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TRIVIAL RING EXTENSIONS

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Dedication

To my father, mother, brothers, and sisters.

Acknowledgement

The beginning of all thanks and gratitude to God Almighty for the completion of this work.

I acknowledge Hebron University for the opportunity they gave me to get my Masters in Mathematics.

I would like to thank and appreciate the efforts made by my supervisor Dr. Ali Abdel-Mohsin Altawaiha. Indeed, his confidence prompted me to offer my best efforts to make this work a success.

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Abstract

Let R be a commutative ring and M an R-module. The trivial ring extension of R by M is the ring $R \ltimes M$ with coordinate-wise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. This construction was introduced by Nagata in 1962 in order to facilitate interaction between rings and their modules. The ring $R \ltimes M$ is also called the idealization of M over R. The trivial ring extension can be used to extend results about ideals to modules and to provide interesting examples of commutative rings with zero divisors. The main discussed results deal with how properties of $R \ltimes M$ are related to those of R and M. For example, $R \ltimes M$ is Noetherian if and only if R is Noetherian and M is finitely generated, $R \ltimes M$ is a Manis valuation ring if and only if R is a valuation ring on R_S and $M = M_S$, and $R \ltimes M$ is a Prüfer ring if and only if for each finitely generated ideal I of R with $I \cap S \neq \emptyset$, I is invertible, and $M = M_S$, where $S = R - (Z(R) \cup Z_R(M))$.

ملخص الرسالة

ليكن R حلقه تبديليه و M هو مقياس-R، فان التمديد الحلقي البديهي هو $R \ltimes M$ مع الجمع العادي للازواج المرتبه والضرب $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$، وهذا التركيب الجبري الذي عرَّفه أولاً ناجاتا في سنه 1962 من اجل استخدام التفاعل بين الحلقهR والمقياس M في نقل نتائج حول المثاليات الى المقاييس حيث أن $0 \ltimes M$ هو مثاليه في $R \ltimes M$، ومن اجل بناء امثله على حلقات تبديليه تضم قواسم صفريه.

Introduction

Throughout, all rings considered in this thesis are commutative with unity. Let R be a ring and M an R module. Then $R \ltimes M = R \oplus M$ with coordinate-wise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity called the idealization of M or the trivial ring extension of R by M. The name comes from the fact that if N is a submodule of M, then $0 \ltimes N$ is an ideal of $R \ltimes M$. This construction was first introduced, in 1962, by Nagata [21] in order to facilitate interaction between rings and their modules and also to give either examples or counterexamples of commutative rings with zero divisors. Some general references are Gilmer [13] and Kaplansky [17]. An excellent introduction to idealization and commutative rings with zero divisors can be found in Huckaba's book [14], and also D.D. Anderson and M. Winders survey paper [6].

This MS. thesis consists of four chapters. In chapter one we review some basic definitions and facts from ring and module theory that will be needed in the next chapters. Chapter two consists of two sections: section 2.1 is devoted to study the structure of the elements and the ideals of the trivial ring extension, namely, we will discuss the maximal, prime, radical, and primary ideals of $R \ltimes M$ as well as the units, idempotents, zero divisors, and nilpotents. Section 2.2 is devoted to study the interaction between the trivial ring extension and some constructions such as localization and taking the integral closure. Some of these constructions commutes with the trivial ring extension, for example, $(R \ltimes M)[x]$ is naturally isomorphic to $R[x] \ltimes M[x]$.

Chapter three consists of three sections: section 3.1 is about the transfer of the notions Noetherian and Artinian rings in trivial ring extension. Also we will use the idealization to construct a new examples of Noetherian (Artinian) rings or non-Noetherian (non-Artinian rings). In Section 3.2 we investigate the transfer the notion of Prüfer ring and some related concepts such as valuation, chained, and arithmetical rings in trivial ring extension. Section 3.3 is devoted to study the atomic rings and the ascending condition on principal ideals via trivial ring extension, and we use the idealization to give some examples of rings with zero divisors with certain factorization properties. Chapter four consists of two sections. Section 4.1 determines the structure theory for Boolean-like rings using idealization. Section 4.2 is devoted to study clean and nil-clean rings via trivial ring extension, and we use trivial ring extension to give a class of non-Boolean clean (nil-clean) rings, also we will study weakly clean (weakly nil-clean) rings via trivial ring extension.

Chapter 1

Preliminaries

In this chapter we review some basic definitions and facts from ring and module theory that will be needed in the next chapters.

Theorem 1.0.1 ([15]). *(The correspondence Theorem). If I is an ideal in a ring R, then there is a one-to-one correspondence between the set of all ideals J of R which contain I and the set of all ideals of R/I, given by $J \mapsto J/I$. Hence every ideal of R/I is of the form J/I, where J is an ideal of R which contains I.*

Definition 1.0.2 ([7]). (Prime and maximal ideals). Let I be an ideal in a ring R with $I \neq R$.

1. I is called a prime ideal if for all $a, b \in R$, $ab \in I$ implies $a \in I$ or $b \in I$.

2. I is called a maximal ideal if there is no ideal J with $I \subsetneq J \subsetneq R$.

Definition 1.0.3 ([7]). (Jacobson radical). The Jacobson radical of $J(R)$ of a ring R is defined to be the intersection of all the maximal ideals of R.

Definition 1.0.4 ([7]). Let R be a ring.

1. If I is any ideal of R, the radical of I is

$$\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$$

   If $\sqrt{I} = I$, then I is called a radical ideal.

2. The ideal $\sqrt{0} = \{a \in R \mid a^n = 0 \text{ for some } n \in \mathbb{N}\}$ is called the nilradical of R and its denoted by $nil(R)$.

Definition 1.0.5 ([7]). (Primary ideals). An ideal I in a ring R is called a primary ideal of R if $I \neq R$ and if

$$ab \in I \text{ implies } a \in I \text{ or } b \in \sqrt{I}.$$

Example 1.0.6 ([7]). The primary ideals of $\mathbb{Z}$ are $\{0\}$ and $p^n\mathbb{Z}$, where p is prime and $n \geq 1$.

Proposition 1.0.7 ([7]). Let I be a primary ideal in a ring R. Then $\sqrt{I}$ is the smallest prime ideal containing I.

Proof. See [7, Proposition 4.1]. □

Definition 1.0.8 ([7]). If I is a primary ideal of R and $P = \sqrt{I}$, then I is said to be P-primary.

Proposition 1.0.9 ([7]). The radical of an ideal I is the intersection of the prime ideals which contain I.

Proof. See [7, Proposition 1.14]. □

Definition 1.0.10 ([7]). (Modules). Let R be a ring. An R-module is a set M together with two operations

$$+ : M \times M \to M \qquad \text{and} \qquad \cdot : R \times M \to M$$

(an "addition" in M and a "scalar multiplication" with elements of R) such that for all $m, n \in M$ and $a, b \in R$ we have:

1. $(M, +)$ is an Abelian group.
2. $(a + b) \cdot m = a \cdot m + b \cdot m$ and $a \cdot (m + n) = a \cdot m + a \cdot n$.
3. $(ab) \cdot m = a \cdot (b \cdot m)$.
4. $1 \cdot m = m$.

An R-module M is also called a module over R.

Example 1.0.11 ([7]).

1. If R is a ring and I is an ideal of R, then I, R, and R/I are modules over R.
2. If F is a field, then an F-module is the same as an F-vector space.

3. A $\mathbb{Z}$-module is just the same as an Abelian group.

Definition 1.0.12 ([15]). (Submodules and quotients). Let R be a ring and M an R-module.

1. A submodule of M is a nonempty subset $N \subset M$ satisfying $m + n \in N$ and $am \in N$ for all $m, n \in N$ and $a \in R$.

2. If N is a submodule of M, then the set

$$M/N = \{x + N \mid x \in M\}$$

of equivalence class modulo N is again an R-module called the quotient module of M modulo N.

Definition 1.0.13 ([15]). Let M be an R-module.

1. For any subset $S \subset M$ the set

$$RS = \{a_1 m_1 + \cdots + a_n m_n \mid n \in \mathbb{N}, a_i \in R, m_i \in S\} \subset M$$

of all finite R-linear combinations of elements of S is the smallest submodule of M that contains S. If $S = \{m_1, ..., m_n\}$ is finite, we write $RS = Rm_1 + \cdots + Rm_n$.

2. The module M is called finitely generated if $M = RS$ for a finite set $S \subset M$, and its called cyclic if $M = Rm$ for some $m \in M$.

Definition 1.0.14 ([15]). (Primary submodules). Let R be a ring, M an R-module.

A proper submodule N of M is primary provided that

$$r \in R,\ m \notin N \text{ and } rm \in N \text{ implies } r^n M \subseteq N \text{ for some } n \in \mathbb{N}.$$

Theorem 1.0.15 ([15]). *Let R be a ring and N a primary submodule of an R-module M. Then $(N : M) = \{r \in R \mid rM \subseteq N\}$ is a primary ideal in R.*

Definition 1.0.16 ([15]). Let P be a prime ideal in a ring R and M an R-module. A primary submodule N of M is said to be a P-primary submodule of M if $P = \sqrt{(N : M)} = \{r \in R \mid r^n M \subseteq N \text{ for some } n \in \mathbb{N}\}$.

Proof. See [15, Theorem 3.2]. ∎

Definition 1.0.17 ([7]). (R-module homomorphisms). Let M and N be R-modules.

1. An R-module homomorphism from M to N is a map $\varphi : M \to N$ such that

$$\varphi(m+n) = \varphi(m) + \varphi(n) \qquad \varphi(am) = a\varphi(m)$$

   for all $m, n \in M$ and $a \in R$.

2. An R-module homomorphism $\varphi : M \to N$ of R-modules is called an isomorphism if it is bijective. In this case, the map $\varphi^{-1} : N \to M$ is a homomorphism of R-modules. We call M and N isomorphic (written $M \cong N$) if there is an isomorphism between them.

Remark 1.0.18 ([7]). (Images and kernels of R-module homomorphisms). Let $\varphi : M \to N$ be a homomorphism of R-modules.

1. The kernel of φ is the set

$$\ker \varphi = \{x \in M \mid \varphi(x) = 0\}$$

   and is a submodule of M.

2. The image of φ is the set

$$Im(\varphi) = \varphi(M)$$

   and is a submodule of N.

Proposition 1.0.19 ([7]). (Isomorphism theorems).

1. For any homomorphism $\varphi : M \to N$ of R-modules, there is an isomorphism $\psi : M/\ker\varphi \to Im(\varphi)$ given by

$$\psi(m + \ker\varphi) = \varphi(m).$$

2. For R-submodules $N' \subseteq N \subseteq M$ we have

$$\frac{M/N'}{N/N'} \cong \frac{M}{N}.$$

3. For two submodules N, N' of an R-module M we have

$$(N + N')/N' \cong N/(N \cap N').$$

Definition 1.0.20 ([15]). Let I be an ideal in a ring R and let M be an R-module. We set

$$IM = \langle \{am \mid a \in I, m \in M\} \rangle = \{a_1 m_1 + \cdots + a_n m_n \mid n \in \mathbb{N}, a_i \in I, m_i \in M\}.$$

Remark 1.0.21. If M is an R-module and I is an ideal of R, then IM is a submodule of M, and M/IM is an R/I-module with scalar multiplication $(r+I)(m+IM) = rm + IM$.

Definition 1.0.22 ([7]). (localization of rings). Let R be a ring.

1. A subset $S \subset R$ is called multiplicatively closed if $1 \in S$, $0 \notin S$, and $ab \in S$ for all $a, b \in S$.

2. Let $S \subset R$ be a multiplicatively closed set. Then

$$(a, s) \sim (b, t) \text{ if and only if there is an element } u \in S \text{ such that } u(at - bs) = 0$$

is an equivalence relation on $R \times S$. We denote the equivalence class of a pair $(a, s) \in R \times S$ by $\frac{a}{s}$. The set of all equivalence classes

$$R_S = \{\frac{a}{s} \mid a \in R, s \in S\}$$

is called the localization of R at the multiplicatively closed set S.

Lemma 1.0.23. *Let R be a ring and $S \subset R$ a multiplicatively closed set. The the localization R_S of R at S is a ring together with the addition and multiplication*

$$\tfrac{a}{s} + \tfrac{b}{t} = \tfrac{at+sb}{st} \text{ and } \tfrac{a}{s}\tfrac{b}{t} = \tfrac{ab}{st}.$$

for all $a, b \in R$ and $s, t \in S$.

Remark 1.0.24. Let S be a multiplicatively closed subset of a ring R. There is a ring homomorphism $\varphi : R \to R_S$, $a \mapsto \frac{a}{1}$. However, φ is only injective if S does not contain zero divisors, as by definition $\frac{a}{1} = \frac{0}{1}$ implies the existence of an element $u \in S$ with $u(a \cdot 1 + 0 \cdot 1) = ua = 0$.

Example 1.0.25. (Standard examples of localization). Let R be a ring.

1. Let $S = R - Z(R)$, where $Z(R) = \{a \in R \mid ab = 0 \text{ for some } 0 \neq b \in R\}$ is the set of all zero divisors of R. The S is a multiplicatively closed. In this case the localization R_S of R at S is called the total quotient ring of R, denoted by $T(R)$. Since S does not contain zero divisors, the map $\varphi : R \to T(R)$, $a \mapsto \frac{a}{1}$ is injective. Of particular importance is the case when R is

an integral domain. Then $S = R - \{0\}$, and every nonzero element $\frac{a}{s}$ is a unit in R_S, with inverse $\frac{s}{a}$. Hence R_S is a field called the quotient field of R, denoted by $Quot(R)$. So if R is an integral domain, then $T(R) = Quot(R)$.

2. Let P be a prime ideal of R. Then $S = R - P$ is multiplicatively closed since $1 \notin P$, $0 \in P$, and for $a, b \notin P$, we have $ab \notin P$. The localization R_S of R is usually denoted by R_P and called the localization of R at the prime ideal P.

Remark 1.0.26. 1. If R is a ring, then every element in the total quotient ring $T(R)$ of R is either a zero divisor or a unit.

