Boolean near ring.

Note that any e such that $e \wedge x' = x'$ is a left identity for $(B, +, .)$ but $(B, +, .)$ has no right identity unless $x = 0$.

□

Boolean near ring of the types defined in the above theorem will be called special.

Example 2.3.3. *Let G be an additive group (not necessarily abelain). Define for each $x \in G, xy = x$, for each $y \in G$. Then $(G, +, .)$ is a boolean near ring.*

2.4 Matrix near ring

Matrix near rings are introduced in Meldrum and Van der Walt (1986). The matrix near ring of a given near ring N is denoted by $M_n(N)$.

Let N be a near ring and $n \in \mathbb{N}$, the set of all natural numbers. The direct sum of n copies of the group $(N, +)$ is denoted by N^n.

Note that $(N, +)$ is not necessarily abelain. The elements of N^n are throught of as culumn vectors. but for reason, we write them in transposed form with pointed brackets. For example, $< x_1, x_2, \cdots, x_n > \in N^n$.

The $n \times n$ matrices will be defined as function from N^n to N^n.

First we recall the following familiar embedding which will be used to define these matrices.

Let N be a near ring with identity. N can be embedded into the near ring $M(N)$ of all mappings of $(N,+)$ into it self, by means of the rule $x \to f^x$, where $f^x(y) = xy,\ \forall y \in N$.

Remark 2.4.1. *[15] Let N be a near ring. The symbol ∞ can be joined to the group $(N,+)$ and we can obtain the group with infinity $(N_\infty,+)$, then $M(N_\infty)$, the set of all functions from N_∞ to N is a near ring under point wise addition and composition. We can now embed N into $M(N_\infty)$, be means of the rule $x \to f^x$ where*

$$f^x(y) = \begin{cases} xy & \textit{if } y \in (\mathbb{R}) \\ \\ x & \textit{if } y = \infty \end{cases}$$

we are now able to introduce the functions

$$F^x_{ij} : N^n \to N^n$$

where $f^x_{ij} = I_j f^x \pi_j$

for $1 \le i,j \le n, x \in \mathbb{R}$ and the symbols $I_j \& \pi_j$ denote the j^{th} coordinate injection and projection, respectively, such that the injection mapping $I_j : N \Rightarrow N^n$ is defined as $I_j(a) = (0, ..., a_{ith}, ..., 0)$ and the projection mapping $\pi_j : N^n \Rightarrow N$ is defined as $\pi_j(a_1, ..., a_n) = a_j$.

Definition 2.4.1. *[15] The near ring of $n \times n$ matrices over N, denoted by $M_n(N)$ is the subnear ring of $M(N^n)$ generated by the set $\Big\{ f^x_{ij} : x \in N,\ 1 \le i,j \le n \Big\}$*

The elements of $M_n(N)$ will be referred to as $n \times n$ matrices over N.

Remark 2.4.2. *[15] We use this definition even when N does not have an identity. So N is not necessarily embedded in $M_n(N)$.*

Note that, for reasons, we use the symbol $\Big[x; i, j\Big]$ for f_{ij}^x

Result:[15]

For all $i, j, k, 1 \in \{1, 2, \cdots, n\}$ and $x, y, z, x_1, \cdots, x_n \in N$, we have

A. $\Big[x; , i, j\Big] + \Big[y; i, j\Big] = \Big[x + y; i, j\Big].$

B. $\Big[x; i, j\Big] + \Big[y; k, 1\Big] = \Big[y; k, 1\Big] + \Big[x; i, j\Big]$ if $i \neq k$

C. $\Big[x; i, j\Big].\Big[y; k, 1\Big] = \begin{cases} \Big[xy; i, 1\Big] & \text{if } j = k \\ \Big[x0; i, 1\Big] & \text{if } j \neq k \end{cases}$

D. $-\Big[x; i, j\Big] = \Big[-x; i, j\Big]$

Theorem 2.4.1. *[15]*

The matrix near ring $M_n(N)$ is a right near ring with identity.

Proof. Since $M_n(N)$ is the subnear ring of the right near ring $M(N^n)$, we have that $M_n(N)$ is a right near ring. Since $f_{ii}^1 \in M_n(N)$ it follows that $f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1 \in M_n(N)$ and for any $(a_1, \cdots, a_n) \in N^n$, we have

$$\Big(f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1\Big)(a_1, \cdots, a_n) = f_{11}^1(a_1, \cdots, a_n) + \cdots + f_{nn}^1(a_1, \cdots, a_n)$$
$$= (a_1, \cdots, a_n) = I(a_1, \cdots, a_n),$$

where $I : N^n \to N^n$ is the identity map. This implies $I = f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1$

and it acts as an identity in $M_n(N)$. Hence, $M_n(N)$ is a right near ring with identity □

Theorem 2.4.2. *[5] If N is a ring with identity, then $M_n(N) = M(N^n)$.*

Proof. Suppose N is a ring. Let N^1 be the ring of $n \times n$ matrices over N. Then we may consider every element of N^1 as a mapping from $N^n \to N^n$. Since $M(N^n)$ is a near ring and N^1 is a ring with respect to the same operations, we have that N^1 is a subnear ring of $M(N^n)$. Therefore, $N^1 \subseteq M(N^n)$.

Now each f_{ij}^r is a matrix of order $n \times n$, and we have $f_{ij}^r \in N^1$, $\forall r \in N \;\; 1 \leq i, j \leq n$. Therefore, $\{f_{ij}^r : r \in N,\; 1 \leq i, j \leq n\} \subseteq N^1$.

Therefore, $M_n(N) = \{f_{ij}^r : r \in N, 1 \leq i, j \leq n\} \subseteq N^1$ then $M_n(N) \subseteq N^1$,so$M_n(N) \subseteq M(N^n)$. Let $A = \Big[a_{ij}\Big]_{n\times n} \in N^1$, then

$$A = \Big[a_{ij}\Big]_{n\times n} = \sum f_{ij}^{a_{ij}} \in \{f_{ij}^r : r \in N\} \subseteq M_n(N)$$

Therefore $M(N^n) \subseteq M_n(N)$. Hence $M(N^n) = M_n(N)$ □

Lemma 2.4.3. *[15] Let N be a near ring with identity and $x, y \in N$, if $[x; i, j] = [y; i, j]$, then $x = y$ where $1 \leq i, j \leq n$.*

Proof. Let$[x; i, j] = [y; i, j]$. Then $[x; i, j]e_j = [y; i, j]e_j$, where $e_j =< 0, \cdots, 1, \cdots, 0 >$ with 1 is in the j^{th} place. So $< 0, \cdots, x, \cdots, 0 >=< 0, \cdots, y, \cdots, 0 >$ with x and y in the i^{th} place. Hence $x = y$. □

Theorem 2.4.4. *[15] Let N be a zero symmetric near ring with identity. If $n > 1$, then $M_n(N)$ can not be integral near ring.*

Proof. Presuming the theorem to be false, we choose any two non-zero elements say x and y, of N. Since $[x; i, j].[y; k, 1] = 0$ if $j \neq k$ then $x0 = 0$, therefore either $[x; i, j] = 0$ or $[y; k, 1] = 0$, as $M_n(N)$ is an integral near ring. In other words, $[x; i, j] = [0; i, j]$ or $[y; k, 1] = [0; k, 1]$. Hence $x = 0$ or $y = 0$, by lemma (12.4.3). This leads a contradiction since x and y are non-zero elements. □

2.5 $N-$ Groups

In this section we recall the definition of N-groups, N-homomorphism and difference operator in a near-ring and illustrate with examples.

Definition 2.5.1. *[6] Let $(N, +, .)$ be a near ring, and $(G, +)$ be a group with additive identity 0. G is said to be an N-group if there exists a mapping $N \times G \to G$ (the image of $(n, g) \in N \times G$ is denoted by ng), satisfying the following conditions:*

$$(n + m)(g) = ng + mg,$$

$$(nm)g = n(mg).$$

for all $g \in G$ and $n, m \in N$. We denote this N-group by ${}_NG$.

Definition 2.5.2. *[5] If $ng = 0$, for all $g \in G$ is true if $n = 0$,then We say that N operates faithfully on G (or that G is a faithful N-group).*

Remark 2.5.1. *[6] An $N-$ group can be described by a homomorphism from the near ring N into $M(G)$, which is an embedding iff the operation is faithful.*

Example 2.5.1. *[5]*

i: *Let N be a near ring, then the mapping $N \times N \to N$ (multiplication in N) with respect to the group $(N, +)$ becomes an N- group. This $N-$ group is denoted by ${}_N N$*

ii: *each (left) module M over a ring R is an $N-$ group where $N = R$*

Definition 2.5.3. *[6] Let N be a near ring:*

1. *A subgroup H of an $N-$ group G is called an $N-$ subgroup (written as $H \leq {}_N G$) if it is closed under the operation of N, i.e $n\delta \in H$ for all $n \in N$, $\delta \in H$.*

2. *Let G_1 and G_2 be two $N-$ groups, $h : G_1 \to G_2$ is called $N-$ homomorphism if for all $p, q \in G_1$ and for all $n \in N$:*

    i. $h(p + q) = h(p) + h(q)$

    ii. $h(np) = nh(p)$.