2. If every element in a ring R is either a zero divisor or a unit, then R is a total quotient ring.

Proof. 1. By the last example, $T(R) = \{\frac{a}{b} \mid a \in R, b \in R - Z(R)\}$. If $\frac{a}{b} \in T(R)$ is not a zero divisor, then a is not a zero divisor of R, that is, $a \in R - Z(R)$. So $\frac{b}{a} \in T(R)$ and $\frac{a}{b} \cdot \frac{b}{a} = 1$. This means that $\frac{a}{b}$ is a unit of $T(R)$.

2. Let R be a ring. If every element in R is either a zero divisor or a unit, then $R - Z(R) = U(R)$. So $R = T(R)$ is a total quotient ring.

□

Example 1.0.27. (Some localizations of $\mathbb{Z}$). Consider the ring of integers $\mathbb{Z}$. The localization of $\mathbb{Z}$ at $\mathbb{Z} - \{0\}$ is the quotient field $Quot(\mathbb{Z}) = \mathbb{Q}$. If $p \in \mathbb{Z}$ is a prime number, then the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ is

$$\mathbb{Z}_{p\mathbb{Z}} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, p \nmid b\}$$

Proposition 1.0.28 ([15])**.** (Ideals in localizations). Let S be a multiplicatively closed subset of a ring R. Then

1. The proper ideals of the ring R_S are of the form $IR_S = I_S = \{\frac{a}{s} \mid a \in I, s \in S\}$ with I is an ideal of R and $I \cap S = \emptyset$.

2. The prime ideals in R_S are of the form $PR_S = P_S$ where P is a prime ideal of R and $P \cap S = \emptyset$.

Definition 1.0.29 ([7])**.** (Local rings) A ring R is called local if it has exactly one maximal ideal.

Example 1.0.30 ([7, Example 1, page 38])**.** Let p be a prime ideal in a ring R. Then R_P is local with maximal ideal $\mathrm{m} = PR_P = \{\frac{a}{s} \mid a \in P, s \notin P\}$.

Definition 1.0.31 ([7])**.** (Saturations). For a multiplicatively closed subset S of a ring R we call

$$\overline{S} = \{s \in R \mid as \in S \text{ for some } a \in R\}$$

the saturation of S.

Remark 1.0.32. Let S, T be two multiplicatively closed subsets of a ring R.

1. $S \subseteq \overline{S}$.

2. If $S \subseteq T$, then $\overline{S} \subseteq \overline{T}$.

Proof. 1. If $s \in S$, then $s \cdot 1 \in S$. So $s \in \overline{S}$.

2. If $x \in \overline{S}$, then $xy \in S$ for some $y \in R$. So $xy \in T$ for some $y \in R$. This means that $x \in \overline{T}$.

$\square$

Definition 1.0.33 ([7])**.** A multiplicatively closed subset S of a ring R is called saturated if $\overline{S} = S$.

Example 1.0.34. Let P be a prime ideal of a ring R, then $S = R - P$ is saturated. Indeed, if $s \in \overline{S}$, then $sa \in S = R - P$ for some $a \in R$, so $s \notin P$, and hence $s \in R - P = S$.

Remark 1.0.35 ([7])**.** Let S be a multiplicatively closed subset of R. Then S is saturated if and only if $S = R - \bigcup_{i \in I} P_i$ where $\{P_i\}_{i \in I}$ is the set of prime ideals of R such that $P_i \cap S = \emptyset$ for each $i \in I$.

Proof. ($\Rightarrow$). Suppose that S is saturated. Let $x \notin S$. Then $Rx \cap S = \emptyset$. By Zorn's Lemma, we can find a prime ideal P with $Rx \subseteq P$ such that $P \cap S = \emptyset$. So $x \in \bigcup_{i \in I} P_i$ where P_i is a prime ideal of R with $P_i \cap S = \emptyset$ for each $i \in I$. Hence $R - S \subseteq \bigcup_{i \in I} P_i$, or equivalently, $R - \bigcup_{i \in I} P_i \subseteq S$. For the other inclusion, let $x \in \bigcup_{i \in I} P_i$. Then $x \in P_i$ for some prime ideal P_i of R with $P_i \cap S = \emptyset$. So $x \notin S$, or $x \in R - S$. It follows that $\bigcup_{i \in I} P_i \subseteq R - S$, or equivalently, $S \subseteq R - \bigcup_{i \in I} P_i$.
($\Leftarrow$). Let $x \in \overline{S}$, then $xy \in S$ for some $y \in R$. Since $xy \in S$, then $xy \notin P_i$ for each i. If $x \notin S$, then $x \in P_i$ for some i, but then $xy \in P_i$ for some i, a contradiction. So, we have $x \in S$, hence $S = \overline{S}$ and S is saturated. $\square$

Definition 1.0.36 ([7])**.** (localization of modules). Let S be a multiplicatively closed subset of a ring R, and let M be an R-module. Then

$$(m, s) \sim (n, t) \text{ if and only if there is an element } u \in S \text{ such that } u(tm - sn) = 0$$

is an equivalence relation on $M \times S$. We denote the equivalence class of a pair $(m, s) \in M \times S$ by $\frac{m}{s}$. The set of all equivalence classes

$$M_S = \{\frac{m}{s} \mid m \in M, s \in S\}$$

is called the localization of M at S. It is an R_S-module together with the addition and scalar multiplication

$$\tfrac{m}{s} + \tfrac{n}{t} = \tfrac{tm+sn}{st} \text{ and } \tfrac{a}{s}\tfrac{m}{t} = \tfrac{am}{st}$$

for all $a \in R$, $m, n \in M$, and $s, t \in S$.

In the case when $S = R - P$ for a prime ideal P of R, we will write M_S as M_P.

Definition 1.0.37 ([18])**.** (Integrally closed). Let R be a subring of a ring T and let $a \in T$. If there are elements $b_0, ..., b_{n-1} \in R$ such that $b_0 + b_1 a + \cdots + b_{n-1}a^{n-1} + a^n = 0$, we say that a is integral over R. If the elements of R are the only elements of T which are integral over R, we say that R is integrally closed in T. If R is integrally closed in its total quotient ring, we say simply that R is integrally closed.

Proposition 1.0.38 ([18])**.** Let R be a subring of a ring T and let

$$R' = \{a \in T \mid a \text{ is integral over } R\}.$$

Then R' is a subring of T and $R \subseteq R'$.

Proof. See [18, Proposition 4.3]. $\square$

In the notations of this proposition, R' is called the integral closure of R in T.

Chapter 2

Properties of trivial ring extension

Let R be a ring and M an R-module. The trivial ring extension of R by M is the ring $R \ltimes M = \{(r, m) \mid r \in R \text{ and } m \in M\}$ where addition is given by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication is given by $(r, m)(s, n) = (rs, rn + sm)$.

Remark 2.0.1 ([6])**.** Let R be a commutative ring with unity 1 and M an R-module.

1. $R \ltimes M$ is a commutative ring with unity $(1, 0)$.

2. R naturally embeds into $R \ltimes M$ via $r \mapsto (r, 0)$.

3. If N is a submodule of M, then $0 \ltimes N$ is an ideal of $R \ltimes M$.

4. $0 \ltimes M$ is an ideal of $R \ltimes M$ with $(0 \ltimes M)^2 = 0$.

5. M and $0 \ltimes M$ are isomorphic as R-modules.

Proof. 1. Let $(a, m), (b, n) \in R \ltimes M$. Then

$$(a, m)(b, n) = (ab, an + bm) = (ba, bm + an) = (b, n)(a, m).$$

So $R \ltimes M$ is commutative. $(1, 0)$ is the unity of $R \ltimes M$ because $(r, m)(1, 0) = (r1, r0 + 1m) = (r, m)$ for all $(r, m) \in R \ltimes M$.

2. Since for each $a, b \in R$, we have $(a + b, 0) = (a, 0) + (b, 0)$, $(ab, 0) = (a, 0)(b, 0)$, and $(a, 0) = (0, 0) \Leftrightarrow a = 0$, then the map $R \to R \ltimes M$ $(r \mapsto (r, 0))$ is an injective ring homomorphism.

3. Assume that N is a submodule of M. If $n_1, n_2 \in N$, then $n_1 + n_2 \in N$. So $(0, n_1) + (0, n_2) = (0, n_1 + n_2) \in 0 \ltimes N$. Next, if $(r, m) \in R \ltimes M$ and $(0, n) \in 0 \ltimes N$, we have $rn \in N$. So $(r, m)(0, n) = (0, rn) \in 0 \ltimes N$. Hence $0 \ltimes N$ is an ideal of R.

4. By (2), $0 \ltimes M$ is an ideal of $R \ltimes M$. Now, if $m_1, m_2 \in M$, then $(0, m_1)(0, m_2) = (0, 0)$. So $(0 \ltimes M)^2 = 0$.

5. Since by (2), the map $R \to R \ltimes M$ $(r \mapsto (r, 0))$ is a ring homomorphism and $0 \ltimes M$ is an $R \ltimes M$-module, then $0 \ltimes M$ is an R-module with scalar multiplication $r(0, m) = (r, 0)(0, m) = (0, rm)$. Now define $f : M \to 0 \ltimes M$ by $f(m) = (0, m)$. Then clearly, f is bijection. Let $m_1, m_2 \in M$. Since $(0, m_1 + m_2) = (0, m_1) + (0, m_2)$, then $f(m_1 + m_2) = f(m_1) + f(m_2)$. Next, let $r \in R$ and $m \in M$, then $f(rm) = (0, rm) = (r, 0)(0, m) = rf(m)$. Thus, f is an R-module isomorphism.

□

2.1 Ideals and distinguished elements of $R \ltimes M$

This section is devoted to study the structure of the elements and the ideals of the trivial ring extension. Namely, we will see the maximal, prime, radical, and primary ideals of $R \ltimes M$. Regarding the elements, we will discuss the units, idempotents, zero divisors, and nilpotents of $R \ltimes M$.

The following result describes a special kind of ideals of $R \ltimes M$, those ideals that could be constructed from an ideal of R and a submodule of M.

Theorem 2.1.1 ([6]). *Let R be a ring, I an ideal of R, M an R-module and N a submodule of M. Then:*

1. *$I \ltimes N$ is an ideal of $R \ltimes M$ if and only if $IM \subseteq N$.*

2. *If $I \ltimes N$ is an ideal of $R \ltimes M$, then $\frac{M}{N}$ is an $\frac{R}{I}$-module and $\frac{R \ltimes M}{I \ltimes N} \cong \frac{R}{I} \ltimes \frac{M}{N}$.*

Proof. (1) and (2). Suppose that $I \ltimes N$ is an ideal of $R \ltimes M$. Then

$$I \ltimes N = (R \ltimes M)(I \ltimes N) = RI \ltimes (RN + IM) = I \ltimes (N + IM).$$

So $N = N + IM$ and hence $IM \subseteq N$. Conversely, suppose that $IM \subseteq N$. We know that $\frac{M}{IM}$ is an $\frac{R}{I}$-module with scalar multiplication $(r + I)(m + IM) = rm + IM$. So $\frac{M}{N} \cong \frac{M/IM}{N/IM}$ is an $\frac{R}{I}$-module with scalar multiplication $(r+I)(m+N) = rm+N$. Next, define

$$\varphi : R \ltimes M \to \frac{R}{I} \ltimes \frac{M}{N}$$

by $\varphi((r,m)) = (r+I, m+N)$, then we know that φ is a surjective group homomorphism with $\ker\varphi = I \ltimes N$. Now, let $(r_1,m_1),(r_2,m_2) \in R \ltimes M$. Then

$$\begin{aligned}
\varphi((r_1,m_1)(r_2,m_2)) &= \varphi(r_1r_2, r_1m_2 + r_2m_1) \\
&= (r_1r_2 + I, r_1m_2 + r_2m_1 + N) \\
&= ((r_1+I)(r_2+I), (r_1+I)(m_2+N) + (r_2+I)(m_1+N)) \\
&= (r_1+I, m_1+N)(r_2+I, m_2+N) \\
&= \varphi((r_1,m_1))\varphi((r_2,m_2)).
\end{aligned}$$

So φ is a ring homomorphism. Hence $I \ltimes N = \ker\varphi$ is an ideal of $R \ltimes M$. The First Isomorphism Theorem gives that

$$\frac{R \ltimes M}{I \ltimes N} \cong \frac{R}{I} \ltimes \frac{M}{N}.$$

□

The following is an illustrative example for Theorem 2.1.1

Example 2.1.2. Let $R = \mathbb{Z}$, $M = \mathbb{Z}/12\mathbb{Z}$, and $N = 6\mathbb{Z}/12\mathbb{Z}$. If $I = 6\mathbb{Z}$, then $IM = (6\mathbb{Z})(\mathbb{Z}/12\mathbb{Z}) = 6\mathbb{Z}/12\mathbb{Z} = N$ and so by Theorem 2.1.1, $I \ltimes N$ is an ideal of $R \ltimes M$. But if $I = 4\mathbb{Z}$, then $IM = (4\mathbb{Z})(\mathbb{Z}/12\mathbb{Z}) = 4\mathbb{Z}/12\mathbb{Z} \nsubseteq 6\mathbb{Z}/12\mathbb{Z} = N$ (as $4\mathbb{Z} \nsubseteq 6\mathbb{Z}$). Hence by Theorem 2.1.1, $I \ltimes N$ is not an ideal of $R \ltimes M$.

Remark 2.1.3. Theorem 2.1.1 does not describe all the ideals of $R \ltimes M$ in general. In the following example we provide a ring R, an R-module M, and an ideal of $R \ltimes M$ which is not in the form $I \ltimes N$.

Example 2.1.4 ([6])**.** Let $R = \mathbb{Z}_4$, $M = \mathbb{Z}_2$, and $J = (R \ltimes M)(\bar{2},\bar{1}) = \{(\bar{0},\bar{0}),(\bar{2},\bar{1})\}$. Then J is an ideal of $R \ltimes M$ such that J does not have the form $I \ltimes N$. For if $J = (R \ltimes M)(\bar{2},\bar{1}) = I \ltimes N$, then $\bar{2} \in I$ and $\bar{1} \in N$. But since $I \ltimes N$ has two elements, then either $I = 0$ or $N = 0$, a contradiction.

The following is a straightforward corollary of Theorem 2.1.1. It takes its importance from the fact that $0 \ltimes M \cong M$ as R-modules.

Corollary 2.1.5 ([6]). *Let R be a ring, I an ideal of R, and M an R-module. Then*

1. $\frac{R\ltimes M}{I\ltimes M} \cong \frac{R}{I}$.

2. $\frac{R\ltimes M}{0\ltimes M} \cong R$.

3. *The ideals of $R \ltimes M$ containing $0 \ltimes M$ are of the form $J \ltimes M$ for some ideal J of R.*

Proof. 1. This follows by Theorem 2.1.1 (2) with $N = M$.

2. This follows by Theorem 2.1.1 (2) with $I = 0$ and $N = M$.

3. This follows by part (2) and the Correspondence Theorem.

□

For a ring R, let $Spec(R)$ denote the set of all prime ideals of R and $Max(R)$ denote the set of all maximal ideals of R.

The next result characterizes the prime, maximal, and radical ideals of $R \ltimes M$.

Theorem 2.1.6 ([14]). *Let R be a ring and M an R-module. Then*

1. *The prime ideals of $R \ltimes M$ have the form $P \ltimes M$ where P is a prime ideal of R. That is, $Spec(R \ltimes M) = \{P \ltimes M \mid P \in Spec(R)\}$.*

2. *The maximal ideals of $R \ltimes M$ have the form $\mathrm{m} \ltimes M$ where m is a maximal ideal of R. That is, $Max(R \ltimes M) = \{\mathrm{m} \ltimes M \mid \mathrm{m} \in Max(R)\}$.*

3. *The radical ideals of $R \ltimes M$ have the form $I \ltimes M$ where I is a radical ideal of R.*

Proof. 1. Let A be a prime ideal of $R \ltimes M$. Then, $(0 \ltimes M)^2 = 0 \subseteq A$ implies $0 \ltimes M \subseteq A$. So by Corollary 2.1.5 (2), $A = J \ltimes M$ for some ideal J of R. Since $\frac{R\ltimes M}{J\ltimes M} \cong \frac{R}{J}$, then we have $A = J \ltimes M$ is a prime ideal of $R \ltimes M$ if and only if J is a prime ideal of R.

2. The proof of this part is similar to the proof of part (1) using the fact that every maximal is prime.

3. Let A be a radical ideal of $R \ltimes M$. Then

$$A = \sqrt{A} = \cap\{Q \mid Q \in Spec(R \ltimes M),\ A \subseteq Q\}.$$

By part (1), $0 \ltimes M \subseteq A$ and hence by Corollary 2.1.5 (2), $A = J \ltimes M$ for some ideal J of R. Since $\frac{R \ltimes M}{J \ltimes M} \cong \frac{R}{J}$, then we have $A = J \ltimes M$ is a radical ideal of $R \ltimes M$ if and only if J is a radical ideal of R.

□

All the following facts are consequences of Theorem 2.1.6.

Corollary 2.1.7 ([6])**.** *Let R be a ring and M an R-module. Then*

1. *$R \ltimes M$ is local if and only if R is local.*

2. *The Jacobson radical of $R \ltimes M$ is $J(R \ltimes M) = J(R) \ltimes M$.*

3. *If J is an ideal of $R \ltimes M$, then $\sqrt{J} = \sqrt{I} \ltimes M$ where $I = \{r \in R \mid (r, b) \in J \text{ for some } b \in M\}$ is an ideal of R. In particular, If I is an ideal of R and N is a submodule of M, then $\sqrt{I \ltimes N} = \sqrt{I} \ltimes M$.*

Proof. 1. By Theorem 2.1.6 (2), the map $Max(R) \to Max(R \ltimes M)$ $(\mathrm{m} \mapsto \mathrm{m} \ltimes M)$ is bijection. So, $R \ltimes M$ is local if and only if $|Max(R \ltimes M)| = 1$ if and only if $|Max(R)| = 1$ if and only if R is local.

2. By Theorem 2.1.6 (2), $Max(R \ltimes M) = \{\mathrm{m} \ltimes M \mid \mathrm{m} \in Max(R)\}$. So

$$J(R \ltimes M) = \bigcap_{\mathrm{m} \in Max(R)} (\mathrm{m} \ltimes M) = (\bigcap_{\mathrm{m} \in Max(R)} \mathrm{m}) \ltimes M = J(R) \ltimes M.$$

3. Suppose that J is an ideal of $R \ltimes M$. Then $\sqrt{J}$ is a radical ideal of $R \ltimes M$. So by Theorem 2.1.6 (3), $\sqrt{J} = K \ltimes M$ for some radical ideal K of R. Let $I = \{r \in R \mid (r, b) \in J \text{ for some } b \in M\}$. Consider the surjective ring homomorphism $\varphi : R \ltimes M \to R$, $\varphi((r, m)) = r$. Then by definition of I, $I = \varphi(J)$. Since J is an ideal of $R \ltimes M$, I is an ideal of R. We claim that $K = \sqrt{I}$. Now, let $x \in K$, then $(x, 0) \in K \ltimes M = \sqrt{J}$, so $(x^n, 0) = (x, 0)^n \in J$ for some $n \in \mathbb{N}$, hence $x^n \in I$ for some $n \in \mathbb{N}$ and this means $x \in \sqrt{I}$. Conversely, let $x \in \sqrt{I}$, then $x^n \in I$ for some $n \in \mathbb{N}$, so $(x^n, b) \in J$ for some $b \in M$. Since $J \subseteq \sqrt{J}$, then $(x^n, b) \in \sqrt{J} = K \ltimes M$. So $x^n \in K$ and hence $x \in \sqrt{K}$, but since K is radical, $\sqrt{K} = K$. So $x \in K$. Thus, $x \in K$ if and only if $x \in \sqrt{I}$, this means $K = \sqrt{I}$. Therefore, $\sqrt{J} = \sqrt{I} \ltimes M$.
In particular, if $J = I \ltimes N$ where I is an ideal of R and N is a submodule of M. Then by

the first paragraph of this part, we have

$$\sqrt{J} = \sqrt{\varphi(J)} \ltimes M = \sqrt{\varphi(I \ltimes N)} \ltimes M = \sqrt{I} \ltimes M.$$

$\square$

Recall that the *Krull dimension* of a ring R, denoted by $\dim R$ is the supremum length n of a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of prime ideals of R, and if there is no upper bound on the length of such a chain, then $\dim R = \infty$. Hence by Theorem 2.1.6 (1), $\dim R \ltimes M = \dim R$.

The following result characterizes the primary ideals of $R \ltimes M$.

Theorem 2.1.8 ([14])**.** *Let R be a ring, I an ideal of R, M an R-module, and N a submodule of M. Then $I \ltimes N$ is primary if and only if either*

1. *$N = M$ and I is a primary ideal of R or*
2. *$N \subsetneq M$, $IM \subseteq N$, and I and N are P-primary where $P = \sqrt{I}$.*

In either case, $I \ltimes N$ is $\sqrt{I} \ltimes M$-primary.

Proof. If $N = M$, then since $\frac{R \ltimes M}{I \ltimes M} \cong \frac{R}{I}$, we have $I \ltimes M$ is primary if and only if I is primary. So assume that $N \subsetneq M$. For $I \ltimes N$ to be an ideal of $R \ltimes M$, we must have $IM \subseteq N$. Now we prove that $I \ltimes N$ is a primary ideal of $R \ltimes M$ if and only if I is a P-primary ideal of R and N is a P-primary submodule of M where $P = \sqrt{I}$. First, suppose that $I \ltimes N$ is a primary ideal of $R \ltimes M$. Let $a, b \in R$ and assume that $ab \in I$. Then $(a, 0)(b, 0) = (ab, 0) \in I \ltimes N$. So either $(a, 0) \in I \ltimes N$ or $(b, 0) \in \sqrt{I \ltimes N} = \sqrt{I} \ltimes M$. Hence $a \in I$ or $b \in \sqrt{I} = P$. So I is a P-primary ideal of R. Next, let $r \in R$ and $m \in M$ and assume that $rm \in N$. Then $(r, 0)(0, m) = (0, rm) \in I \ltimes N$. So either $(0, m) \in I \ltimes N$ or $(r, 0) \in \sqrt{I \ltimes N} = \sqrt{I} \ltimes M$. Hence $m \in N$ or $r \in \sqrt{I}$. Since $IM \subseteq N$, $I \subseteq (N : M)$, so $r \in \sqrt{I} \subseteq \sqrt{(N : M)}$. Hence N is a primary submodule of M. We claim that $\sqrt{I} = \sqrt{(N : M)}$. First, take $m \in M - N$. Now, let $r \in \sqrt{(N : M)}$, so $r^n M \subseteq N$, but then $r^n m \in N$. So $r^n \in \sqrt{I}$ and hence $r \in \sqrt{I}$. It follows that N is a P-primary submodule of M. Conversely, suppose that I is a P-primary ideal of R and N is a P-primary submodule of M where $P = \sqrt{I}$. Let $(a, m), (b, n) \in R \ltimes M$ and assume that $(a, m)(b, n) \in I \ltimes N$. Then $ab \in I$ and $an + bm \in N$. Now since I is a P-primary ideal of R, then either $a \in I$ or $b \in \sqrt{I}$. Case 1: If $a \in I$. Then $an \in IM \subseteq N$. So $bm = (an + bm) - an \in N$. Since N is a P-primary submodule of

M, then either $m \in N$ or $b \in P = \sqrt{I}$. So $(a,m) \in I \ltimes N$ or $(b,n) \in \sqrt{I} \ltimes M = \sqrt{I \ltimes N}$. Case 2: If $b \in \sqrt{I}$. Then $(b,n) \in \sqrt{I} \ltimes M = \sqrt{I \ltimes N}$. Hence in either case, $I \ltimes N$ is a primary ideal of $R \ltimes M$. The last statement follows since $\sqrt{I \ltimes N} = \sqrt{I} \ltimes M$. $\square$

The following is an illustrative example for Theorem 2.1.8.

Example 2.1.9. Let p be a prime number. Then for all $n \in \mathbb{N}$, $p^n\mathbb{Z}$ is a $p\mathbb{Z}$-primary ideal of $\mathbb{Z}$ and $(p\mathbb{Z})[x]$ is a $p\mathbb{Z}$-primary submodule of $\mathbb{Z}[x]$. So by Theorem 2.1.8, $p^n\mathbb{Z} \ltimes (p\mathbb{Z})[x]$ is a primary ideal of $\mathbb{Z} \ltimes \mathbb{Z}[x]$ for all $n \in \mathbb{N}$.

Let R be a ring and M an R-module. Notice that for $a \in R$ and $m \in M$, we have always that $(R \ltimes M)(a,m) \subseteq Ra \ltimes (Rm + aM)$, but the reverse inclusion is not always true. For example, if $R = \mathbb{Z}_4$ and $M = \mathbb{Z}_2$, then $R\bar{2} \ltimes (R\bar{1} + \bar{2}M) = R\bar{2} \ltimes R\bar{1} = R\bar{2} \ltimes M \neq (R \ltimes M)(\bar{2}, \bar{1})$ (see Example 2.1.4). In the next theorem we provide some equivalent conditions that makes the reverse inclusion true.

Theorem 2.1.10 ([6])**.** *Let R be a ring, M an R-module, and $(R \ltimes M)(a,m)$ a principal ideal of $R \ltimes M$. Then the following conditions are equivalent:*

1. $(R \ltimes M)(a,m) = Ra \ltimes (Rm + aM)$.

2. $(a,0) \in (R \ltimes M)(a,m)$.

3. *There is $x \in R$ such that $xa = a$ and $xm \in aM$.*

Proof. (1) $\Rightarrow$ (2). Assume that $(R \ltimes M)(a,m) = Ra \ltimes (Rm + aM)$. Then since $(a,0) \in Ra \ltimes (Rm + aM)$, we have $(a,0) \in (R \ltimes M)(a,m)$.

(2) $\Rightarrow$ (3). Assume that $(a,0) \in (R \ltimes M)(a,m)$. Then $(a,0) = (x,n)(a,m)$ for some $(x,n) \in R \ltimes M$. Now $(a,0) = (x,n)(a,m)$ if and only if $a = xa$ and $0 = xm + an$ if and only if $xa = a$ and $xm = a(-n) \in aM$. Hence, there is $x \in R$ such that $xa = a$ and $xm \in aM$.

(3) $\Rightarrow$ (1). Suppose that there is $x \in R$ such that $xa = a$ and $xm \in aM$. So $xm = am'$ for some $m' \in M$. Let $r, s \in R$ and $n \in M$. Then

$$
\begin{aligned}
(ra, sm + an) &= (rxa, sm + xan) \\
&= (rxa, xan) + (0, sm) \\
&= (r, n)(xa, 0) + (s, 0)(0, m) \\
&= (r, n)(xa, xm - am') + (s, 0)[(a, m) - (a, 0)] \\
&= (r, n)(x, -m')(a, m) + (s, 0)(a, m) - (s, 0)(x, -m')(a, m) \\
&\in (R \ltimes M)(a, m).
\end{aligned}
$$

Hence $Ra \ltimes (Rm + aM) \subseteq (R \ltimes M)(a, m)$. Since the other inclusion is always true, we have $(R \ltimes M)(a, m) = Ra \ltimes (Rm + aM)$. $\square$

The following are illustrative examples for Theorem 2.1.10.

Example 2.1.11.

1. Let $R = \mathbb{Z} \ltimes \mathbb{Z}$. Consider $(2, 2) \in R$. Then $(2, 0) = (1, -1)(2, 2) \in R(2, 2)$. So by Theorem 2.1.10, $R(2, 2) = 2\mathbb{Z} \ltimes (2\mathbb{Z} + 2\mathbb{Z}) = 2\mathbb{Z} \ltimes 2\mathbb{Z}$.

2. Let $R = \mathbb{Z} \ltimes 2\mathbb{Z}$. Consider $(2, 2) \in R$. Then $(2, 0) \notin R(2, 2)$. For if $(2, 0) \in R(2, 2)$, then $(2, 0) = (a, 2b)(2, 2) = (2a, 2a + 4b)$ for some $a, b \in \mathbb{Z}$. But then $a = 1$ and $b = -1/2$ and so $b \notin \mathbb{Z}$, a contradiction (since $b \in \mathbb{Z}$). Hence by Theorem 2.1.10, $R(2, 2) \neq 2\mathbb{Z} \ltimes (2\mathbb{Z} + 4\mathbb{Z}) = 2\mathbb{Z} \ltimes 2\mathbb{Z}$.