3. *If H is the kernal of an $N-$ homomorphism, then it is called an $N-$ normal subgroup and we write $H \trianglelefteq_N G$.*

Example 2.5.2. *[19] Consider the near ring $(\mathbb{R}, +, .)$,where $\mathbb{R}$ denotes the set of real numbers,then $\mathbb{Q}$ (the set of rationals) is a $N-$ subgroup of $\mathbb{R}$*

Remark 2.5.2. *If G is an abelain group, N is a ring and G is a module over this ring N, then the concepts of $N-$ subgroup and submodule are one and the same.*

Example 2.5.3. *[5] Let $(H, +)$ be a subgroup of a group $(G, +)$. Then $M_H(G) = \{f \in M(G) | f(H) \subseteq H\}$ is a subnear ring of $M(G)$.*

Solution:

First we show that $M_H(G)$ is a subgroup of $M(G)$. Let $f, g \in M_H(G)$ and $x \in H$. Clearly, $f(x) \in f(H) \subseteq H$ and $g(x) \in g(H) \subseteq H$. Now $(f-g)(x) = f(x)-g(x) \in H$. Therefore $f-g \in M_H(G)$. This shows that $M_H(G)$ is a subgroup of $(M(G), +)$. Now we show that $M_H(G)M_H(G) \subseteq M_H(G)$. Let $f_1, f_2 \in M_H(G)$ and $x \in H$. Now $(f_1f_2)(x) = f_1(f_2(x)) \in f_1(f_2(H)) \subseteq f_1(H) \subseteq H$. This is true for all $x \in H$. Therefore $(f_1f_2)(H) \subseteq H$. Hence, $f_1f_2 \in M_H(G)$, for all $f_1, f_2 \in M_H(G)$. Hence, $M_H(G)$ is a subnear ring of $M(G)$.

Theorem 2.5.1. *[5] Let G be an $N-$ group. Then*

i. $0x = 0, \ \forall x \in G$

ii. $(-n)x = -nx, \ \forall x \in G, \ n \in N$

iii. $n0 = 0, \ \ \forall n \in N_0$

iv. $nx = n0, \ \forall x \in G$ *and* $n \in N_c$.

Proof. i. Let $x \in G$.

Consider $0x = (0+0)(x)$. Now $0x = 0x+0x \Rightarrow 0+0x = 0x+0x$,so $0x = 0$.

ii. From (i), $0 = 0x = (-n+n)x$. This implies $(-n)x + nx = 0$, and hence $(-n)x = -nx$.

iii. Take $n \in N_0$. Then $n0 = n(0+0) = (n0)(0) = 0.0 = 0$ (since $n \in N_0$)

iv. Let $x \in G$ and $n \in N_c$. Then $n0 = n$. Now $nx = (n0)x = n(0x) = n0$

Therefore, $nx = n0$ for all $n \in N_c$ and $x \in G$.

$\square$

Definition 2.5.4. *[5] Let G be an $N-$ group. Then G is said to be a unitary $N-$ group if:*

i. N is a near ring with unity 1.

ii. $1.x = x$, for all $x \in G$

Example 2.5.4. *[5] Every unitary module M over ring R with unity is a unitary $N-$ group.*

Definition 2.5.5. *[6] Let N be a near ring and G an $N-$ group. For $f \in N$, and $x, a \in G$, we define*

$$\triangle fxa = -fx + f(x+a).$$

and we call it the difference of f at x in direction a.

Theorem 2.5.2. *[6] With the notation of the definition we have*

$$\triangle fx(a+b) = \triangle fxa + \triangle f(x+a)b \quad \textit{(quasi-linearity)} \tag{2.3}$$

$$\triangle f(x+a)b = -\triangle fxa + \triangle fx(a+b) \quad \textit{(translation-rule)} \tag{2.4}$$

Proof. We show the second equation:

$$-\triangle fxa + \triangle fx(a+b) = -f(x+a) + fx - fx + f(x+a+b)$$

$$= \triangle f(x+a)b.$$

□

Theorem 2.5.3. *[6] Continuing with the above notation and with $g \in N$, we have*

$$\triangle(f+g)xa = gx + \triangle fxa + gx + \triangle gxa,$$

$$\triangle(-f)xa = -(fx + \triangle fxa - fx)$$

Proof. We show the second equation:

$\triangle(-f)xa = -(-f)x + (-f)(x+a) = fx - f(x+a) = -(f(x+a) - fx) = -(fx - fx + f(x+a) - fx) = -(fx + \triangle fxa - fx)$ □

Remark 2.5.3. *[6] This (in general) $\triangle$ is not linear in the first argument unless G is abelian.*

Theorem 2.5.4. *[6] The operator $\triangle$ fulfills the following chain rule:*

$$\triangle(fg)xa = \triangle f(gx)\triangle gxa.$$

Proof.

$$\triangle(fg)xa = -fgx + fg(x+a) = -fgx + f(gx + \triangle gxa)$$

$$= \triangle f(gx)\triangle gxa.$$

□

The difference operator can be iterated in the following way. The definition is motivated by the formalism $\triangle^{n+1}f = \triangle(\triangle^n f)$ such that n is positive integer.

Definition 2.5.6. *[6] Let G be an $N-$group, $f \in N, x, a, b \in G$. Then we define the higher order difference operator as*

$$\triangle^{n+1} fxba = -\triangle^n fxa + \triangle^n f(x+b)a.$$

In particular,

$$\begin{aligned}\triangle^2 fxba &= \triangle(\triangle f)xba \\ &= -\triangle fxa + \triangle f(x+b)a. \\ &= -f(x+a) + fx - f(x+b) + f(x+b+a).\end{aligned}$$

Theorem 2.5.5. *[6] $\triangle^2$ fulfills the following chain rule:*

$$\triangle^2(fg)xba = \triangle^2 f(gx)(\triangle gxa)(\triangle gxb) + \triangle f(g(x+b) + \triangle gxa)\triangle^2 gxab$$

Proof. We compute

$$\begin{aligned}\triangle^2(fg)xba &= -\triangle(fg)xa + \triangle(fg)(x+b)a \\ &= -\triangle f(gx)(\triangle gxa) + \triangle f(g(x+b))(\triangle gxa) \\ &- \triangle f(g(x+b))(\triangle gxa) + \triangle f(g(x+b))(\triangle g(x+b)a). \\ &= \triangle^2 f(gx)(\triangle gxb)(\triangle gxa) + \triangle f(g(x+b) + \triangle gxa)(\triangle^2 gxba).\end{aligned}$$

The last stage has used the translation rule with

$$x = g(x+b) \text{ and}$$

$$a = \triangle gxa$$

$\square$

2.6 Derivation in near ring

The aim of this section is to investigate some results of near rings satisfying certain identities involving generalized derivations.

Definition 2.6.1. *[13]*

i: Let N be a near ring. An additive mapping $D : N \to N$ is said to be a derivation if $D(xy) = xD(y) + D(x)y, \ \forall x, y \in N$

ii: An additive mapping $F : N \to N$ is said to be right generalized derivation associated with D if

$$F(xy) = f(x)y + xD(y), \quad \forall x, y \in N$$

and is said to be a left generalized derivation associated with D if

$$F(xy) = xF(y) + D(x)y$$

iii: F is said to be a generalized derivation with associated derivation D on N if it is both a right and a left generalized derivation on N with associated derivation D

Lemma 2.6.1. *[3] A near ring N is zero symmetric if and only if N admits a derivation.*

Remark 2.6.1. *[3] As a consequence of Lemma* (2.6.1)*, the zero map is not a derivation on a non zero constant near ring.*

Theorem 2.6.2. *[13] A near ring N is zero symmetric if and only if N admits a generalized derivation.*

Proof. Suppose N is zero symmetric, then the zero map is a generalized derivation on N. Conversely, if N admits a generalized derivation, then it admits a derivation. By Lemma (2.6.1) N is zero symmetric.

□

For any near ring $N \neq \{0\}$ with zero multiplication, the identity map is a non-zero generalized derivation on N associated with any derivation on N (any additive mapping of $(N, +)$ is a derivation on N). The following example shows that any zero symmetric near ring in which multiplication is not the zero multiplication admits no generalized derivation.

Example 2.6.1. *Let N be a zero symmetric near ring in which multiplication is not the zero multiplication, Define $f : N \to N$ by $f(x) = cx$, where $c \in N$.*
such that $cN \neq 0$.
Observe that f is a non zero generalized derivation on N associated with the zero derivation.

Remark 2.6.2. *[2] Let N be a near ring with a derivation d. Then $xd(y) + d(x)y = (dx)y + xd(y)$ for all $x, y \in N$.*

Theorem 2.6.3. *[2]*

i: Let N be a near ring with a derivation d, if $z \in Z(N)$ then $d(z) \in Z(N)$

ii: *(The partial distributive Law) Let N be a near ring and d be a derivation on N. For all $x, y, z \in Z$ we have:*

$$z\big(xd(y) + d(x)y\big) = zxd(y) + zd(x)y.$$

Proof. i: Let N be a near ring with a derivation d,for any $z \in Z(N)$ and any $x \in N$ we have $xd(z) + zd(x) = xd(z) + d(x)z = d(xz) = d(zx) = d(z)x + zd(x)$. It follows that $xd(z) = d(z)x$, that is, $d(z) \in Z(N)$

ii: By calculating $d(zxy)$ in two different ways, we see that $d((zx)y) = zxd(y) + d(zx)y$ and $d(z(xy)) = zd(xy) + d(z)xy = z(xd(y) + d(x)y) + d(z)xy$

Hence we have $d(zx)y = (zd(x) + d(z)x)y = zd(x)y + d(z)xy$ and also $z\big(xd(y) + d(x)y\big) = zxd(y) + zd(x)y$.

□

Chapter 3

Ideals in near ring

3.1 Definitions and Examples

In this section we introduce some definitions that include Ideal, direct (internal) sum, direct summand, direct complement, simple near ring, and maximal ideal,

Definition 3.1.1. *[17] Let N be a near ring, a normal subgroup I of $(N, +)$ is called*

i. a right ideal if $IN \subseteq I$..

ii. a left ideal if $n(m + i) - nm \in I$, $\forall n, m \in N$ and $\forall i \in I$

iii. an ideal if it is both right and left ideal

Example 3.1.1. *[5] Let $N = \{0, 1, 2, 3, 4, 5\}$*

Define addition and multiplication as follows

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	1	4	0	2
4	4	2	5	0	3	1
5	5	3	4	1	2	0

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	1	1	1	2	1	2
3	0	0	0	3	0	3
4	0	0	0	4	0	4
5	1	1	1	5	1	5

Then $(N, +, .)$ is a near ring and $I = \{0, 3, 4\}$ is an ideal of N.

Remark 3.1.1. *Every ideal in near ring is a subnear ring, but the converse is not true in general.*

Example 3.1.2. *[19] Consider the two near rings $(\mathbb{Z}, +, .)$ and $(\mathbb{Q}, +, .)$, where $\mathbb{Z}$ and $\mathbb{Q}$ are the set of integers and the set of rational numbers, respectively. $\mathbb{Z}$ is a subnear ring of $\mathbb{Q}$, but $\mathbb{Z}$ is not an ideal of the near ring $\mathbb{Q}$.*

Remark 3.1.2. *[5] If I and J are ideals of a near ring N, then*

1. $I+J=\{a+b : a\in I,\ b\in J\}$ *is an ideal of* N

2. $I\cap J$ *is an ideal of* N

3. $I\cup J$ *is an ideal of N, provided* $I\subseteq J$*or* $J\subseteq I$

Definition 3.1.2. *[19] Let* N *be a near ring, Let* $\Big\{I_k\Big\}_{k\in K}$*(here* K *is an index set) be the collection of all ideals of* N*. Their sum* $\sum\limits_{k\in K} I_k$ *is called an (internal) direct sum if each element of* $\sum\limits_{k\in K} I_k$ *has a unique representation as a finite sum of elements of different* I_k'*s.*

In this case we write for the sum $\sum\limits_{k\in I} I_k$ *or* $(I_1+I_2+I_3+\cdots)$*.*

Theorem 3.1.1. *[5] Let* N *be a near ring. Let* $\Big\{I_k\Big\}_{k\in K}$ *be the collection of all ideals of the near ring* N*. The following are equivalent*

i. *the sum of the ideals* I_k *is direct.*

ii. *the sum of the normal subgroups* $(I_k,+)$ *is direct*

iii. *for all* $k\in K:\quad I_k\bigcap\Bigg(\sum\limits_{i\in K,\ i\neq k} I_i\Bigg)=\{0\}$

Proof. Suppose the sum of ideals I_k is direct. Then the sum of the normal subgroups $(I_k,+)$ is direct. Therefore, the condition $(i)\Rightarrow(ii)$ is proved.

Assume (ii) Let $x\in I_k\bigcap\Bigg(\sum\limits_{L\in K} I_L\Bigg)$; now x can be written as $x=x_{L_1}+x_{L_2}+\cdots+x_{L_n}$ where $L_1,L_2,\cdots,L_n$ are different from k and $x_{L_i}\in I_i$.

So, we can write x as $x=x_{l_1}+x_{L_2}+\cdots+x_{L_n}+0_k$. Also $x=0_{L_1}+0_{L_2}+\cdots+0_{L_n}+x$. Since the sum is direct, it follows that these two k representations must be the same. So, $x=0$which in turn implies that $I_k\bigcap\sum I_L=\{0\}$. Therefore, condition $(ii)\Rightarrow(iii)$ is proved. Assume the condition (iii) holds Now, we have to show that the sum of $I_k's$ is direct. Suppose some x in the sum has the following representations

$$x = x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$$

(without Loss of generality, we may assume that indices are the same for both representations by adding zeros if necessary, at appropriate places).

Now $(x_1 - y_1) = (y_2 + y_3 + \cdots + y_n) - (x_2 + x_3 + \cdots + x_n)$

$\Rightarrow x_1 - y_1 = y_2 - x_2 + y_3 - x_3 + \cdots + y_n - x_n \in I_1 \bigcap \sum_{L \in K,\ L \neq 1} I_L = \{0\}$.

This implies $x_1 = y_1$. Similarly, we can prove that $x_i = y_i,\ \forall i$. Therefore, x has a unique representation. Hence, the sum is direct. □

Definition 3.1.3. *[5] An ideal I of a near ring N is called a direct summand of N if there exists an ideal J of N such that $N = I \oplus J$. In this case, the ideal J is called a direct complement of I in N.*

Example 3.1.3. *[5] Consider the near ring $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Then $I = \{0, 3\}$ and $J = \{0, 2, 4\}$ are ideals of $\mathbb{Z}_6$*

It is clear that $I \cap J = \{0\}$ and $I + J = \{0, 2, 3, 4, 5, 1\} = \mathbb{Z}_6$. Hence, the ideal I is direct summand, and the ideal J is the complement of I in the near ring $\mathbb{Z}_6$. Also, the sum $I + J$ is a direct sum.

Remark 3.1.3. *[5] Let I be an ideal of a near ring N. If I is a direct summand, then each ideal of I is an ideal of N.*

Remark 3.1.4. *[5] Let N be a near ring and I an ideal of N,* *i: Define a relation* $\sim$ *on* N *as* $a \sim b$ *if and only if* $a - b \in I, \ \forall a, b \in N$.

Now we verify that this relation is an equivalance relation:

Reflexive: Since $a - a = 0 \in I$, *we have* $a \sim a$.

Symmetric: Suppose $a \sim b$, *then* $a - b \in I \Rightarrow b - a \in I \Rightarrow b \sim a$

Transitive: Suppose $a \sim b$ *and* $b \sim c$. *Then* $a - b \in I$ *and* $b - c \in I \Rightarrow$ $a - c = (a - b) + (b - c) \in I$, *which implies that* $a \sim c$.

Therefore, the relation $\sim$ *is an equivalence relation on* N.

ii: Write $a + I = \{a + x | x \in I\}$ *for* $a \in N$.

Now $a + I$ *is the equivalance class containing* a.

iii: let $N/I = \{a + I | a \in N\}$ *be the set of all equivalence classes.*

iv: Define $(+)$ *and* $(.)$ *on* N/I *as* $(a+I)+(b+I) = (a+b)+I$ *and* $(a+I).(b+I) = (a.b) + I$.

Then it follows that $(N/I, +, .)$ *is a near ring. In this near ring* $(0 + I)$*is the additive identity and* $(-a + I)$ *is the additive inverse of* $a + I$. *The near ring* N/I *is called the quotient near ring of* N *modulo* I.

Theorem 3.1.2. *Let* I *be an ideal of a near ring* N. *Then* N *is nilpotent if and only if* I *and* N/I *are nilpotent .*

Proof. Suppose that N is nilpotent, Since $I \subseteq N$ and N is nilpotent, I is also nilpotent. Also, since N is nilpotent, there exist a positive integer k such that $N^k = \{0\}$. Now $(N/K)^k = N^k/I = \{0 + I\}$ (since $N^k = \{0\}$). Therefore, N/I is nilpotent.

Conversely, suppose that I and N/I are nilpotent. Since N/I is nilpotent, then there exist a positive integer t such that $(N/I)^t = \{0 + I\}$, that is $N^t \subseteq I$. Since I is nilpotent, there exists a positive integer s such that $I^s = \{0\}$. Therefore, $N^{st} = \{0\}$, So, N is nilpotent. □

Definition 3.1.4. *[5]*

1. *A near ring N is called simple if N has no nontrivial ideals.*
2. *A proper ideal I of N is called maximal if $I \subseteq J \subseteq N$ and J is an ideal of N implies that either $I = J$ or $J = N$.*

Example 3.1.4. *[5]*

i: Let $N = \{0, 1, 2\}$. Define addition$(+)$ and multiplication $(.)$ as follows:

+	0	1	2
o	0	1	2
1	1	2	0
2	2	0	1

.	0	1	2
0	0	0	0
1	0	1	1
2	0	0	2

Then $(N, +, .)$ is a near ring. It is a simple near ring.

ii: For any prime number p, the near ring $(\mathbb{Z}_p, +, .)$ is a simple near ring.

Theorem 3.1.3. *[5] An ideal of a near ring N is maximal if and only if N/I is simple.*

Proof. Suppose I is a proper ideal of N, and suppose I is a maximal ideal in N. Let K be an ideal of N/I. To show N/I is a simple, we have show that either $K = (0)$ or $K = N/I$. Then $K = J/I$, for some ideal J of N such that $I \subseteq J \subseteq N$. Since $I \subseteq J \subseteq N$ and I is a maximal ideal of N, we have that $I = J$ or $J = N$. This implies that $J/I = I$ or $J/I = N/I$. Therefore, $K = J/I = 0 + I$, the zero element in N/I or $k = N/I$. Hence, N/I is a simple near ring.

Converse. Suppose tht N/I is a simple. We prove that I is a maximal ideal of N. Let J be any ideal of N such that $I \subseteq J$. This implies that J/I is an ideal of N/I. Since N/I is a simple we have either $J/I = I$ or $J/I = N/I$. Next we show that $J/I = I \Rightarrow J = I$. Suppose $J/I = I$. It is obvious that $I \subseteq J$. Take $x \in J$. This implies $x + I \in J/I = I \Rightarrow x \in I$. Now we verified that $J/I = I \Rightarrow J = I$. Next we show that $J/I = N/I \Rightarrow J = N$. Suppose that $J/I = N/I$. Since J is an ideal of N, we have $J \subseteq N$. Take $n \in N$. Now $n + I \in N/I = J/I \Rightarrow n + I = i_1 + I$ for some $i_1 \in J$. This implies that $n - i_1 \in I \subseteq J$ and hence $n - i_1 + i_1 \in J$. So, $n \in J$. Now we verified that either $J = I$ or $J = N$. Hence, I is a maximal ideal of N. □

3.2 Homomorphism and Isomorphism on near rings

In this section, we define homomorphism between near ring, and we present fundamental homomorphism theorems.