Recall that a module M over a ring R is called *divisible* if $rM = M$ for all $r \in R - Z(R)$, where $Z(R)$ is the set of all zero divisors of R.

Corollary 2.1.12. *Let R be an integral domain and M an R-module. Then $(R \ltimes M)(a, m) = Ra \ltimes (Rm + aM)$ for all $a \in R$ and $m \in M$ if and only if M is divisible.*

Proof. $(\Rightarrow)$. Let $0 \neq a \in R$ and $m \in M$. Then by hypothesis, $(R \ltimes M)(a, m) = Ra \ltimes (Rm + aM)$. So by Theorem 2.1.10, there is an element $x \in R$ such that $xa = a$ and $xm \in aM$. But since R is an integral domain and $a \neq 0$, then $x = 1$. So $m = 1m = xm \in aM$. Hence $M = aM$. Therefore, M is divisible.

$(\Leftarrow)$. Let $a \in R$ and $m \in M$. If $a = 0$, then $(a, 0) = (0, 0) \in (R \ltimes M)(a, m)$. So by Theorem 2.1.10, $(R \ltimes M)(a, m) = Ra \ltimes (Rm + aM)$. If $a \neq 0$, then M is divisible gives that $M = aM$. So $1 \in R$ satisfies $1a = a$ and $1m \in aM$. Hence by Theorem 2.1.8, $(R \ltimes M)(a, m) = Ra \ltimes (Rm + aM)$. $\square$

Definition 2.1.13. Let R be a ring, and let I, J be two ideals of R. We say that I and J are comparable if either $I \subseteq J$ or $J \subseteq I$.

The following theorem determines all the ideals of $R \ltimes M$ when R is an integral domain and M is a divisible R-module.

Theorem 2.1.14 ([6])**.** *Let R be an integral domain and M an R-module. Then the following conditions are equivalent:*

1. *Every ideal of $R \ltimes M$ is comparable to $0 \ltimes M$.*

2. *Every ideal of $R \ltimes M$ has the form $I \ltimes M$ or $0 \ltimes N$ for some ideal I of R or submodule N of M.*

3. *M is divisible.*

Proof. (1) $\Rightarrow$ (2). Let J be an ideal of $R \ltimes M$, then either $J \supseteq 0 \ltimes M$ or $J \subseteq 0 \ltimes M$. If $J \supseteq 0 \ltimes M$, then by Corollary 2.1.4 (2), $J = I \ltimes M$ for some ideal I of R. If $J \subseteq 0 \ltimes M$, then the pre-image of J, say N, under the R-module homomorphism $M \to 0 \ltimes M$ $(m \mapsto (0, m))$, is a submodule of M such that $J = 0 \ltimes N$.

(2) $\Rightarrow$ (3). Let $0 \neq a \in R$. Then $(R \ltimes M)(a, 0)$ is an ideal of $R \ltimes M$ such that $(R \ltimes M)(a, 0) \neq 0 \ltimes N$ for any submodule N of M (since $a \neq 0$). By (2), $(R \ltimes M)(a, 0) = I \ltimes M$ for some ideal I of R. But $(R \ltimes M)(a, 0) = Ra \ltimes (R0 + aM) = Ra \ltimes aM$ (by Theorem 2.1.10). Hence $M = aM$ and thus M is divisible.

(3) $\Rightarrow$ (1). Assume that M is divisible and let J be an ideal of $R \ltimes M$. We show that either $J \supseteq 0 \ltimes M$ or $J \subseteq 0 \ltimes M$. Suppose that $J \nsubseteq 0 \ltimes M$, then there exists $(a, b) \in J$ such that $(a, b) \notin 0 \ltimes M$. So $a \neq 0$ in R. Let $m \in M$. Since M is divisible, then $m = am'$ for some $m' \in M$. So $(0, m) = (a, b)(0, m') \in J$. Thus $J \supseteq 0 \ltimes M$. $\square$

The following are illustrative examples for Theorem 2.1.14.

Example 2.1.15.

1. $\mathbb{R}[x]$ is a divisible $\mathbb{Z}$-module since if $f(x) \in \mathbb{R}[x]$, then $f(x) = aa^{-1}f(x) \in a\mathbb{R}[x]$ for all $0 \neq a \in \mathbb{Z}$. So by Theorem 2.1.14, the ideals of $\mathbb{Z} \ltimes \mathbb{R}[x]$ are $\{n\mathbb{Z} \ltimes \mathbb{R}[x] \mid n \in \mathbb{Z}\} \cup \{0 \ltimes N \mid N \text{ is a submodule of } \mathbb{R}[x]\}$.

2. $\mathbb{Z}[x]$ is not a divisible $\mathbb{Z}$-module since $2\mathbb{Z}[x] \neq \mathbb{Z}[x]$. So $\mathbb{Z} \ltimes \mathbb{Z}[x]$ contains an ideal J such that $J \neq I \ltimes \mathbb{Z}[x]$ and $J \neq 0 \ltimes N$ for any ideal I of $\mathbb{Z}$ and submodule N of $\mathbb{Z}[x]$. In fact,

$J = (\mathbb{Z} \ltimes \mathbb{Z}[x])(2,0) = 2\mathbb{Z} \ltimes 2\mathbb{Z}[x]$ is an ideal of $\mathbb{Z} \ltimes \mathbb{Z}[x]$ such that $J \neq I \ltimes \mathbb{Z}[x]$ and $J \neq 0 \ltimes N$ for any ideal I of $\mathbb{Z}$ and submodule N of $\mathbb{Z}[x]$ (since $2 \neq 0$ and $2\mathbb{Z}[x] \neq \mathbb{Z}[x]$).

For an R-module M, let $Z_R(M) = \{r \in R \mid rm = 0 \text{ for some } 0 \neq m \in M\}$ denote the set of all zero divisors of M with respect to R.

Let A be $R \ltimes M$-module. Since the map $R \to R \ltimes M$ $(r \mapsto (r,0))$ is a ring homomorphism, then A is an R-module with scalar multiplication $ra = (r,0)a$.

The following theorem determines the relation between $Z_R(A)$ and $Z_{R\ltimes M}(A)$ where A is an $R \ltimes M$-module.

Theorem 2.1.16 ([6]). *Let R be a ring, M an R-module, and A an $R \ltimes M$-module. Then*

$$Z_{R\ltimes M}(A) = Z_R(A) \ltimes M.$$

Proof. Let A be an $R \ltimes M$-module. Then A is an R-module with $ra = (r,0)a$. Let $(r,m) \in Z_{R\ltimes M}(A)$. Since $Z_{R\ltimes M}(A)$ is a union of prime ideals of $R \ltimes M$, $(r,m) \in P \ltimes M \subseteq Z_{R\ltimes M}(A)$ for some prime ideal P of R. So $(r,0) = (r,m) - (0,m) \in P \ltimes M \subseteq Z_{R\ltimes M}(A)$. Hence there exists $0 \neq a \in A$ such that $(r,0)a = 0$. But then $ra = (r,0)a = 0$. So $r \in Z_R(A)$. Thus, $(r,m) \in Z_R(A) \ltimes M$. Conversely, let $(r,m) \in Z_R(A) \ltimes M$, then $r \in Z_R(A)$. So there exists $0 \neq a \in A$ such that $0 = ra = (r,0)a$. This implies $(r,0) \in Z_{R\ltimes M}(A)$. Hence, as before, $(r,m) \in Z_{R\ltimes M}(A)$. Therefore, $Z_{R\ltimes M}(A) = Z_R(A) \ltimes M$. □

The following corollary determines the zero divisors and the regular elements (non-zero divisors) of $R \ltimes M$.

Corollary 2.1.17 ([14]). *Let R be a ring and M an R-module. Then*

1. *$Z(R \ltimes M) = (Z(R) \cup Z_R(M)) \ltimes M$.*

2. *$S \ltimes M$ where $S = R - (Z(R) \cup Z_R(M))$ is the set of regular elements of $R \ltimes M$.*

Proof. 1. From Theorem 2.1.16 with $A = R \ltimes M$, we have

$$Z(R \ltimes M) = Z_{R\ltimes M}(R \ltimes M) = Z_R(R \ltimes M) \ltimes M = Z_R(R \oplus M) \ltimes M = (Z(R) \cup Z_R(M)) \ltimes M.$$

2. The set of regular elements of $R \ltimes M$ is $R \ltimes M - Z(R \ltimes M)$. Now

$$R \ltimes M - Z(R \ltimes M) = R \ltimes M - [(Z(R) \cup Z_R(M)) \ltimes M] = [R - (Z(R) \cup Z_R(M))] \ltimes M.$$

So the set of regular elements of $R \ltimes M$ is $S \ltimes M$ where $S = R - (Z(R) \cup Z_R(M))$.

□

The following are illustrative examples for Corollary 2.1.17.

Example 2.1.18. 1. Let $R = \mathbb{Z} \ltimes \mathbb{Q}/\mathbb{Z}$. By Corollary 2.1.17, the set of zero-divisors of R is

$$Z(R) = (Z(\mathbb{Z}) \cup Z_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})) \ltimes \mathbb{Q}/\mathbb{Z} = (\{0\} \cup (\mathbb{Z} - \{1, -1\})) \ltimes \mathbb{Q}/\mathbb{Z} = (\mathbb{Z} - \{1, -1\}) \ltimes \mathbb{Q}/\mathbb{Z}.$$

2. Let R be a ring and P a prime ideal of R. Let $A = R \ltimes R/P$. If $x \in Z_R(R/P)$, then $x(y+P) = P$ for some $P \neq y+P \in R/P$. But then $xy + P = P$ for some $y \notin P$, so $xy \in P$ for some $y \notin P$. Thus, $x \in P$. Conversely, if $x \in P$, then $x(1+P) = x+P = P$. This implies that $x \in Z_R(R/P)$. It follows that $Z_R(R/P) = P$. By Corollary 2.1.17 (1),

$$Z(A) = (Z(R) \cup Z_R(R/P)) \ltimes R/P = (Z(R) \cup P) \ltimes R/P.$$

For a ring R, let $nil(R)$ denote the set of all nilpotents of R, $U(R)$ denote the set of all units of R, and $Id(R)$ denote the set of all idempotents of R. .

The following theorem determines the nilpotents, units, and idempotents of $R \ltimes M$.

Theorem 2.1.19 ([14]). *Let R be a ring and M an R-module. Then*

1. $nil(R \ltimes M) = nil(R) \ltimes M$.
2. $U(R \ltimes M) = U(R) \ltimes M$.
3. $Id(R \ltimes M) = Id(R) \ltimes 0$.

Proof. 1. By Corollary 2.1.7 (3), $\sqrt{0 \ltimes 0} = \sqrt{0} \ltimes M$. So

$$nil(R \ltimes M) = \sqrt{0 \ltimes 0} = \sqrt{0} \ltimes M = nil(R) \ltimes M.$$

2. Let $(u, m) \in U(R \ltimes M)$, then there exists $(v, n) \in R \ltimes M$ such that $(u, m)(v, n) = (1, 0)$. So $uv = 1$. This means that $u \in U(R)$. Hence $(u, m) \in U(R) \ltimes M$. Conversely, let $(u, m) \in U(R) \ltimes M$, then $u \in U(R)$. So there exists $u^{-1} \in R$ such that $uu^{-1} = 1$. But then the element $(u^{-1}, -u^{-2}m)$ belongs to $R \ltimes M$, and $(u, m)(u^{-1}, -u^{-2}m) = (uu^{-1}, -u^{-1}m + u^{-1}m) = (1, 0)$. Hence $(u, m) \in U(R \ltimes M)$.

3. Let $(e,m) \in Id(R \ltimes M)$. Then (e,m) is an idempotent. This means that $(e,m)^2 = (e,m)$. So $(e,m) = (e,m)^2 = (e^2, 2em)$, which implies $e^2 = e$ and $m = 2em$. So e is an idempotent of R, or $e \in Id(R)$. Now, since $m = 2em$, then $em = 2e^2m$, so $em = 2em$, hence $em = 0$, but then $m = 2em = 0$. Thus, $(e,m) = (e,0) \in Id(R) \ltimes 0$. Conversely, let $e \in Id(R)$, then e is an idempotent of R. So $e^2 = e$. Hence, $(e,0)^2 = (e^2,0) = (e,0)$. This shows that $(e,0)$ is an idempotent of R. That is, $(e,0) \in Id(R \ltimes M)$.

□

Recall that a ring R is called reduced (respectively Boolean) if $nil(R) = 0$ (respectively $Id(R) = R$).

Corollary 2.1.20. *Let R be a ring and M an R-module.*

1. *$R \ltimes M$ is an integral domain if and only if R is an integral domain and $M = 0$.*
2. *$R \ltimes M$ is a reduced ring if and only if R is a reduced ring and $M = 0$.*
3. *$R \ltimes M$ is a field if and only if R is a field and $M = 0$.*
4. *$R \ltimes M$ is a Boolean ring if and only if R is a Boolean ring and $M = 0$.*

Recall that an ideal I of a ring R is called *regular* if it contains a regular element (non-zero divisor) of R.

Theorem 2.1.21 ([6])**.** *Let R be a ring, M an R-module, and $S = R - (Z(R) \cup Z_R(M))$. Then the following conditions are equivalent:*

1. *Every regular ideal of $R \ltimes M$ has the form $I \ltimes M$ where I is an ideal of R with $I \cap S \neq \emptyset$.*
2. *$sM = M$ for all $s \in S$ or equivalently, $M = M_S$.*

Proof. (1) $\Rightarrow$ (2). Let $s \in S$. Then $(R \ltimes M)(s,0)$ is a regular ideal of $R \ltimes M$. So by (1), $(R \ltimes M)(s,0) = I \ltimes M$ where I is an ideal of R with $I \cap S \neq \emptyset$. By Theorem 2.1.10, $(R \ltimes M)(s,0) = Rs \ltimes sM$. Hence $sM = M$.