Definition 3.2.1. *[5] Let N and N_1 be near rings. A mapping $h : N \to N_1$ is called*

i: a homomorphism (or near ring homomorphism) if

$$h(m+n) = h(m) + h(n) \text{ and } h(mn) = h(m).h(n), \ \forall m, n \in N$$

moreover if $N = N_1$, then a homomorphism h is called an endomorphism.

ii: a monomorphism if h is a one-to-one homomorphism .

iii: an epimorphism if h is an onto homomorphism .

iv: an isomorphism if h is a a homomorphism which is one-one, and onto. Moreover, if $N = N_1$, then an isomorphism h is called an automorphism.

Theorem 3.2.1. *[5](Homomorphism Theorem for near rings.)*

i: If I is an ideal of a near ring N, then the mapping $\Pi : N \to N/I$ (defined by $\Pi(n) = n + I$) is a near ring epimorphism. Also, N/I is a homomorphic image of N.

ii: Conversely, if $h : N \to N_1$, is an epimorphism, then $\ker(h)$ is an ideal of N, and $N/\ker(h) \cong N_1$.

Proof. i Suppose I is an ideal of N. Define the mapping $\Pi : N \to N/I$ by $\Pi(n) = n + I$. Now we show that Π is an epimorphism. Let $n_1, n_2 \in N$ Now $\Pi(n_1 + n_2) = (n_1 + n_2) + I = (n_1 + I) + (n_2 + I) = \Pi(n_1) + \Pi(n_2)$. And $\Pi(n_1 n_2) = (n_1 n_2) + I = (n_1 + I).(n_2 + I) = \Pi(n_1).\Pi(n_2)$. Therefore, Π is a near ring homomorphism. Let $x \in N/I$. Then their exists $n_1 \in N$ such that $x = n_1 + I$ Now $\Pi(n_1) = n_1 + I = x$. So Π is onto. Thus , we verified that $\Pi : N \to N/I$ is a near ring epimorphism. Hence, N/I is a homomorphic image of N.

ii Suppose $h : N \to N_1$ is an epimorphism.

Part 1: Now we show that $\ker(h)$ is an ideal of N and $N/\ker \cong N_1$. First we show that $\ker(h)$ is a normal subgroup of N. Clearly, $0 \in \ker(h)$, and so $\ker(h) \neq \phi$. Let $n \in N$, and $x \in \ker(h)$.

Now

$$
\begin{aligned}
h(n + x - n) &= h(n) + h(x) - h(n). \quad (\text{Since } \ h \text{ is a homomorphism}) \\
&= h(n) + 0 - h(n), \ (\text{Since } \ h(x) = 0) \\
&= h(n) - h(n) = 0
\end{aligned}
$$

This shows that $n + x - n \in \ker(h)$.

Part 2: Now we show that $\ker(h)$ is a right ideal of N. Let $n \in N$, and $x \in \ker(h)$. Now $h(xn) = h(x)h(n) = 0h(n) = 0$. Therefore, $xn \in \ker(h)$. Hence $\Big(\ker(h)\Big)N \subseteq \ker(h)$.

Part 3: In this part, we show that $\ker(h)$ is a left ideal of N.

Let $n, n_1 \in N$, and $x \in \ker(h)$.

Now

$$h(n(n_1 + x) - nn_1) = h(n)h(n_1 + x) - h(n)h(n_1)$$

$$= h(n)(h(n_1) + h(x)) - h(n)h(n_1)$$

$$= h(n)(h(n_1) + 0) - h(n)h(n_1)$$

$$= h(n)h(n_1) - h(n)h(n_1) = 0$$

Therefore, $n(n_1 + x) - nn_1 \in \ker(h)$. Hence, $\ker(h)$ is an ideal of N.

<u>Part 4:</u> Define a mapping $\Phi : N/\ker(h) \to N_1$ as follows:

$$\Phi(n + \ker(h)) = h(n) \text{ for all } n + \ker(h) \in N/\ker(h).$$

Now we show that Φ is a near ring isomorphism. Let $n_1 + \ker(h)$, $n_2 + \ker(h) \in N/\ker(h)$.

Now, $\Phi(n_1 + \ker(h)) = \Phi(n_2 + \ker(h))$

if and only if $h(n_1) = h(n_2)$

if and only if $h(n_1) - h(n_2) = 0$

if and only if $h(n_1 - n_2) = 0$

if and only if $(n_1 - n_2) \in \ker(h)$

if and only if $(n_1 + \ker(h) = n_2 + \ker(h)$

Therefore Φ is one-one and will defined. Let $n_1 \in N_1$, since h is onto, there exists $n \in N$ such that $h(n) = n_1$. Now $\Phi(n + \ker(h)) = h(n) = n_1$, and so Φ is onto.

Part 5: Let $n_1 + \ker(h),\ n_2 + \ker(h) \in N/\ker(h)$ Now

$$\begin{aligned}
\Phi\Big((n_1+\ker(h))+(n_2+\ker(h))\Big) &= \Phi((n_1+n_2)+\ker(h)).\\
&= h(n_1+n_2)\\
&= h(n_1)+h(n_2)\\
&= \Phi(n_1+\ker(h))+\Phi(n_2+\ker(h))
\end{aligned}$$

Also,

$$\begin{aligned}
\Phi\Big((n_1+\ker(h)).(n_2+\ker(h))\Big) &= \Phi((n_1n_2)+\ker(h)).\\
&= h(n_1n_2)\\
&= h(n_1)h(n_2)\\
&= \Phi(n_1+\ker(h)).\Phi(n_2+\ker(h))
\end{aligned}$$

So, Φ is a near ring homomorphism. Hence $\Phi : N/\ker(h) \to N_1$ is a near ring isomorphism

□

Theorem 3.2.2. *[5](First Isomorphism Theorem)*

If I and J are ideals of N, then $I \cap J$ is an ideal of J and $(I+J)/I \cong J/I \cap J$

Proof. We know that $I \cap J$ is an ideal of J. Define a mapping $\Phi : I + J \to J/I \cap J$ s follows:

$$\Phi(a+b) = b + (I \cap J), \forall a \in I, b \in J$$

Clearly Φ is an epimorphism, and $\ker(\Phi) = I$ Then by "Homomorphism Theorem for near rings"

$$(I+J)/I \cong J/I \cap J$$

$\square$

Remark 3.2.1. *[5] Let* $f : (N_1, +, .) \to (N_2, +', .')$ *be a near ring homomorphism,*

1. *If* I *is an ideal of the near ring* N_1*, then* $f(I)$ *is an ideal of* N_2*.*

2. *If* J *is an ideal of the near ring* N_2*, then* $f^{-1}(J)$ *is an ideal of* N_1

3.3 Prime and semiprime ideals

In this section we present the concepts of prime and semiprime ideals.

Definition 3.3.1. *[5] Let* P *be an ideal of the near ring* N*,* P *is called:*

i: a prime ideal if for all ideals I *and* J *of* N*,* $IJ \subseteq P$ *implies* $I \subseteq P$ *or* $J \subseteq P$*.*

ii: a semiprime ideal if for all ideals I *of* N*;* $I^2 \subseteq P$ *implies* $I \subset P$*.*

Definition 3.3.2. *[5] Let* N *be a near ring,* N *is called:*

i: a prime near ring if $\{0\}$ *is a prime ideal.*

ii: a semiprime near ring if $\{0\}$ *is a semiprime ideal.*

Theorem 3.3.1. *[5] For an ideal* P *of a near ring* N*, and ideals* I, J *of* N*, the following conditions are equivalent:*

i: P *is a prime ideal.*

ii: $a \notin P$ *and* $b \notin P \Rightarrow (a)(b) \nsubseteq P$*, for all* $a, b \in N$*.*

iii: If I contains P properly and J contains P properly, then $IJ \nsubseteq P$.

iv: $I \nsubseteq P$ and $J \nsubseteq P \Rightarrow IJ \nsubseteq P$.

Proof. To prove $(i) \Rightarrow (ii)$, assume P is a prime ideal. Suppose $a \notin P$, and $b \notin P$. Then $(a) \nsubseteq P$ and $(b) \nsubseteq P$. If $(a)(b) \subseteq P$, since P is a prime ideal, it follows that either $(a) \subseteq P$ or $(b) \subseteq P$, which is contradiction. Hence $(a)(b) \nsubseteq P$. The proof for $(i) \Rightarrow (ii)$ is complete.

To prove $(ii) \to (i)$, ssume (ii) Suppose I, J are ideals of N such that $IJ \subseteq P$. If possible, assume that $I \nsubseteq P$ and $J \nsubseteq P$. Then by (ii) $IJ \nsubseteq P$. which is a contradiction. Therefore, $I \subseteq P$ or $J \subseteq P$ then P is a prime ideal. The proof of $(ii) \Rightarrow (i)$ is complete.

To prove $(ii) \Rightarrow (iii)$, assume (ii) Suppose I and J are ideals of N such that $I \supset P$ and $J \supset P$. choose $a \in I/P$ and $b \in J/P$. This means that $a \notin P$ and $b \notin P$. By condition (ii), we get $(a)(b) \nsubseteq P$. Hence $IJ \nsubseteq P$. This proves $(ii) \Rightarrow (iii)$.

To prove $(iii) \Rightarrow (iv)$, assume (iii). Suppose $I \nsubseteq P$ and $J \nsubseteq P$, take $a \in I/P$ and $b \in J/P$. Then $(a) + P$ contains P properly, and $(b) + P$ contains P properly. By condition (iii), it follows that $\Big((a) + P\Big)\Big((b) + P\Big) \nsubseteq P$. Therefore, there exist $a_1 \in (a), b_1 \in (b),\ p, p_1 \in P$ such that $(a_1 + p)(b_1 + p_1) \notin P$. This implies thta $a_1(b_1 + p_1) + p(b_1 + p_1) \notin P$. Therefore, $a_1(b_1 + p_1) - a_1b_1 + p(b_1 + p_1) \notin P$. Since P is an ideal, we have $a_1(b_1 + p_1) - a_1b_1 \in P, p(b_1 + p_1) \in P$, and $a_1b_1 \notin P$. Hence $IJ \nsubseteq P$. The proof for $(iii) \Rightarrow (iv)$ is complete. $\square$

Theorem 3.3.2. *[5] If I is an ideal of a near ring N that is a direct summand,*

and P is a prime ideal of N, then $P \cap I$ is a prime ideal in I.

Proof. Suppose J_1 and J_2 are ideals of I (by considering I as anear ring) Such that $J_1J_2 \subseteq P \cap I$. This implies that $J_1J_2 \subseteq P$ and $J_1J_2 \subseteq I$.

Since I is a direct summand of N,by remark (3.1.3) it follows that J_1, J_2 are ideals of N. Since P is a prime ideal of N and $J_1J_2 \subseteq P$, it follows that $J_1 \subseteq P$ or $J_2 \subseteq P$. Therefore $J_2 \subseteq P \cap I$ or $J_2 \subseteq P \cap I$. Hence $P \cap I$ is a prime ideal in I.

□

Definition 3.3.3. *[5] A near ring N is called a zero near ring if $NN = \{0\}$.*

Definition 3.3.4. *[5] Let N be a near ring*

i The intersection of all prime ideals of N is called the prime radical of N and is denoted by $P(N)$

ii For any proper ideal I of N, the intersection of all prime ideals of N containing I is called the prime radical of I and is denoted by $P(I)$.

Example 3.3.1. *[5] Every integral near ring is a prime near ring.*

Solution:

Suppose N is an integral near ring. To show that N is a prime near ring, it is enough to show (0) is a prime ideal. Let I and J be ideals of N such that $IJ \subset (0)$. If either $I = (0)$ or $J = (0)$, then there is nothing to prove. If possible, suppose that $I \neq (0)$ or $J \neq (0)$, then we can choose $0 \neq a \in I$ and $0 \neq b \in J$ such that $ab = 0$, which is a contradiction to the fact that N is integral. Therefore either $I = (0)$ or $J = (0)$. Thus we have proved that (0) is a prime ideal of N. Hence, N is a prime near ring.

Theorem 3.3.3. *[5] If the near ring N is simple, then either N is prime or N is a zero near ring.*

Proof. Suppose N is not a zero near ring. Then $NN \neq \{0\}$. we have to prove that (0) is a prime ideal of N. Suppose I and J are ideals of N such that $IJ \subseteq (0)$. since I and J are ideal of N, and N is simple, it follows that $I, J \in \{(0), N\}$. If $I = N$ and $J = N$, and then $NN \subseteq IJ \subseteq (0)$. Which is a contradiction. Therefore, $I = (0)$ or $J = (0)$. Thus, (0) is a prime ideal of N. Hence, N is a prime near ring. □

Lemma 3.3.4. *[5] Let I be an ideal of N. Then I is a prime ideal if and only if N/I is a prime near ring.*

Theorem 3.3.5. *[5] If I is a maximal ideal of N, then either I is a prime ideal or $N^2 \subseteq I$.*

Proof. Suppose I is a maximal ideal of N, then N/I is simple. Therefore the only ideals of N/I are $0 + I = I$ or N/I then by Theorem(3.3.3), N/I is a prime near ring or zero near ring. If N/I is a prime near ring, then I is a prime ideal if N/I is a zero near ring $(N/I)(N/I) = I$ and this implies that $N^2 \subseteq I$. The proof is complete . □

Theorem 3.3.6. *[13] If N is a prime near ring and F a generalized derivation* *on N associated with D of N, then*

$$a(bF(c) + D(b)c) = abF(c) + aD(b)c, \quad \forall a, b, c \in N.$$

Proof. Clearly $F(a(bc)) = aF(bc) + D(a)bc = a(bF(c) + D(b)c) + D(a)bc$, and, also we obtain $F((ab)c) = abF(c) + D(ab)c = abF(c) + aD(b)c + D(a)bc$. Comparing these two expressions for $F(abc)$ gives the desired conclusion. □

Definition 3.3.5. *[13] A non empty subset U of a near ring N will be called a semigroup right ideal (resp., semigroup left ideal) if $UN \subset U$ $(NU \subset U)$, and U is called a semigroup ideal if it is a right as well as a left semigroup ideal.*

Lemma 3.3.7. *[13] Let N be a prime near ring and let $U \neq \{0\}$ be a semigroup ideal of N. If $U \subseteq Z(N)$, then N is commutative.*

Lemma 3.3.8. *[12] Let N be a prime near ring. If N admits a non zero derivation D for which $D(N) \subset Z(N)$, then N is a commutative ring.*

Remark 3.3.1. *[11] If N is a prime near ring if $xNy = 0$, for $x, y \in N$, we have $x = 0$ or $y = 0$*

Notation:

For any a, b in the near ring N, we write $[x, y] = xy - yx$.

Theorem 3.3.9. *[1] Let N be a prime near ring. If N admits a non zero derivation d such that $d\Big([x, y]\Big) = [x, y]$ for all $x, y \in N$, then N is a commutative near ring.*

Proof. assume that

$$d\Big([x,y]\Big) = [x,y] \text{ for all } x,y \in N \tag{3.1}$$

Replacing y by xy in (3.1), because of $[x,yx] = [x,y]x$, we get

$$[x,y]x = d\Big([x,y]x\Big),\ \forall x,y \in N$$

Since $d\Big([x,y]x\Big) = [x,y]d(x) + d[x,y]x$, then according to (3.1) we obtain

$$[x,y]x = [x,y]d(x) + [x,y]x$$

and therefore $[x,y]d(x) = 0$. Hence

$$(xy - yx)d(x) = 0,\ \forall x,y \in N. \tag{3.2}$$

Substituting yz for y in (3.2), we obtain $(xz - zx)yd(x) = 0$ which leads to

$$(xz - zx)Nd(x) = 0,\ \forall x,z \in N \tag{3.3}$$

Since N is prime, Equation (3.3) reduces to

$$[x,z] = 0 \text{ or } d(x) = 0,\ \forall x,z \in N$$

It follows that for each fixed $x \in N$ we have

$$x \in Z(N) \text{ or } d(x) = 0 \tag{3.4}$$

But $x \in Z(N)$ also implies that $d(x) \in Z(N)$ $\forall x \in N$ Therefore $d(N) \subset Z(N)$. Then by last Lemma we conclude that N is a commutative near ring. This completes the proof of our Theorem.

□

The following example proves that the primeness hypothesis in the last theorem is necessary even in the case of arbitrary ring.

Example 3.3.2. *[1] Let R be a commutative ring which is not zero ring and*

$$\text{Consider } N = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \Big| x, y \in R \right\}$$

If we define $d : N \to N$ by $d \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$, then it is straight forward to check that d is a non zero derivation of N.
On the other hand, if $a = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}$, where $0 \neq r$, then $aNa = 0$
Which proves that N is not prime. Moreover, d satisfies the condition

$$d\Big([A, B]\Big) = [A, B], \text{ for all } A, B \in N$$

but N is non commutative ring.

Theorem 3.3.10. *[12] Let N be a prime near ring which admitss a non zero derivation d. Then the following assertions are equivalent.*

i. $d\Big([x, y]\Big) = [d(x), y]$ *for all* $x, y \in N$.

ii. $[d(x), y] = [x, y]$ *for all* $x, y \in N$.

iii. *N is a commutative ring.*

Proof. It easy to verify that $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$

To prove $(i) \Rightarrow (iii)$ assume that

$$d\Big([x, y]\Big) = [d(x), y] \text{ for all } x, y \in N \tag{3.5}$$

Replacing y by yx in (3.5), we get

$$[d(x), yx] = d\Big([x, y]x\Big),\ \forall x, y \in N$$

by definition of d we get

$$xyd(x) = yd(x)x,\ \forall x, y \in N. \tag{3.6}$$

Substituting yz for y in (3.6) where $z \in N$, we obtain $[x, y]zd(x) = 0$ which leads to $[x, y]Nd(x) = 0, \forall x, y \in N$. Therefore $d(x) = 0$ or $[x, y] = 0$, Since N is prime. Then for each fixed $x \in N$ we have $d(x) = 0$ or $x \in Z(N)$. But $x \in Z(N)$ also implies that $d(x) \in Z(N)$. Then $d(N) \subset Z(N)$ and using Lemma, we conclude that N is a commutative ring.

To prove $(ii) \Rightarrow (iii)$ Suppose that

$$[d(x), y] = [x, y] \text{ for all } x, y \in N \tag{3.7}$$

Replacing x by xy in (3.7), because of $[xy, y] = [x, y]y$, we get

$$[d(xy), y] = [x, y]y = [d(x), y]y \text{ for all } x, y \in N$$

Now, by the partial distributive Law the last equation can be written ass

$$d(x)y^2 + xd(y)y - yxd(y) - yd(x)y = d(x)y^2 - yd(x)y,$$

so that

$$xd(y)y = yxd(y) \text{ for all } x, y \in N \tag{3.8}$$

Since Equation (3.8) is the same as Equation (3.6) arguing as in the proof of $(i) \to (iii)$ we find that N is a commmutative ring. $\square$

3.4 Weakly Prime ideal in near ring

In this section we obtain equivalent conditions for an ideal to be a weakly prime ideal.