(2) $\Rightarrow$ (1). Let J be a regular ideal of $R \ltimes M$. Then $J \cap (S \ltimes M) \neq \emptyset$. So $(s,m) \in J$ for some $s \in S$ and $m \in M$. Since $sM = M$, then $m \in sM$. So if we take $x = 1 \in R$, then $xs = s$ and $xm \in sM$. Hence by Theorem 2.1.10, $(R \ltimes M)(s,m) = Rs \ltimes (Rm + sM) = Rs \ltimes (Rm + M) = Rs \ltimes M$. So $0 \ltimes M \subseteq (R \ltimes M)(s,m) \subseteq J$. Thus, $J = I \ltimes M$ for some ideal I of R. Since $(s,m) \in J = I \ltimes M$, we have $s \in I$ and this means $I \cap S \neq \emptyset$. □

The following example determines the regular ideals of the ring $\mathbb{Z} \ltimes \mathbb{R}[x]$.

Example 2.1.22. Let $R = \mathbb{Z}$ and $M = \mathbb{R}[x]$. Then $S = R - (Z(R) \cup Z_R(M)) = R - \{0\} = \mathbb{Z} - \{0\}$. Since M is a divisible R-module, then $sM = M$ for all $s \in S$. So by Theorem 2.1.21, every regular ideal of $R \ltimes M$ has the form $I \ltimes M$ where I is an ideal of R with $I \cap S \neq \emptyset$. Hence, the regular ideals of $\mathbb{Z} \ltimes \mathbb{R}[x]$ are $\mathbb{Z} \ltimes \mathbb{R}[x]$ where $0 \neq n \in \mathbb{Z}$.

2.2 Some ring constructions of $R \ltimes M$

In this section we determine the saturated multiplicatively closed subsets of $R \ltimes M$, the localizations of $R \ltimes M$, the integral closure of $R \ltimes M$ in $T(R \ltimes M)$, and the ring of polynomials of one variable over $R \ltimes M$.

The following theorem determines the saturated multiplicatively closed subsets of $R \ltimes M$.

Theorem 2.2.1 ([6]). *Let R be a ring and M an R-module.*

1. *There is a one-to-one correspondence between the saturated multiplicatively closed subsets of R and those of $R \ltimes M$ given by $S \leftrightarrow S \ltimes M$.*

2. *If S is a multiplicatively closed subset of R and N is a submodule of M, then $S \ltimes N$ is a multiplicatively closed subset of $R \ltimes M$ with saturation $\overline{S \ltimes N} = \overline{S} \ltimes M$.*

Proof. 1. Let A be a saturated multiplicatively closed subset of $R \ltimes M$. Then

$$A = R \ltimes M - \bigcup_{i \in I}(P_i \ltimes M) = (R - \bigcup_{i \in I} P_i) \ltimes M$$

where $\{P_i \ltimes M\}_{i \in I}$ is the set of prime ideals of $R \ltimes M$ such that $(P_i \ltimes M) \cap A = \emptyset$ for each $i \in I$. Let $S = R - \bigcup_{i \in I} P_i$. Then $P_i \cap S = \emptyset$ for each $i \in I$. So S is saturated . Thus, $A = S \ltimes M$ where S is saturated. Conversely, let S be a saturated multiplicatively closed subset of R, then $S \ltimes M$ is a multiplicatively closed subset of $R \ltimes M$ and $S = R - \bigcup_{i \in I} P_i$ where $\{P_i\}_{i \in I}$ is the set of prime ideals of R such that $P_i \cap S = \emptyset$ for each $i \in I$. But then

$$S \ltimes M = (R - \bigcup_{i \in I} P_i) \ltimes M = R \ltimes M - \bigcup_{i \in I}(P_i \ltimes M)$$

and $(P_i \ltimes M) \cap (S \ltimes M) = (P_i \cap S) \ltimes M = \emptyset$ for each $i \in I$. So $S \ltimes M$ is saturated.

2. Let S be a multiplicatively closed subset of R, and let N be a submodule of M. Then $0 \notin S$, $1 \in S$, and S is closed under multiplication. But then $(0,0) \notin S \ltimes N$, $(1,0) \in S \ltimes N$, and for $(a,m),(b,n) \in S \ltimes N$, we have $(a,m)(b,n) = (ab, an+bm) \in S \ltimes N$. So $S \ltimes N$ is a multiplicatively closed subset of $R \ltimes M$. Next, $\overline{S \ltimes N}$ is a saturated multiplicatively closed subset of $R \ltimes M$, so by (1), $\overline{S \ltimes N} = T \ltimes M$ for some saturated multiplicatively closed subset T of R. Since T is saturated, $\overline{T} = T$. Now $S \ltimes N \subseteq \overline{S \ltimes N} = T \ltimes M$. So $S \subseteq T$ and this implies $\overline{S} \subseteq \overline{T} = T$. Hence $\overline{S} \ltimes M \subseteq T \ltimes M = \overline{S \ltimes N}$. Since $\overline{S} \ltimes M$ is saturated

multiplicatively closed subset of $R \ltimes M$, then $\overline{\overline{S} \ltimes M} = \overline{S} \ltimes M$. So $\overline{S \ltimes N} \subseteq \overline{\overline{S} \ltimes M} = \overline{S} \ltimes M$. Thus $\overline{S \ltimes N} = \overline{S} \ltimes M$.

□

The following theorem determines the localizations of $R \ltimes M$.

Theorem 2.2.2 ([14]). *Let R be a ring and M an R-module.*

1. *Let S be a multiplicatively closed subset of R and N a submodule of M. Then $(R \ltimes M)_{S \ltimes N}$ is naturally isomorphic to $R_S \ltimes M_S$. In the case where $N = 0$, the isomorphism is simply $(r, m)/(s, 0) \mapsto (r/s, m/s)$.*
2. *Let P be a prime ideal of R. Then $(R \ltimes M)_{P \ltimes M} \cong R_P \ltimes M_P$.*
3. *The total quotient ring $T(R \ltimes M)$ of $R \ltimes M$ is naturally isomorphic to $R_S \ltimes M_S$ where $S = R - (Z(R) \cup Z_R(M))$.*

Proof. 1. Let S be a multiplicatively closed subset of R and N a submodule of M. Then $S \ltimes N$ is a multiplicatively closed subset of $R \ltimes M$. For N, either $N = 0$ or $N \neq 0$.

Case 1: $N = 0$. Define $f : (R \ltimes M)_{S \ltimes 0} \to R_S \ltimes M_S$ by $f((r, m)/(s, 0)) = (r/s, m/s)$. If $x = (r_1, m_1)/(s_1, 0), y = (r_2, m_2)/(s_2, 0) \in (R \ltimes M)_{S \ltimes 0}$. Then $x + y = (r_1 s_2 + s_1 r_2, s_2 m_1 + s_1 m_2)/(s_1 s_2, 0)$ and $xy = (r_1 r_2, r_1 m_2 + r_2 m_1)/(s_1 s_2, 0)$. Now

$$\begin{aligned} f(x+y) &= (\frac{r_1 s_2 + s_1 r_2}{s_1 s_2}, \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}) \\ &= (\frac{r_1}{s_1} + \frac{r_2}{s_2}, \frac{m_1}{s_1} + \frac{m_2}{s_2}) \\ &= (\frac{r_1}{s_1}, \frac{m_1}{s_1}) + (\frac{r_2}{s_2}, \frac{m_2}{s_2}) \\ &= f(x) + f(y). \end{aligned}$$

Also

$$\begin{aligned} f(xy) &= (\frac{r_1 r_2}{s_1 s_2}, \frac{r_1 m_2 + r_2 m_1}{s_1 s_2}) \\ &= (\frac{r_1}{s_1}\frac{r_2}{s_2}, \frac{r_1}{s_1}\frac{m_2}{s_2} + \frac{r_2}{s_2}\frac{m_1}{s_1}) \\ &= (\frac{r_1}{s_1}, \frac{m_1}{s_1})(\frac{r_2}{s_2}, \frac{m_2}{s_2}) \\ &= f(x)f(y). \end{aligned}$$

So f is a ring homomorphism. Next, let $x = (r,m)/(s,0) \in \ker f$. Then $(0/1, 0/1) = f(x) = (r/s, m/s)$. So there is $u, v \in S$ such that $ur = 0$ and $vm = 0$. But then $uv \in S$ and $(uv, 0)(r,m) = (vur, uvm) = (0,0)$. So $x = 0$. Hence f is injective. Finally, if $y = (r/s, m/t) \in R_S \ltimes M_S$, then $y = (rt/st, sm/st) = f((st, sm)/(st,0)) = f(x)$ where $x = (st, sm)/(st, 0) \in (R \ltimes M)_{S\ltimes 0}$. So f is surjective. It follows that f is an isomorphism.

Case 2: $N \neq 0$. Since for $(r,m)/(s,n) \in (R \ltimes M)_{S\ltimes N}$, we have

$$\frac{(r,m)}{(s,n)} = \frac{(s,-n)(r,m)}{(s,-n)(s,n)} = \frac{(sr, sm - rn)}{(s^2, 0)}.$$

Then $f : (R \ltimes M)_{S\ltimes N} \to R_S \ltimes M_S$ given by $f((r,m)/(s,n)) = (r/s, (sm - rn)/s^2)$ is an isomorphism.

2. This follows immediately from (1) with $S = R - P$ and $N = M$.

3. Let $S = R - (Z(R) \cup Z_R(M))$. Then by Corollary 2.1.17 (2), $S \ltimes M$ is the set of regular elements of $R \ltimes M$. So the total quotient ring of $R \ltimes M$ is $T(R \ltimes M) = (R \ltimes M)_{S\ltimes M}$. By (1), $T(R \ltimes M) \cong R_S \ltimes M_S$.

□

For an integral domain R, let $Quot(R)$ denote the quotient field of R.

Example 2.2.3. Let $R = \mathbb{Z} \ltimes \mathbb{Z}$, and let p be a prime number. Then

1. By Theorem 2.2.2 (2), the localization of R at $P = p\mathbb{Z} \ltimes \mathbb{Z}$ is $R_P \cong \mathbb{Z}_{p\mathbb{Z}} \ltimes \mathbb{Z}_{p\mathbb{Z}}$.

2. By Theorem 2.2.2 (3), the total quotient ring of R is $T(R) \cong \mathbb{Z}_S \ltimes \mathbb{Z}_S$ where $S = \mathbb{Z} - (Z(\mathbb{Z}) \cup Z(\mathbb{Z})) = \mathbb{Z} - \{0\}$. So $\mathbb{Z}_S = Quot(\mathbb{Z}) = \mathbb{Q}$. Hence $T(R) \cong \mathbb{Q} \ltimes \mathbb{Q}$.

Next, we determine the integral closure of $R \ltimes M$, but first we need the following lemma.

Lemma 2.2.4. *Let R be a ring, M an R-module, and $r \in T(R)$. Then r is integral over R if and only if $(r, 0)$ is integral over $R \ltimes M$.*

Proof. Suppose that r is integral over R. Then there is $a_0, ..., a_{n-1} \in R$ such that $r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0$. But then $(r,0)^n + (a_{n-1}, 0)(r,0)^{n-1} + \cdots + (a_0, 0) = (0,0)$. So $(r,0)$ is integral over $R \ltimes M$. Conversely, suppose that $(r, 0)$ is integral over $R \ltimes M$, then there is $(a_0, m_0), ..., (a_{n-1}, m_{n-1}) \in R \ltimes M$ such that $(r,0)^n + (a_{n-1}, m_{n-1})(r,0)^{n-1} + \cdots + (a_0, m_0) = (0,0)$. So $(r^n + a_{n-1}r^{n-1} + \cdots$

$\cdot + a_0, 0 + r^{n-1}m_{n-1} + \cdots + m_0) = (0,0)$. Hence $r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0$. It follows that r is integral over R. $\square$

The following theorem determines the integral closure of $R \ltimes M$ in $T(R \ltimes M)$.

Theorem 2.2.5 ([14])**.** *Let R be a ring, M an R-module, and $S = R - (Z(R) \cup Z_R(M))$. If R' is the integral closure of R in $T(R)$, then $(R' \cap R_S) \ltimes M_S$ is the integral closure of $R \ltimes M$ in $T(R \ltimes M)$.*

Proof. First, note that $R \ltimes M \subseteq (R' \cap R_S) \ltimes M_S \subseteq R_S \ltimes M_S = T(R \ltimes M)$ (since $R \subseteq R'$, $R \subseteq R_S$, and $M \subseteq M_S$). Now, let $(r,b) \in (R' \cap R_S) \ltimes M_S$, then $r \in R' \cap R_S$ and $b \in M_S$. So r is integral over R. By the last lemma, $(r,0)$ is integral over $R \ltimes M$. Since $(0,b)^2 = (0,0)$, $(0,b)$ is integral over $R \ltimes M$. So $(r,b) = (r,0) + (0,b)$ is integral over $R \ltimes M$. This means $(r,b) \in (R \ltimes M)'$. Thus, $(R' \cap R_S) \ltimes M_S \subseteq (R \ltimes M)'$. Conversely, let $(r,b) \in (R \ltimes M)'$. Since $(0,b) \in (R \ltimes M)'$, then $(r,0) = (r,b) - (0,b) \in (R \ltimes M)'$. So $r \in R'$. Hence $(r,b) \in (R' \cap R_S) \ltimes M_S$. Thus $(R \ltimes M)' \subseteq (R' \cap R_S) \ltimes M_S$. $\square$

Corollary 2.2.6 ([14])**.** *Let R be a ring, M an R-module, and $S = R - (Z(R) \cup Z_R(M))$.*

1. *If R is integrally closed, then $R \ltimes M_S$ is the integral closure of $R \ltimes M$ in $T(R \ltimes M)$.*

2. *If $Z_R(M) \subseteq Z(R)$, then $R \ltimes M_S$ is integrally closed if and only if R is integrally closed.*

Proof.

1. Suppose that R is integrally closed, then $R = R'$ and so $R' \cap R_S = R \cap R_S = R$. Hence by Theorem 2.2.5, the integral closure of $R \ltimes M$ in $T(R \ltimes M)$ is $(R' \cap R_S) \ltimes M_S = R \ltimes M_S$.