Definition 3.4.1. *[14] Let N be a near ring, Let P be a proper ideal of N, P is called weakly prime ideal if $0 \neq AB \subseteq P$, A and B are ideals of N implies $A \subseteq P$ or $B \subseteq P$.*

Remark 3.4.1. *[14] Every prime ideal is weakly Prime and $\{0\}$ is always weekly prime ideal of N.*

The following examples shows that a weakly prime ideal need not be a prime ideal in general.

Example 3.4.1. *[14] Let $N = \{0, a, b, c, 1, 2, 3\}$. Define addition and multiplication on N as follows:*

+	0	1	2	3	a	b	c	d
0	0	1	2	3	a	b	c	d
1	1	2	3	0	d	c	a	b
2	2	3	0	1	b	a	d	c
3	3	0	1	2	c	d	b	a
a	a	d	b	c	2	0	1	3
b	b	c	a	d	0	2	3	1
c	c	a	d	b	1	3	0	2
d	d	b	c	a	3	1	2	0

.	0	1	2	3	a	b	c	d
0	0	0	0	0	0	0	0	0
1	0	1	2	3	a	b	c	d
2	0	2	0	2	2	2	0	0
3	0	3	2	1	b	a	c	d
a	0	a	2	b	a	b	c	d
b	0	b	2	a	b	a	c	d
c	0	c	0	c	0	0	0	0
d	0	d	0	d	2	2	0	0

Then $(N, +, .)$ *is a near ring. Here* $\{0, c\}$ *is a weakly prime ideal, but not prime, since* $\{0, 2\}^2 \subseteq \{0, c\}$.

Example 3.4.2. $(\mathbb{Z}_6, +, .)$ *is anear ring.*

$\{0\}$ *is a weakly prime ideal, but not prime ideal.*

Theorem 3.4.1. *[14] Let N be a near ring and P a weakly prime ideal of N. If P is not prime then $P^2 = 0$.*

Proof. Suppose that $P^2 \neq 0$. We show that P is prime. Let A and B be ideals of N such that $AB \subseteq P$. If $AB \neq 0$, then $A \subseteq P$ or $B \subseteq P$. So assume that $AB = 0$. Since $P^2 \neq 0$, there exist $p_0, q_0 \in P$ such thta $< p_0 > + < q_0 > \neq 0$. Then $(A+ < p_0 >)(B+ < q_0 >) \neq 0$. Suppose $(A+ < p_0 >)(B+ < q_0 >) \nsubseteq P$. Then there exist $a \in A; b \in B$ and $p_0' \in < p_0 >; q_0' \in < q_0 >$ such that $(a+p_0')(b+q_0') \notin P$. which implies $a(b + q_0') \notin P$, but $a(b + q_0') = a(b + q_0') - ab \in P$. Since $AB = 0$, a contradiction.So $0 \neq (A+ < p_0 >)(B+ < q_0 >) \subseteq P$ Which implies $A \subseteq P$ or $B \subseteq P$. □

Corollary 3.4.1.1. *[14] Let N be a near ring and P an ideal of N. If $P^2 \neq 0$, then P is prime if and only if P is weakly prime.*

Corollary 3.4.1.2. *[14] Let P be a weakly prime ideal of N, let $P(N)$ be the prime radical of N. Then either $P \subseteq P(N)$ or $P(N) \subseteq P$. If $P \subseteq P(N)$, then P is not prime, while if $P(N) \subseteq P$, then P is prime.*

Theorem 3.4.2. *[14] Let N be a near ring and P an ideal of N. Then the following are equivalent*

i. P is a weakly prime ideal

ii. for any ideals I, J of N with $P \subset I$ and $P \subset J$, we have either $IJ = 0$ or $IJ \nsubseteq P$

iii. for any ideals I, J of N with $I \not\subseteq P$ and $J \not\subseteq P$, we have either $IJ = 0$ or $IJ \not\subseteq P$.

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are clear.

To prove $(ii) \Rightarrow (iii)$. Let I, J be ideals of N with $I \not\subseteq P$ and $J \not\subseteq P$. Then there exist $i_1 \in I$ and $j_1 \in J$ such that $i_1, j_1 \notin P$. Suppose that $< i >< j > \neq 0$ for some $i \in I$ and some $j \in J$. Then $(P+ < i > + < i_1 >)(P+ < j > + < j_1 >) \neq 0$. And $P \subset P+ < i > + < i_1 >$; $P \subset P+ < j > + < j_1 >$. By hypothesis, $(P+ < i > + < i_1 >)(P+ < j > + < j_1 >) \not\subseteq P$.. Which implies$< i > (P+ < j > + < j_1 >) < i_1 > (P+ < j > + < j_1 >) \not\subseteq P$. So there exit $i' \in < i >$; $i'_1 \in < i_1 >$, $j', j'' \in < j >$; $j'_1, j''_1 \in < j_1 >$ and $p_1, p_2 \in P$ such that:

$$i'(p_1 + j' + j'_1) + i'_1(p_2 + j'' + j''_1) \notin P.$$

Therefore $i'(p_1+j'+j'_1)-i'(j'+j'_1)+i'(j'+j'_1)+i'_1(p_2+j''+j''_1)-i'_1(j''+j''_1)+i'_1(j''+j''_1) \notin P$. But since $i'(p_1+j'+j'_1)-i'(j'+j'_1) \in P$, and $i'_1(p_2+j''+j''_1)-i'_1(j''+j''_1) \in P$, we have P does not contain either $i'(j' + j'_1)$ or $i'_1(j'' + j''_1)$which shows that $IJ \not\subseteq P$. $\square$

Definition 3.4.2. *[14] Let N be a near ring*

i A subset M of N is called $m-$ system if $a, b \in M$, then there exist $a_1 \in < a >$ and $b_1 \in < b >$ such that $a_1 b_1 \in M$.

ii A subset M of N is called weakly $m-$ system if $M \cap A \neq \phi$ and $M \cap B \neq \phi$ for any ideals A, B of N, then either $AB \cap M \neq \phi$ or $AB = 0$.

Remark 3.4.2. *[14] Clearly every $m-$ system is a weakly $m-$ system, but a weekly m-system need not be $m-$ system, since in example* 3.4.1,

$M = \{1, 2, 3, a, b, d\}$ *is a weakly $m-$ system but not an $m-$ system since $x_1x_2 \notin M$ for all $x_1, x_2 \in < 2 >$.*

Theorem 3.4.3. *[14] Let Nbe a near ring, let $M \subseteq N$ be a non-void weakly $m-$ system in N and I an ideal of N with $I \cap M = \phi$. Then I is contained in a weakly prime ideal $P \neq N$ with $P \cap M = 0$*

Proof. Let $A^* = \{J : J \text{ is an ideal of } N \text{ with } J \cap M = \phi\}$. Clearly $I \in A^*$. Then by Zorn's Lemma. A^* contains a maximal element (say) P wth $P \cap M = \phi$. We show that P is a weakly prime ideal of N.

Let A and B be an ideals of N with $P \subset A$ and $P \subset B$. Then by maximality of A^*, $A \cap M \neq \phi$ and $B \cap M \neq \phi$ since M is weakly $m-$ system, we have $AB = 0$ or $AB \cap M \neq \phi$; that is $AB = 0$ or $AB \nsubseteq P$ since $P \cap M = \phi$. So by Last Theorem, P is a weakly prime ideal of N and also containing I. □

3.5 Semi-symmetric ideal in near ring

In this section we introduce Semi-symmetric ideal in near ring and prove that the prime radical of Semi-symmetric Ideal is completely semiprime.

Definition 3.5.1. *[14] Let N be a near ring, an ideal P of N is called completely prime (completely Semi-prime) if $ab \in P$ implies $a \in P$ or $b \in P$ ($a^2 \in P$ implies $a \in P$)*

Definition 3.5.2. *[20] An ideal I of a near ring N is said to be semi-symmetric if $x^n \in I$ for some positive integer n implies $< x >^n \subseteq I$.*

Remark 3.5.1. *[20] It is evident that every completely semiprime ideal is semi-symmetric.*

The following example shows that there exist simi-symmetric ideals which are not completely semiprime.

Example 3.5.1. *[20] Let $N = \{0, 1, 2, 3\}$ be the additive group on integers module 4. Define multiplication as follows.*

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	2	0	2
3	0	3	0	3

Then $(N, +, .)$ is a near ring.

Clearly $\{0\}$ is a semi-symmetric ideal, but not a completely semiprime ideal.

Theorem 3.5.1. *[20] If I is a semi symmetric ideal of a near ring N, then the prime radical of I, $P(I)$ is completely semiprime.*

Proof. Let $a^2 \in P(I)$. Then there exists an integer $k \geq 1$ such that $(a^2)^k \in I$. Since I is semi-symmetric, $< a >^{2k} \subseteq I \subseteq P(I)$ and hence $a \in P(I)$. Thus $P(I)$ is completely semiprime. $\square$

Corollary 3.5.1.1. *[20] If I is a semi-symmetric ideal of a near ring N, then*

$$P(I) = \{x \in N | x^k \in I \text{ for some positive integer } k\}.$$

Definition 3.5.3. *[20] A near ring N is said to be semi-symmetric if $\{0\}$ is a semi-symmetric ideal of N.*

Near rings without nonzero nilpotent elements, commutative near rings and Boolean near rings are examples of semi-symmetric near rings.

Notation:

for any subset S of a near ring N, we write $A(S) = \{x \in N | xS = \{0\}\}$. Clearly $A(S)$ is a left ideal of N.

Theorem 3.5.2. *[20]*

Suppose a near ring N satisfies one of the following conditions:

i. for all $a, b \in N : ab = 0$ implies $ba = 0$.

ii. every nilpotent element in N is central.

Then N is a semi-symmetric near ring.

Proof. Assume (i). Clearly for every subset S of N, $A(S)$ is an ideal of N. Let $a \in N$ and $a^n = 0$ for some positive integer n, then $a \in A(a^{n-1})$ and hence $< a > a^{n-1} = \{0\}$. So by our assumption, $a < a > a^{n-2} = \{0\}$ and hence $a \in A\Big(< a > a^{n-2}\Big)$.

Again by the above argument,we have$< a >< a > a^{n-2} = \{0\}$ .Continuing this process, we obtain that $< a >^n = \{0\}$. Thus N is a semi-symmetric near ring. Similarly, we can prove that if every nilpotent element is central then N is a semi-symmetric near ring. $\square$

Corollary 3.5.2.1. *[20] Suppose N is a semi-symmetric near ring*

Then the prime radical of N is completely semiprime. Moreover, the set of all nilpotent elements of N is an ideal of N.

Corollary 3.5.2.2. *[20] Let N be a near ring with the property that $ab = 0$ implies $ba = 0$. Then the set of all nilpotent elements is an ideal.*

Definition 3.5.4. *A minimal ideal of a near ring N is an ideal which is minimal in the set of all non zero ideals.*

Lemma 3.5.3. *[20] If J is completely semiprime ideal of a near ring N, then it has the following property*

$$x_1 \cdots x_n \in J \quad \textit{implies} \; < x_1 > \cdots < x_n > \in J, \; \forall x_1, \cdots, x_n \in N.$$

We now prove our main Theorem.

Theorem 3.5.4. *[20] If I is a semi-symmetric ideal of a near ring N. ,then every minimal prime ideal of I is completely prime.*

Proof. Let P be a minimal prime ideal of I and Let S be the multiplicative subsemigroup of N generated by N/P. Now we claim that $P(I) \cap S = \phi$. If not, choose an element x in $P(I) \cap S$. Since $x \in S$, there exist $x_1, \cdots, x_n$ in N/P such that $x = x_1 \cdots x_n \in P(I)$, then $P(I)$ is completely semiprime and hence by last

Lemma $< x_1 > \cdots < x_n >\subseteq P(I) \subseteq P$. Thus, $< x_i >\subseteq P$ for some i and $x_i \in P$, a contradiction. Therefore $P(I) \cap S = \phi$.

Write $L = \{J/J$ is an ideal of N such that $I \subseteq J$ and $J \cap S = \phi\}$.

Note thta $L \neq \phi$, since $P(I) \in L$. By Zorn's Lemma L contains a maximal element, say Q. Now Q is an ideal and $Q \subseteq N/S$. We show that Q is a prime ideal. Otherwise, there exist ideals A and B such that $AB \subseteq Q$, $A \nsubseteq Q$ and $B \nsubseteq Q$. choose $x \in A/Q$ and $y \in B/Q$. By the maximality of Q, we have that $(Q+ < x >) \cap S \neq \phi$ and $(Q+ < y >) \cap S \neq \phi$. Let $a \in (Q+ < x >) \cap S$ and $b \in (Q+ < y >) \cap S$. Clearly $ab \in S$. Aslo $a = c + d$ and $b = e + f$ for some $c, e \in Q$, $d \in< x >$ and $f \in< y >$. Therefore $ab = (c + d)(e + f) = c(e + f) + d(e + f) - df + df \in Q$ because $df \in< x >< y >\subseteq AB \subseteq Q$. Thus $ab \in Q \cap S$, a contradiction. Hence Q is a prime ideal. But $I \subseteq Q \subseteq N/S \subseteq P$ and P is a minimal prime ideal of I so $Q = N/S = P$. Since S is a semigroup, P is a completely prime. $\square$

Corollary 3.5.4.1. *[20] An ideal P of a near ring N is completely prime if and only if it is prime and Semi-symmetric .*

3.6 $3-$ prime ideal with respect to an element of a near ring

In this section ,we introduce the notions of 3- prime ideal with respect to an element x of a near ring denoted by ($x-3-$prime ideal) and the 3-prime ideal near ring with respect to an element x denoted by ($x - 3-$ prime ideal near ring) ,and we study the image and inverse image of x-3-prime ideals under epimomorphisms .

Definition 3.6.1. *[16] An ideal I of a near ring N is called a $3-$ prime ideal if for all $a, b \in N$, $a.N.b \subseteq I$ implies $a \in I$ or $b \in I$.*

Definition 3.6.2. *[16] An ideal I of near ring N is called $3-$ prime ideal with respect to an element x of the near ring denoted by ($x - 3-$ prime ideal) if for all $a, b \in N$*

$$x.(a.N.b) \subseteq I \rightarrow x.a \in I \text{ or } x.b \in I$$

Example 3.6.1. *[16] Let $N = \{0, a, b, c\}$ be the near ring with addition and multiplication defined by the following tables*

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	a	0	a
a	0	a	a	0
b	0	a	b	c
c	0	a	c	b

Let $I = \{0, a\}$. Then I is a $c-3-$ prime ideal since

$$c.(a.N.b) \subseteq I \rightarrow c.a \in I \text{ or } c.b \in I.$$

Theorem 3.6.1. *[16] Let $\{I_j\}_{j\in J}$ be a family of $x-3-$ prime ideals of a near ring N for all $j \in J$,where $x \in N$. Then $\bigcap_{j\in J} I_j$ is a x-3 prime ideal of N.*

Proof. Let $a, b, x \in N \bigcap_{j\in J} I_j$be an ideal since I_j is $x-3-$ prime ideals of N, $I_j \neq \phi, I_j \subseteq N$ Let $x.a.N.b \subseteq \bigcap_{j\in J} I_j$,then $x.a.N.b \subseteq I_j, \forall j \in J$ since I_j is $x-3-$ prime ideal of N, $I_j \neq \phi,\ I_j \subseteq N$. Let $x.a.N.b \subseteq \bigcap_{j\in J} I_j$ then $x.a.N.b \subseteq I_j,\ \ \forall j \in J$. Since I_j is $x-3-$ prime ideal $x.a \in I_j$ or $x.b \in I_j$,$\forall j \in J$. Therefore, $x.a \in \bigcap_{j\in J} I_j$ or $x.b \in \bigcap_{j\in J} I_j$ then $\bigcap_{j\in J} I_j$ is an $x-3-$ prime ideal of N. □

Remark 3.6.1. *[16] If I_1 and I_2 are two $x-3-$ prime ideals of a near ring Nsuch that $I_1 \nsubseteq I_2$ and $I_2 \nsubseteq I_1$, then $I_1 \cup I_2$ may fail to be $x-3-$ prime ideal.*

Example 3.6.2. *[16] Consider the set $N = \{0, a, b, c\}$ be a near ring with addition and multiplication defined as follows:*

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Let $I_1 = \{0, a\}$ *and* $I_2 = \{0, b\}$*. Then they are* $b-3-$ *prime ideals since*

$$b.(a.N.b) \subseteq I_1 \rightarrow b.a \in I_1 \text{ or } b.b \in I_1$$

$$b.(a.N.b) \subseteq I_2 \rightarrow b.a \in I_2 \text{ or } b.b \in I_2$$

but $I_1 \cup I_2 = \{a, b\}$ *is not an ideal of the near ring* N*.*

Remark 3.6.2. *[16] If* I_1 *and* I_2 *are two* $x-3-$ *prime ideals of a near ring* N*, then* $I_1 I_2$ *may fail to* $x-3-$ *prime ideal of* N*.*

Example 3.6.3. *[16] Consider the near ring* $N = \{0, a, b, c\}$ *with addition and multiplication defined as follows:*

Let $I_1 = \{0, a\}$ *and* $I_2 = \{0, b\}$ *which are* $c-3-$ *prime ideals of* N *,but*

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

$I_1.I_2 = \{0\}$ *is not* $c - 3-$ *prime ideal of* N *since*

$$c.(b.N.a) = \{0\} \subseteq I_1.I_2 \text{ but}$$

$$c.b \notin I_1.I_2 \text{ and } c.a \notin I_1.I_2.$$

Definition 3.6.3. *[16] The near ring N is called $3-$prime ideal near ring with respect to an element x denoted by ($x - 3-$ prime ideal near ring), if every ideal of the near ring N is an $x - 3-$ prime ideal of N, where $x \in N$.*

Example 3.6.4. *[16] Consider the near ring $N = \{0, a, b, c\}$ with addition and multiplication defined as follows:*

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

N is $c-3-$ prime ideal near ring since all ideals of N, $I_1 = \{0\}$ and $I_2 = N$ are $c-3-$ prime ideals.

Theorem 3.6.2. *[16] Let N be a near ring with multiplicative identity e', then I is $e'-3-$ prime ideal of N if and only if I is a $3-$ prime ideal of N.*

Proof. Let $y, z \in N, e'$ is the identity. Suppose that I is $e'-3-$ prime ideal of N. Let $y.N.z \subseteq I$, so $e'.(y.N.z) \subseteq I$. Since I is $e'-3-$ prime ideal of N then $e'.y \in I$ or $e'.z \in I$ implies $y \in I$ or $z \in I$. Therefore I is $3-$ prime ideal of N.
Conversely, Let $y, z \in N$, e' is the identity. $e'.(y.N.z) \subseteq I$ implies that $y.N.z \subseteq I$. Since I is $3-$ prime ideal of N then $y \in I$ or $z \in I$ which implies $e'.y \in I$ or $e'.z \in I$. Therefore I is $e'-3-$ prime ideal of N. □

Remark 3.6.3. *[16] In general not all $x-3-$ prime ideals are $3-$ prime ideals.*

Example 3.6.5. *Consider the near ring $N = \{0, 1, 2, 3\}$ with addition and multiplication defined as follows:*

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	0	0	0
3	0	1	2	3

Let $I = \{0, 1\}$ be the $2 - 3-$ prime ideal but I not $3-$ prime ideal, since:

$$2.N.3 \subseteq I \text{ but } 2 \notin I \text{ and } 3 \notin I.$$

Theorem 3.6.3. *[16] Let $(N_1, +, .)$ and $(N_2, +', .')$ be two near rings, $f : N_1 \to N_2$ be an epimomorphism and I be $x-3-$ prime ideal of N_1, then $f(I)$ is $f(x)-3-$ prime ideal of N_2*

Proof. Let I be $x - 3-$ prime ideal of N_1, then $f(I)$ is an ideal of N_2.