2. Suppose that $R \ltimes M_S$ is integrally closed. Now since $Z_R(M) \subseteq Z(R)$, then $S = R - Z(R)$, so $R_S = T(R)$, hence $T(R \ltimes M_S) = T(R) \ltimes M_S$. To show that R is integrally closed, let $r \in T(R)$ be integral over R. Then $(r,0) \in T(R) \ltimes M_S = T(R \ltimes M_S)$ is integral over $R \ltimes M_S$. But $R \ltimes M_S$ is integrally closed gives that $(r,0) \in R \ltimes M_S$. So $r \in R$. Hence R is integrally closed. Conversely, suppose that R is integrally closed, then by (1), we have $(R \ltimes M_S)' = R \ltimes M_S$. So $R \ltimes M_S$ is integrally closed.

$\square$

Example 2.2.7. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}^2$. Then R is integrally closed in $T(R) = \mathbb{Q}$. So by Corollary 2.2.6 (1), the integral closure of $R \ltimes M$ is $R \ltimes M_S = \mathbb{Z} \ltimes \mathbb{Q}^2$. Also by Corollary 2.2.6 (2), $R \ltimes M_S = \mathbb{Z} \ltimes \mathbb{Q}^2$ is integrally closed since $Z_R(M) = 0 \subseteq Z(R)$.

Proposition 2.2.8 ([6])**.** Let R be a ring, M an R-module, and $S = R - (Z(R) \cup Z_R(M))$. If $R \ltimes M$ is integrally closed, then $M = M_S$.

Proof. Suppose that $R \ltimes M$ is integrally closed. Let $m \in M$ and $s \in S$. Then $(0, m/s) = (0, m)/(s, 0) \in T(R \ltimes M)$. Now, $(0, m/s)^2 = (0, m)^2/(s, 0)^2 = (0, 0)$. This implies that $(0, m/s)$ is a root of $f(x) = x^2 \in (R \ltimes M)[x]$. But since $R \ltimes M$ is integrally closed, then $(0, m/s) \in R \ltimes M$. So $m/s \in M$ and hence $M = M_S$. $\square$

In general, $R \ltimes M_S$ integrally closed does not imply that R is integrally closed. The following example provides an arbitrary non-integrally closed R and an R-module M such that $R \ltimes M_S$ is integrally closed.

Example 2.2.9 ([14], page 166)**.** Let R be a non-integrally closed and $M = \bigoplus\{R/P \mid P \in Spec(R)\}$. By Example 2.1.18 (2), we have $Z_R(R/P) = P$ for all $P \in Spec(R)$. So

$$Z_R(M) = \bigcup_{P \in Spec(R)} Z_R(R/P) = \bigcup_{P \in Spec(R)} P.$$

Since

$$R - U(R) = \bigcup_{\mathrm{m} \in Max(R)} \mathrm{m}.$$

Then $R - U(R) \subseteq Z_R(M)$. We claim that $R \ltimes M$ is a total quotient ring. To do this, let $(u, m) \notin Z(R \ltimes M) = (Z(R) \cup Z_R(M)) \ltimes M$. Then $u \notin Z(R) \cup Z_R(M)$. So $u \notin R - U(R)$. Hence $u \in U(R)$ and $(u, m) \in U(R \ltimes M)$. So every non-zero divisor of $R \ltimes M$ is a unit or equivalently, every element of $R \ltimes M$ is either a zero divisor or a unit. So $R \ltimes M$ is a total quotient ring and hence $R \ltimes M$ is integrally closed. By the last proposition, $M = M_S$ where $S = R - (Z(R)) \cup Z_R(M)$. Therefore, $R \ltimes M_S$ is integrally closed.

Next, we determine the inverse of $I \ltimes M$ where I is an ideal of a ring R, and M is an R-module with $M = M_S$ where $S = R - (Z(R) \cup Z_R(M))$. We first recall the following definition.

Definition 2.2.10 ([8])**.** Let R be a commutative ring with total quotient ring $T(R)$. An ideal I of R is invertible if $II^{-1} = R$, where $I^{-1} = \{x \in T(R) \mid xI \subset R\}$.

Proposition 2.2.11. Let R be a ring, M an R-module, and $S = R - (Z(R) \cup Z_R(M))$. If I is an ideal of R and $M = M_S$, then the inverse of $I \ltimes M$ is $(I \ltimes M)^{-1} = (I^{-1} \cap R_S) \ltimes M$.

Proof. First, note that $(I \ltimes M)^{-1} \subseteq T(R \ltimes M) = R_S \ltimes M_S = R_S \ltimes M$. Now, let $(a, b) \in (I \ltimes M)^{-1}$, then $a \in R_S$ and $b \in M$. Let $x \in I$ be an arbitrary, then $(x, 0) \in I \ltimes M$. But $(a, b) \in (I \ltimes M)^{-1}$, so $(ax, xb) = (a, b)(x, 0) \in R \ltimes M$. Hence, $ax \in R$. Since x is arbitrary, then $aI \subseteq R$ and this means

$a \in I^{-1}$. Thus, $(a,b) \in (I^{-1} \cap R_S) \ltimes M$. For the other inclusion, let $(a,b) \in (I^{-1} \cap R_S) \ltimes M$. Then $a \in I^{-1}$, $a \in R_S$, and $b \in M$. So, $aI \subseteq R$. Let $(x,m) \in I \ltimes M$ be an arbitrary. Then $x \in I$. Since $aI \subseteq R$, $ax \in R$. But then $(a,b)(x,m) = (ax, am + xb) \in R \ltimes M_S = R \ltimes M$. Hence, $(a,b)(I \ltimes M) \subseteq R \ltimes M$ and this means $(a,b) \in (I \ltimes M)^{-1}$. Thus, $(I \ltimes M)^{-1} = (I^{-1} \cap R_S) \ltimes M$. $\square$

Corollary 2.2.12. *Let R be a ring, M an R-module, and $S = R - (Z(R) \cup Z_R(M))$. If I is an ideal of R with $I \cap S \neq \emptyset$ and $M = M_S$, then $(I \ltimes M)^{-1} = I^{-1} \ltimes M$; Hence $I \ltimes M$ is invertible if and only if I is invertible.*

Proof. Suppose that I is an ideal of R such that $I \cap S \neq \emptyset$, and assume that $M = M_S$. Choose $s \in I \cap S$. We claim that $I^{-1} \subseteq R_S$. Now, let $x \in I^{-1}$. Then $xI \subseteq R$. So, $xs = r$ for some $r \in R$. Hence, $x = rs^{-1} \in R_S$. Thus, $I^{-1} \subseteq R_S$ and so $I^{-1} \cap R_S = I^{-1}$. By Proposition 2.2.11, $(I \ltimes M)^{-1} = (I^{-1} \cap R_S) \ltimes M = I^{-1} \ltimes M$. Next, assume that $I \ltimes M$ is invertible, then $(I \ltimes M)(I^{-1} \ltimes M) = (I \ltimes M)(I \ltimes M)^{-1} = R \ltimes M$. But then $II^{-1} \ltimes (IM + I^{-1}M) = R \ltimes M$. So $II^{-1} = R$ and hence I is invertible. Conversely, suppose that I is invertible, so $II^{-1} = R$. Now, since $M = M_S$, then $M = sM$ for all $s \in S$ and hence $M = sM$ for all $s \in I \cap S$. Let $m \in M$ be an arbitrary, and let $t \in I \cap S$. Then $m = tm'$ for some $m' \in M$. So $m \in IM$ and hence $M = IM$. Also, since $I^{-1} \subseteq R_S$ and $M = M_S$, then $I^{-1}M \subseteq R_S M \subseteq M_S = M$. It follows that $IM + I^{-1}M = M$. So $(I \ltimes M)(I \ltimes M)^{-1} = (I \ltimes M)(I^{-1} \ltimes M) = II^{-1} \ltimes (IM + I^{-1}M) = R \ltimes M$. Thus, $I \ltimes M$ is invertible. $\square$

Suppose that R_1 and R_2 are rings and M_i is an R_i-module, $i = 1, 2$. Then $M_1 \times M_2$ is an $R_1 \times R_2$-module with action $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$. Conversely, let $R = R_1 \times R_2$ and suppose that M is an R-module. Put $M_1 = (R_1 \times 0)M$ and $M_2 = (0 \times R_2)M$. So M_i is an R_i-module and M is the internal direct sum of M_1 and M_2 and hence $M \cong M_1 \times M_2$.

The following Theorem shows that $(R_1 \times R_2) \ltimes (M_1 \times M_2)$ and $(R_1 \ltimes M_1) \times (R_2 \ltimes M_2)$ are isomorphic.

Theorem 2.2.13 ([6])**.** *let R_1 and R_2 be rings, and let M_i be an R_i-module, $i = 1, 2$. Then $(R_1 \times R_2) \ltimes (M_1 \times M_2) \cong (R_1 \ltimes M_1) \times (R_2 \ltimes M_2)$.*

Proof. Define $\varphi : (R_1 \times R_2) \ltimes (M_1 \times M_2) \to (R_1 \ltimes M_1) \times (R_2 \ltimes M_2)$ by

$$\varphi((r_1, r_2), (m_1, m_2)) = ((r_1, m_1), (r_2, m_2)).$$

Let $a = ((r_1, r_2), (m_1, m_2))$, $b = ((s_1, s_2), (n_1, n_2)) \in (R_1 \times R_2) \ltimes (M_1 \times M_2)$. Then $a + b = ((r_1 + s_1, r_2 + s_2), (m_1 + n_1, m_2 + n_2))$ and $ab = ((r_1s_1, r_2s_2), (r_1n_1 + s_1m_1, r_2n_2 + s_2m_2))$. So

$$\begin{aligned}\varphi(a+b) &= ((r_1 + s_1, m_1 + n_1), (r_2 + s_2, , m_2 + n_2))\\ &= ((r_1, m_1) + (s_1, n_1), (r_2, m_2) + (s_2, n_2))\\ &= ((r_1, m_1), (r_2, m_2)) + ((s_1, n_1), (s_2, n_2))\\ &= \varphi(a) + \varphi(b).\end{aligned}$$

Also

$$\begin{aligned}\varphi(ab) &= ((r_1s_1, r_1n_1 + s_1m_1), (r_2s_2, r_2n_2 + s_2m_2))\\ &= ((r_1, m_1)(s_1, n_1), (r_2, m_2)(s_2, n_2))\\ &= ((r_1, m_1), (r_2, m_2))((s_1, n_1), (s_2, n_2))\\ &= \varphi(a)\varphi(b).\end{aligned}$$

So φ is a ring homomorphism. Clearly, φ is onto. Finally, let $a = ((r_1, r_2), (m_1, m_2)) \in \ker\varphi$, then $((0, 0), (0, 0)) = \varphi(a) = ((r_1, m_1), (r_2, m_2)) \Leftrightarrow (r_1, m_1) = (0, 0)$ and $(r_2, m_2) = (0, 0) \Leftrightarrow r_1 = 0$, $r_2 = 0$, $m_1 = 0$, and $m_2 = 0 \Leftrightarrow a = ((0, 0), (0, 0))$. So $\ker\varphi = 0$. Equivalently, φ is one-to-one. Therefore, φ is an isomorphism. □

Remark 2.2.14. If R_i is a ring and M_i is an R_i-module for each $i = 1, ..., n$, then $\prod_{i=1}^n R_i \ltimes \prod_{i=1}^n M_i \cong \prod_{i=1}^n (R_i \ltimes M_i)$.

The following is an illustrative example for Theorem 2.2.13.

Example 2.2.15. Let p, q be two distinct prime numbers. Then $(\mathbb{Z} \times \mathbb{Z}) \ltimes (\mathbb{Z}_p \times \mathbb{Z}_q) \cong (\mathbb{Z} \ltimes \mathbb{Z}_p) \times (\mathbb{Z} \ltimes \mathbb{Z}_q)$.

The following theorem shows that the ring of polynomials $(R \ltimes M)[x]$ over $R \ltimes M$ is naturally isomorphic to $R[x] \ltimes M[x]$.

Theorem 2.2.16 ([6]). *Let R be a ring and M an R-module. Then*

$$(R \ltimes M)[x] \cong R[x] \ltimes M[x].$$

Proof. Define $\varphi : (R \ltimes M)[x] \to R[x] \ltimes M[x]$ by

$$\varphi(\sum_{i=0}^{k}(r_i, m_i)x^i) = (\sum_{i=0}^{k} r_i x^i, \sum_{i=0}^{k} m_i x^i).$$

Let $f(x) = \sum_{i=0}^{s}(a_i, m_i)x^i$ and $g(x) = \sum_{j=0}^{t}(b_j, n_j)x^j$ be two elements in $(R \ltimes M)[x]$. Then

$$f(x) + g(x) = \sum_{k=0}^{u}[(a_k, m_k) + (b_k, n_k)]x^k = \sum_{k=0}^{u}(a_k + b_k, m_k + n_k)x^k$$

where $u = max\{s, t\}$, $(a_i, m_i) = (0, 0)$ for $i > s$, and $(b_i, n_i) = (0, 0)$ for $i > t$. Now

$$\begin{aligned}
\varphi(f(x) + g(x)) &= (\sum_{k=0}^{u}(a_k + b_k)x^k, \sum_{k=0}^{u}(m_k + n_k)x^k) \\
&= (\sum_{k=0}^{u} a_k x^k, \sum_{k=0}^{u} m_k x^k) + (\sum_{k=0}^{u} b_k x^k, \sum_{k=0}^{u} n_k x^k) \\
&= (\sum_{k=0}^{s} a_k x^k, \sum_{k=0}^{s} m_k x^k) + (\sum_{k=0}^{t} b_k x^k, \sum_{k=0}^{t} n_k x^k) \\
&= \varphi(f(x)) + \varphi(g(x)).
\end{aligned}$$

Next,

$$f(x)g(x) = \sum_{k=0}^{s+t}(c_k, l_k)x^k$$

where $(c_k, l_k) = \sum_{i=0}^{k}(a_{k-i}, m_{k-i})(b_i, n_i)$ for $k = 0, ..., s + t$. So

$$(c_k, l_k) = \sum_{i=0}^{k}(a_{k-i}b_i, a_{k-i}n_i + b_i m_{k-i}) = (\sum_{i=0}^{k} a_{k-i}b_i, \sum_{i=0}^{k} a_{k-i}n_i + \sum_{i=0}^{k} b_i m_{k-i}).$$

Let $l'_k = \sum_{i=0}^{k} a_{k-i}n_i$ and $l''_k = \sum_{i=0}^{k} b_i m_{k-i}$. Then $l_k = l'_k + l''_k$, $k = 0, ..., s + t$. So

$$\begin{aligned}
\varphi(f(x)g(x)) &= (\sum_{k=0}^{s+t} c_k x^k, \sum_{k=0}^{s+t} l_k x^k) \\
&= (\sum_{k=0}^{s+t} c_k x^k, \sum_{k=0}^{s+t} l'_k x^k) + \sum_{k=0}^{s+t} l''_k x^k) \\
&= (\sum_{i=0}^{s} a_i x^i, \sum_{i=0}^{s} m_i x^i)(\sum_{j=0}^{t} b_j x^j, \sum_{j=0}^{t} n_j x^j) \\
&= \varphi(f(x))\varphi(g(x)).
\end{aligned}$$

It follows that φ is a ring homomorphism. Next, let $f(x) = \sum_{i=0}^{s}(a_i, m_i)x^i \in \ker\varphi$. Then $(0,0) = \varphi(f(x)) = (\sum_{i=0}^{s} a_i x^i, \sum_{i=0}^{s} m_i x^i)$, so $a_i = 0$ and $m_i = 0$ for each $i = 0, ..., s$. Hence $f(x) = 0$ and φ is one-to-one. Finally, let $\beta = (\sum_{i=0}^{s} r_i x^i, \sum_{j=0}^{t} m_j x^j) \in R[x] \ltimes M[x]$. If $s = t$, then $\beta = \varphi(\alpha)$ where $\alpha = \sum_{i=0}^{s}(r_i, m_i)x^i$. If $s > t$, then $\beta = \varphi(\alpha)$ where $\alpha = \sum_{i=0}^{t}(r_i, m_i)x^i + \sum_{i=t+1}^{s}(r_i, 0)x^i$. Similarly, to prove that β is an image of some $\alpha \in (R \ltimes M)[x]$ in the case that $s < t$. So φ is onto and therefore, φ is an isomorphism. $\square$

Chapter 3

Transfer results and examples

In this chapter we will study several notions via trivial ring extension, namely, Noetherian, Artinian, Manis valuation, Prüfer, chained, and arithmetical rings. Also, we will study the atomic rings and the ACCP via trivial ring extension. We construct a number of examples about these notions using the trivial ring extension.

3.1 Noetherian and Artinian rings via trivial ring extension

We start this section by recalling some basic definitions and facts about Noetherian and Artinian rings.

Definition 3.1.1. Let R be a ring.

1. We say that R satisfies the ascending chain condition (ACC), or that R is Noetherian if any ascending chain of ideals of R terminates. That is, if we have ideals $I_1 \subset I_2 \subset \cdots$, then there is some $n \in \mathbb{N}$ such that $I_n = I_m$ for all $m \geq n$.

2. We say that R satisfies the descending chain condition (DCC), or that R is Artinian if any descending chain of ideals of R terminates. That is, if we have ideals $I_1 \supset I_2 \supset \cdots$, then there is some $n \in \mathbb{N}$ such that $I_n = I_m$ for all $m \geq n$.

Theorem 3.1.2. *[[15]] (Cohen's Theorem). A ring R is Noetherian if and only if every prime ideal of R is finitely generated.*

Theorem 3.1.3 ([7])**.** *A ring R is Artinian if and only if R is Noetherian and* $\dim R = 0$.

The following theorem establishes the transfer of the Noetherian (Artinian) notions in trivial ring extension.

Theorem 3.1.4 ([6]). *Let R be a ring, M an R-module, and $E = R \ltimes M$.*

1. *E is Noetherian if and only if R is Noetherian and M is finitely generated.*

2. *E is Artinian if and only if R is Artinian and M is finitely generated.*

Proof. 1. Suppose that E is Noetherian. Then $R \cong \frac{E}{0 \ltimes M}$ is Noetherian. Now, the ideal $0 \ltimes M$ of E is finitely generated. So there are $m_1, ..., m_n \in M$ such that $0 \ltimes M = \sum_{i=1}^{n} E(0, m_i)$. Now

$$\begin{aligned} 0 \ltimes M &= \sum_{i=1}^{n} E(0, m_i) \\ &= \sum_{i=1}^{n} (0 \ltimes Rm_i) \qquad (\textit{By Theorem } 2.1.10) \\ &= 0 \ltimes \sum_{i=1}^{n} Rm_i. \end{aligned}$$

So $M = \sum_{i=1}^{n} Rm_i$ is a finitely generated R-module. Conversely, suppose that R is Noetherian and M is finitely generated. We will use Cohen's theorem to prove that E is Noetherian. Let $P \ltimes M$ be a prime ideal of $R \ltimes M$, then P is a prime ideal of R and hence P is finitely generated. But M is finitely generated R-module. So $P \ltimes M$ is a finitely generated ideal of E. Thus by Cohen's theorem, we have E is Noetherian.

2. Suppose that E is Artinian. Then $R \cong \frac{E}{0 \ltimes M}$ is Artinian. Since an Artinian ring is Noetherian, then E is Noetherian and so by (1), M is finitely generated. Conversely, suppose that R is Artinian and M is finitely generated. By Theorem 3.1.3, R is Noetherian with $\dim R = 0$. Since M is finitely generated, then by (1), E is Noetherian. But $\dim E = \dim R = 0$. So, we have E is Noetherian with $\dim E = 0$. Hence by Theorem 3.1.3, E is Artinian.

□

Theorem 3.1.4 can be used to construct a new examples of Noetherian (non-Noetherian) rings and a new examples of Artinian (non-Artinian) rings

The following example gives a class of Noetherian rings using trivial ring extension.

Example 3.1.5. 1. Since every ideal of $\mathbb{Z}$ is principal, then $\mathbb{Z}$ is Noetherian. Now $\mathbb{Z}_p$ is a finitely generated $\mathbb{Z}$-module for any prime number p, so by Theorem 3.1.4, $\mathbb{Z} \ltimes \mathbb{Z}_p$ is Noetherian for any prime p. Hence by Example 2.2.15, the ring $R_{p,q} = (\mathbb{Z} \times \mathbb{Z}) \ltimes (\mathbb{Z}_p \times \mathbb{Z}_q)$ is Noetherian for all distinct primes p and q.

2. Let $\mathbb{F}$ be any field, then $\mathbb{F}$ is Noetherian. Since $\mathbb{F}[x_1, ..., x_n]$ is a finitely generated $\mathbb{F}$-module, then by Theorem 3.1.4, $\mathbb{F} \ltimes \mathbb{F}[x_1, ..., x_n]$ is Noetherian.

The following example gives a new examples of non-Noetherian rings using trivial ring extension.

Example 3.1.6. 1. Since the set of rational numbers $\mathbb{Q}$ is not finitely generated $\mathbb{Z}$-module, then by Theorem 3.1.4, $\mathbb{Z} \ltimes \mathbb{Q}$ is not Noetherian.

2. Let $\mathbb{F}$ be any field, and let $R = \mathbb{F}[x_1, x_2, ...]$. Since

$$Rx_1 \subset Rx_1 + Rx_2 \subset Rx_1 + Rx_2 + Rx_3 \subset \cdots$$

is an infinite strictly increasing chain of ideals of R, then R is not Noetherian. So by Theorem 3.1.4, $R \ltimes M$ is not Noetherian for any R-module M.

Next, we give examples of Artinian rings and examples of non-Artinian rings.

Example 3.1.7. 1. Let $\mathbb{F}$ be any field. Then $R_n = \frac{\mathbb{F}[x]}{x^n\mathbb{F}[x]}$ is an Artinian ring for every positive integer n (This is because R_n is a finite dimensional vector space of dimension n). So if M is a finitely generated R_n-module, then by Theorem 3.1.4, $R_n \ltimes M$ is an Artinian ring.

2. Since the ring of integers $\mathbb{Z}$ is not Artinian, then the ring $R_{p,q}$ given in example 3.1.5 (1), is not Artinian for all distinct primes p and q.

3.2 Characterization of Manis valuation rings, Prüfer rings, chained rings, and arithmetical rings of the form $R \ltimes M$.

We start this section by recalling some basic definitions and facts.

Definition 3.2.1. Let R be an integral domain.

1. R is called a valuation domain if for each $x \in Quot(R)$, either $x \in R$ or $x^{-1} \in R$ [7].

2. R is called a Prüfer domain if every nonzero finitely generated ideal of R is invertible [23].

Definition 3.2.2 ([20])**.** Let T be a ring and R a subring of T. R is called a valuation ring on T if there exists a prime ideal P of R such that for each $x \in T - R$, there exists $x' \in P$ such that $xx' \in R - P$. The pair (R, P) is called a valuation pair of T.

Definition 3.2.3. Let R be a ring. Then:

1. R is called a Manis valuation ring if its a valuation ring on $T(R)$. [20].

2. R is called a Prüfer ring if every finitely generated regular ideal of R is invertible [9].

3. R is called a chained ring if the set of ideals of R is totally ordered by inclusion [12].

4. R is called an arithmetical ring if R_{m} is a chained ring for each maximal ideal m of R [16].

Notice that if R is a local domain, then all the notions in the last definition are coincide.

Remark 3.2.4. Let R be a ring.

1. If R is a chained ring, then R is an arithmetical ring.

2. If R is a chained ring, then R is a Manis valuation ring.

3. If R is an arithmetical ring, then R is a Prüfer ring.

Proof.
1. Let R be a chained ring. Then any two ideals of R are comparable, so R has exactly one maximal ideal, say m. Hence $R_{\mathrm{m}} = R$ is a chained ring. That is, R is an arithmetical ring.

2. Let R be a chained ring. Then R is a local ring. Let P be the unique maximal ideal of R, and let $x = \frac{a}{b} \in T(R) - R$. Since R is chained, then either $Ra \subseteq Rb$ or $Rb \subseteq Ra$. But since $\frac{a}{b} \notin R$, we have $Rb \subseteq Ra$, so $b = ac$ for some $c \in R$. Again since $\frac{a}{b} \notin R$, then $c \notin U(R)$, so $c \in P$ (as $P = R - U(R)$). Now, $xc = \frac{a}{b}c = \frac{ac}{b} = \frac{b}{b} = 1$. So $xc \in R - P$. Hence R is a Manis valuation ring.

3. See [8, Theorem 2.2] and [8, Theorem 2.5].

□

Example 3.2.5.
1. Let R be a principal ideal domain and I a nonzero ideal of R. Write $I = aR$ where $a \in R$. Then $I^{-1} = \frac{1}{a}R$. Now, $II^{-1} = (aR)(\frac{1}{a}R) = R$, so I is invertible. Hence R is Prüfer. Since R is a domain, then R is arithmetical. So R_{pR} is a chained ring for any prime element p of R.

2. Let F be a field, then F has only two ideals, namely 0 and F, and these ideals are comparable, so F is a chained ring. By the remark above, F is a Manis valuation ring. Since a field is a local ring, then F an arithmetical ring. Also, since a field is a PID, then by (1), F is Prüfer.

3. Let R be a total quotient ring, then every element of R is either a zero divisor or a unit. So if I is a regular ideal of R, it contains a regular element, hence I contains a unit of R. This implies $I = R$ is invertible. So R is a Prüfer ring.

4. Let R be a Boolean ring, and let m be a maximal ideal of R. Then R_{m} is a local ring with maximal ideal $\mathrm{m}R_{\mathrm{m}} = \{\frac{a}{b} \mid a \in R, b \notin \mathrm{m}\}$. Let x be a nonzero element of R_{m}. Then $x^2 = x$, so $x(x-1) = 0$. If $x \in \mathrm{m}R_{\mathrm{m}}$, $x-1 \notin \mathrm{m}R_{\mathrm{m}}$, but then $x-1$ is a unit of R_{m}, which implies $x = 0$, a contradiction. So, we have $x \notin \mathrm{m}R_{\mathrm{m}}$, which implies x is a unit of R_{m}. From $x(x-1) = 0$, we have $x = 1$. It follows that $R_{\mathrm{m}} = \{0, 1\} \cong \mathbb{Z}_2$ is a field, and so by (1), its a chained ring. Thus, R is an arithmetical ring.

The following theorem characterizes when $R \ltimes M$ is a valuation ring or Prüfer ring.

Theorem 3.2.6 ([14]). *Let R be a ring, M an R-module, and $S = R - (Z(R) \cup Z_R(M))$.*

1. $R \ltimes M$ is a Manis valuation ring if and only if R is a valuation ring on R_S and $M = M_S$.

2. $R \ltimes M$ is a Prüfer ring if and only if for each finitely generated ideal I of R with $I \cap S \neq \emptyset$, I is invertible, and $M = M_S$.

Proof. 1. If $R \ltimes M$ is a Manis valuation ring, then $(R \ltimes M, P \ltimes M)$ is a valuation pair of $T(R \ltimes M) = R_S \ltimes M_S$ for some prime ideal P of R. Since $R \ltimes M$ is integrally closed, then $M = M_S$. So $T(R \ltimes M) = R_S \ltimes M$. Now, let $x \in R_S - R$, then $(x, 0) \in T(R \ltimes M) - R \ltimes M$. So there exists $(x', m') \in P \ltimes M$ such that $(x, 0)(x', m') \in (R \ltimes M) - (P \ltimes M) = (R - P) \ltimes M$. But then $x' \in P$ and $xx' \in R - P$. Thus, (R, P) is a valuation pair of R_S. That is, R is a valuation ring on R_S. Conversely, suppose that R is a valuation ring on R_S and $M = M_S$. Then (R, P) is a valuation pair of R_S for some prime ideal P of R. Let $(x, m) \in T(R \ltimes M) - R \ltimes M = (R_S - R) \ltimes M$. Then $x \in R_S - R$. So there exists $x' \in P$ such that $xx' \in R - P$. But then $(x', 0) \in P \ltimes M$ and $(x, m)(x', 0) = (xx', x'm) \in (R - P) \ltimes M = (R \ltimes M) - (P \ltimes M)$. Thus, $R \ltimes M$ is a Manis valuation ring.