Let $c, y, z \in N_2$, $\exists a, b \in N_1$ such that $f(a) = y$, $f(x) = c$, $f(b) = z$, hence $f(x).'(f(a).N_2.f(b)) \subseteq f(I)$ since f be an epimomorphism $f(x.a.N_1.b) \subseteq f(I)$ since I is $x - 3-$ prime ideal of N_1 then $x.a.N_1.b \subseteq I$ implies $x.a \in I$ or $x.b \in I$

$$f(x.a) \in f(I) \text{ or } f(x.b) \in f(I)$$

so, $f(x).'f(a) \in f(I)$ or $f(x).'f(b) \in f(I)$. Therefore $f(I)$ is $f(x) - 3-$ prime ideal of N_2. □

Theorem 3.6.4. *[16] Let $(N_1, +, .)$ and $(N_2, +', .')$ be two near rings, and $f : N_1 \to N_2$ be an epimorphism and J be a $f(x) - 3-$ prime ideal of N_2 , then $f^{-1}(J)$ is an $x - 3-$ prime ideal of N_1,.*

Proof. Let $x, a, b \in N_1$, $f^{-1}(J)$ is an ideal. Let $x(a.N_1.b) \subseteq f^{-1}(J)$. so $f(x.(a.N_1.b)) \subseteq J$ then $f(x).'(f(a).'N_2.'f(b)) \subseteq J$ Now, since J is $f(x) - 3-$ prime ideal of N_2

$$f(x).'f(a) \in J \text{ or } f(x).'f(b) \in J$$

then $f(x.a) \in J$ or $f(x.b) \in J$ Therefore, $x.a \in f^{-1}(J)$ or $x.b \in f^{-1}(J)$. Then $f^{-1}(J)$ is an $x - 3-$ prime ideal of N_1. □

Theorem 3.6.5. *[16] Let I be an ideal of the $x-3-$ prime ideal near ring N. Then the quotient near ring N/I is $x+I-3-$ prime ideal near ring.*

Proof. The natural homomorphism $nat_1 : N \to N/I$ which is defined by $nat_1(x) = x+I$, for all $x \in N$, is an epimomorphism.
Now let J be an ideal of the quotient near ring N/I. Then we have $nat_1^{-1}(J)$ is an ideal of the near ring N. So, $nat_1^{-1}(J)$ is an $x-3-$ prime ideal of the near ring N (since N is $x-3-$ prime ideal near ring)
By theorem (54) we have $nat_1(nat_1^{-1}(J)) = J$ is $nat_1(x)-3-$ prime ideal of N/I, then J is $x+I-3-$ prime ideal of the quotient near ring. Then N/I is $x+I-3-$ prime ideal near ring. □

3.7 IFP ideals in near rings

In this section we present the concept of insertion of factors property of near rings and N-groups .

Definition 3.7.1. *[4] A near ring N is said to fulfil the insertion of factors property (IFP) proiveded that for all $a, b, n \in N$, $ab = 0$ implies $anb = 0$. Such near rings are called IFP near rings .*

Example 3.7.1. *Let $(N, +, .)$ be the near ring where $(N, +)$ is the Kleins four group $N = \{0, a, b, c\}$ and ′.′ is defined as follows*
Clearly this is an IFP near ring.

Definition 3.7.2. *[4] Let I be an ideal of a near ring N and N/I is an IFP*

.	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

near ring, then the ideal I is called an IFP ideal of N.

Remark 3.7.1. *I is an IFP ideal of a near ring N iff $a, b \in I$ implies $anb \in I$ for all $a, b, n \in N$.*

Theorem 3.7.1. *[4] If I is an IFP ideal and a $3-$ prime ideal of a near ring N, then I is a completely prime ideal*

Proof. Let $ab \in I$, for $a, b \in N$. since I is an IFP ideal, then $aNb \subseteq I$. Hence $a \in I$ or $b \in I$, since I is a $3-$ prime ideal. Therefore I is a completely prime ideal. □

Theorem 3.7.2. *[4] Let N be a zero-symmetric near ring, Let I be a completely semiprime ideal of N. Then I is an IFP ideal.*

Proof. Assume I is a completely semiprime ideal of N and $ab \in I$ for $a, b \in N$. It is easily seen that $NI \subset I$.

Since N is a zero-symmetric, then $(ba)^2 = baba \in NIN \subseteq I$ and then $ba \in I$ since I is completely semiprime. Hence $(anb)^2 = anbanb \in NIN \subseteq I$ for all $n \in N$, whence $anb \in I$ since I is completely semiprime. Therefore I is an IFP ideal. □

Corollary 3.7.2.1. *[4] Let I be a completely prime ideal of a near ring N. Then I is an IFP ideal.*

Theorem 3.7.3. *[4] Let N be a zero-symmetric Boolean near ring, and Let I be an ideal of N. Then I is an IFP ideal.*

Proof. Assume $ab \in I$ for $a, b \in N$. Since $ba = (ba)^2 = baba \in NIN \subseteq I$, then $anb = (anb)^2 = anbanb \in NIN \subseteq I$ for all $n \in N$. therefore I is an IFP ideal of N. □

Notation:

If A and B be subsets of a near ring N, we denote the set $\left\{n \in N \middle| nB \subseteq A\right\}$ by $(A : B)$. We denote $(A : \{b\})$ by $(A : b)$.

Theorem 3.7.4. *[5] Let Nbe a near ring, the following assertions are equivalent:*

i. N has IFP.

ii. for all $n \in N$, $(0 : n)$ is an ideal of N.

iii. for all $S \subseteq N$, $(0 : S)$ is an ideal of N.

Proof. $(i) \Rightarrow (ii)$: Take $y, x \in (0 : n)$. Then $xn = 0$ and $yn = 0$ by the right distributive Law, $(x + y)n = xn + yn = 0 + 0 = 0$. Therefore, $x + y \in (0 : n)$. Clearly $(0 : n)$ is a normal subgroup of N and a left ideal of N. To show $(0 : n)$ is a right ideal of N, take $n_1 \in (0 : n)$ and $n_2 \in N$. Now $n_1n = 0$. Since N has IFP, we have $n_1mn = 0$ for all $m \in N$. In particular, $n_1n_2n = 0$, where $n_2 \in N$. This implies that $n_1n_2 \in (0 : n)$. Therefore, $(0 : n)N \subseteq N$. Hence, $(0 : n)$ is an ideal of N.

$(ii) \Rightarrow (iii)$: since $(0:S) = \bigcap_{s\in S}(0:s)$ and each $(0:s)$ is an ideal, we have that $(0:S)$ is an ideal of N.

$(iii) \Rightarrow (i)$: Assume that for all $S \subseteq N$, $(0:S)$ is an ideal of N. Let $a, b \in N$. Suppose $ab = 0$. This implies that $a \in (0:b) = (0:\{b\})$. Since $\{b\} \subseteq N$ and $(0:\{b\})$is an ideal (by assumption),we have $an \in (0:\{b\})$ for all $n \in N$. This implies $anb = 0$ for all $n \in N$. Hence, N has IFP. $\square$

Remark 3.7.2. *If N is an IFP near ring. Then for all $x \in N$, $(0:x)$ is an IFP- ideal of N.*

Definition 3.7.3. *[5] An $N-$ group G is said to be an IFP $N-$group if it satisfies the following condition:*

$$n \in N,\ g \in G,\ ng = 0 \Rightarrow nmg = 0 \textit{ for all } m \in N$$

By the definition, it is clear that every IFP near ring N is an IFP $N-$group.

Theorem 3.7.5. *[5] If G is an IFP $N-$group with a torsion-free element $a \in G$, then N is an IFP near ring.*

(Here, an element $0 \neq a \in G$ is said to be torsion-free if $n \in N,\ na = 0 \Rightarrow n = 0$)

Proof. Take $x, y \in N$ such that $xy = 0$. Now $xy = 0 \Rightarrow xya = 0$. Since G has IFP, we have $xnya = 0$ for all $n \in N$. Therefore, $xny \in (0:a) = 0$ (Since a is torsion-free). This is true for all $n \in N$. Hence, N is an IFP near ring. $\square$

Definition 3.7.4. *[5] An $N-$group is called faithful $N-$group if $(0:G) = 0$*

Theorem 3.7.6. *[5] If G is a faithful IFP $N-$group, then N is an IFP near ring.*

Proof. Suppose $x, y \in N$ such that $xy = 0$, then $xyG = 0$. Since G has IFP, we have $xnyG = 0$ for all $n \in N$. This implies that $xny \in (0 : G) = (0)$, since G is faithful. This is true for all $n \in N$. Hence, N is an IFP near ring. $\square$

Theorem 3.7.7. *[5] If G is an IFP $N-$ group and I is an ideal of N such that $I \subseteq (0 : G)$, then G is an IFP $N^{\star}-$ group where $N^{\star} = N/I$ is the quotient near ring.*

Proof. Let $n + I,\ m + I \in N/I,\ a \in G$ with $(n + I)a = 0$. This implies $na = 0$, since G has IFP, we have $nma = 0$. This implies $(n + I)(m + I)a = 0$. This is true for all $m + I \in N/I$. Hence, G is an IFP $(N/I)-$ group. $\square$

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