2. Assume that $R \ltimes M$ is a Prüfer ring. Then $R \ltimes M$ is integrally closed, so $M = M_S$. Let I be a finitely generated ideal of R with $I \cap S \neq \emptyset$. Then by Theorem 2.1.21, $I \ltimes M$ is a regular

ideal of $R \ltimes M$. We claim that if $I = \sum_{i=1}^{n} Ra_i$, then $I \ltimes M = \sum_{i=1}^{n}(R \ltimes M)(a_i, 0)$. Suppose $I = \sum_{i=1}^{n} Ra_i$, then $a_i \in I$ for all $i = 1, ..., n$. So $\sum_{i=1}^{n}(R \ltimes M)(a_i, 0) \subseteq I \ltimes M$. For the reverse inclusion, let $(x, m) \in I \ltimes M$. Then $x \in I$ gives that $x = \sum_{i=1}^{n} r_i a_i$ for some $r_i \in R$, $i = 1, ..., n$. Now since $I \cap S \neq \emptyset$, there exists $s \in I \cap S$. So $s^{-1}m \in M$ since $M = M_S$. Hence

$$(x, m) = (x, 0) + (0, m) = \sum_{i=1}^{n}(r_i, 0)(a_i, 0) + (s, 0)(0, s^{-1}m) \in \sum_{i=1}^{n}(R \ltimes M)(a_i, 0).$$

Thus, $I \ltimes M = \sum_{i=1}^{n}(R \ltimes M)(a_i, 0)$ is a finitely generated regular ideal of $R \ltimes M$. Since $R \ltimes M$ is a Prüfer ring, $I \ltimes M$ is invertible and therefore, by Corollary 2.2.12, I is invertible. Conversely, let J be a finitely generated regular ideal of $R \ltimes M$. Since $M = M_S$, then by Theorem 2.1.21, $J = I \ltimes M$ where I is an ideal of R with $I \cap S \neq \emptyset$. By hypothesis, I is invertible. Hence by Corollary 2.2.12, $J = I \ltimes M$ is invertible. Thus, $R \ltimes M$ is a Prüfer ring.

□

Corollary 3.2.7. *Let R be a ring, M an R-module, and $S = R - (Z(R) \cup Z_R(M))$.*

1. *If $Z_R(M) \subseteq Z(R)$, then $R \ltimes M$ is a Manis valuation ring if and only if R is a Manis valuation ring and M is divisible.*

2. *If $Z_R(M) \subseteq Z(R)$, then $R \ltimes M$ is a Prüfer ring if and only if R is a Prüfer ring and M is divisible.*

Proof. 1. If $Z_R(M) \subseteq Z(R)$, then $S = R - Z(R)$. So $R_S = T(R)$. Hence by Theorem 3.2.6 (1), $R \ltimes M$ is a Manis valuation ring if and only if R is a valuation ring on $R_S = T(R)$ and $M = M_S$ if and only if R is a Manis valuation ring and M is divisible.

2. As in (1), since $Z_R(M) \subseteq Z(R)$, $S = R - Z(R)$. So for any ideal I of R, I is regular if and only if $I \cap S \neq \emptyset$. Hence by Theorem 3.2.6 (2), $R \ltimes M$ is a Prüfer ring if and only if for each finitely generated ideal I of R with $I \cap S \neq \emptyset$, I is invertible, and $M = M_S$ if and only if for each finitely generated regular ideal I of R, I is invertible and M is divisible if and only if R is a Prüfer ring and M is divisible.

□

We next give examples concerning valuation rings and Prüfer rings involving trivial ring extension.

Example 3.2.8. 1. Let $R = \mathbb{Z}_{2\mathbb{Z}}$ and $M = \mathbb{Q} = T(R)$. Then $Z_R(M) = 0 = Z(R)$, so $S = R - \{0\}$. Since $\mathbb{Z}_{2\mathbb{Z}}$ is a valuation ring on $\mathbb{Q}$ and $\mathbb{Q}_S = \mathbb{Q}$. Then by Corollary 3.2.7 (1), $R \ltimes M = \mathbb{Z}_{2\mathbb{Z}} \ltimes \mathbb{Q}$ is a Manis valuation ring.

2. Let $R = \mathbb{Z}$ and $M = \mathbb{R}[x]$. Then from Example 2.1.15 (1), M is divisible R-module. Now since $\mathbb{Z}$ is a PID, it is Prüfer. As $Z_R(M) = 0 = Z(R)$, hence by Corollary 3.2.7 (2), $\mathbb{Z} \ltimes \mathbb{R}[x]$ is a Prüfer ring.

The following example gives a Prüfer ring which is not a Manis valuation ring.

Example 3.2.9. Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$. Then $Z_R(M) = 0 = Z(R)$. Since $\mathbb{Z}$ is a Prüfer ring and $\mathbb{Q}$ is divisible $\mathbb{Z}$-module. So by Corollary 3.2.7 (2), $R \ltimes M = \mathbb{Z} \ltimes \mathbb{Q}$ is a Prüfer ring. Next, since $\mathbb{Z}$ is not a valuation domain (since $x = \frac{2}{3} \in Quot(\mathbb{Z}) = \mathbb{Q}$, but $x \notin \mathbb{Z}$ and $x^{-1} = \frac{3}{2} \notin \mathbb{Z}$). So by Corollary 3.2.7 (1), $\mathbb{Z} \ltimes \mathbb{Q}$ is not a Manis valuation ring.

Notice that in general, $R \ltimes M$ Prüfer does not imply that R is Prüfer. The following example provides a ring R and an R-module M for which $R \ltimes M$ is a Prüfer ring but R is not Prüfer.

Example 3.2.10. Let $R = \mathbb{Q}[x, y]$ and $M = \bigoplus\{R/P \mid P \in Spec(R)\}$. Then as in Example 2.2.9, $R \ltimes M$ is a total quotient ring and hence is a Prüfer ring. We show that R is not a Prüfer ring. Consider the ideal $I = Rx + Ry$ of R. Claim: I is not invertible. On the contrary, suppose I is invertible, then $II^{-1} = R$. We know that $Quot(R) = \mathbb{Q}(x, y)$. Now,

$$I^{-1} = \{f \in Quot(R) \mid fI \subseteq R\} = \{f \in \mathbb{Q}(x, y) \mid xf \in \mathbb{Q}[x, y] \text{ and } yf \in \mathbb{Q}[x, y]\}.$$

Let $f \in I^{-1}$, then $xf \in \mathbb{Q}[x, y]$ and $yf \in \mathbb{Q}[x, y]$, so there are $p, q \in \mathbb{Q}[x, y]$ such that $xf = p$ and $yf = q$, but then $\frac{p}{x} = f = \frac{q}{y}$, which implies $yp = xq$, so $x \mid yp$, but since x and y are distinct primes, then $x \nmid y$, hence $x \mid p$, and so $f = \frac{p}{x} \in R$. This shows that $I^{-1} \subseteq R$. Clearly, $R \subseteq I^{-1}$. It follows that $I^{-1} = R$. We have $II^{-1} = IR = I \neq R$, a contradiction. Therefore, R is not a Prüfer ring.

Lemma 3.2.11. *Let R ba a ring and I an ideal of R. If R is a chained ring, then R/I is a chained ring.*

Proof. If J/I and K/I are two ideals in R/I, then either $J \subseteq K$ or $K \subseteq J$ since R is a chained ring. So either $J/I \subseteq K/I$ or $K/I \subseteq J/I$. Hence the set of ideals of R/I is totally ordered by inclusion. Thus, R/I is a chained ring. $\square$

The following theorem characterizes the chained rings of the form $R \ltimes M$.

Theorem 3.2.12 ([24]). *Let R be a ring and M a non-zero R-module. Then $R \ltimes M$ is a chained ring if and only if R is a valuation domain and M is divisible R-module whose (cyclic) submodules are totally ordered by inclusion.*

Proof. $(\Rightarrow)$. Suppose that $R \ltimes M$ is a chained ring. Then by the lemma above, $R \cong \frac{R \ltimes M}{0 \ltimes M}$ is a chained ring and since M is isomorphic to an ideal of $R \ltimes M$, the submodules of M are totally ordered by inclusion. Let $0 \neq a \in R$ and $m \in M$. Then $0 \ltimes M \subseteq (R \ltimes M)(a, 0)$. So $(0, m) = (a, 0)(r, n)$ for some $r \in R$ and $n \in M$. But then $m = an \in aM$. Hence $M = aM$. Now, let $a, b \in R - \{0\}$. Then $M = aM = abM$. Since $M \neq 0$, then $ab \neq 0$. Hence R is an integral domain and M is divisible.

$(\Leftarrow)$. Suppose that R is a valuation domain and M is a divisible R-module whose (cyclic) submodules are totally ordered by inclusion. Since M is divisible, then by Theorem 2.1.14, every ideal of $R \ltimes M$ has the form $I \ltimes M$ or $0 \ltimes N$ for some ideal I of R or submodule N of M. By hypothesis, it is clearly that the ideals of $R \ltimes M$ are totally ordered by inclusion. Hence $R \ltimes M$ is a chained ring. $\square$

Definition 3.2.13. Let R be a ring and M an R-module. M is arithmetical if the R_{m}-submodules of M_{m} are totally ordered by inclusion for each maximal ideal m of R.

The following corollary characterizes the arithmetical rings of the form $R \ltimes M$.

Corollary 3.2.14 ([6]). *Let R be a ring and M a non-zero R-module. Then $R \ltimes M$ is arithmetical ring if and only if R is an arithmetical ring, M is an arithmetical R-module, and for each maximal ideal m of R with $M_{\mathrm{m}} \neq 0_{\mathrm{m}}$, R_{m} is a valuation domain and M_{m} is a divisible R_{m}-module.*

Proof. $(\Rightarrow)$. Suppose that $R \ltimes M$ is an arithmetical ring. So $R_{\mathrm{m}} \ltimes M_{\mathrm{m}} \cong (R \ltimes M)_{\mathrm{m} \ltimes M}$ is a chained ring for each maximal ideal m of R. By the proof of Theorem 3.2.12, R_{m} is a chained ring and the R_{m}-submodules of M_{m} are totally ordered by inclusion for each maximal ideal m of R. It follows that R is an arithmetical ring and M is an arithmetical R-module. Next, let m be a maximal ideal of R with $M_{\mathrm{m}} \neq 0_{\mathrm{m}}$. Then $R_{\mathrm{m}} \ltimes M_{\mathrm{m}}$ is a chained ring gives that R_{m} is a valuation domain and M_{m} is a divisible R_{m}-module.

$(\Leftarrow)$. Let $\mathrm{m} \ltimes M$ be a maximal ideal of $R \ltimes M$ where m is a maximal ideal of R. Then $(R \ltimes M)_{\mathrm{m} \ltimes M} \cong R_{\mathrm{m}} \ltimes M_{\mathrm{m}}$. Now since R is arithmetical, then R_{m} is a chained ring, and since M is arithmetical, then the R_{m}-submodules of M_{m} are totally ordered by inclusion. If $M_{\mathrm{m}} = 0_{\mathrm{m}}$, then $R_{\mathrm{m}} \ltimes M_{\mathrm{m}}$ is a chained ring. So assume that $M_{\mathrm{m}} \neq 0_{\mathrm{m}}$. Then by hypothesis, R_{m} is a valuation domain and M_{m} is a divisible R_{m}-module. By Theorem 3.2.12, $R_{\mathrm{m}} \ltimes M_{\mathrm{m}}$ is a chained ring. Therefore, $R \ltimes M$ is an arithmetical ring. □

Next, we consider a particular construction using trivial ring extension, $R = F \ltimes V$ where F is a field and V is a vector space over F. The following corollary shows that $F \ltimes V$ is both a Manis valuation ring and a Prüfer ring, and it is arithmetical if and only if $\dim_F V = 1$.

Corollary 3.2.15. *Let F be a field, V a non-zero F-vector space, and $R = F \ltimes V$. Then*

1. *R is a Manis valuation ring.*
2. *R is a Prüfer ring.*
3. *R is arithmetical if and only if $\dim_F V = 1$.*

Proof. 1. (1) and (2). Since F is a field, it is both a Manis valuation ring and a Prüfer ring. Now, if $v \in V$ and $0 \neq a \in F$, then $v = aa^{-1}v \in aV$. So $V = aV$ and V is divisible. Because $Z_F(V) = 0 = Z(F)$, then by Corollary 3.2.7, R is both a Manis valuation ring and a Prüfer ring.

2. (⇒). Suppose that $R = F \ltimes V$ is an arithmetical ring. Since F is a field, it is a local valuation ring. So by Corollary 2.1.7 (1), R is local. So R is a local arithmetical ring, hence it is a chained ring. Let v_1, v_2 be two non-zero vectors of V. Then Fv_1 and Fv_2 are two cyclic subspaces of V. Since R is a chained ring, then by Theorem 3.2.12, either $Fv_1 \subseteq Fv_2$ or $Fv_2 \subseteq Fv_1$. So $v_1 = av_2$ or $v_2 = bv_1$ for some $a, b \in F$. So $\{v_1, v_2\}$ is linearly dependent. Hence $\dim_F V < 2$. But $V \neq 0$, so $\dim_F V = 1$.

   (⇐). Suppose that $\dim_F V = 1$, so there is a vector $0 \neq v_0 \in V$ such that $B = \{v_0\}$ is a basis for V. If $0 \neq v \in V$, $\dim_F Fv = 1$. So $Fv = V$. Hence $F0 = 0$ and $Fv_0 = V$ are the only cyclic subspaces of V. Hence the cyclic subspaces of V are totally ordered by inclusion. It follows by Theorem 3.2.12, that $R = F \ltimes V$ is a chained ring and hence is an arithmetical ring.

□

Next, we provide examples of a Manis valuation ring that is not a chained ring.

Example 3.2.16. 1. Let $F = \mathbb{Q}$, $V = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$, and $R = F \ltimes V$. Then by Corollary 3.2.15, R is a Manis valuation ring. But since $\dim_{\mathbb{Q}} \mathbb{Q}(\sqrt{2}) = 2 \neq 1$, then again by Corollary 3.2.15, R is not an arithmetical ring and hence R is not a chained ring because a chained ring is a local arithmetical ring.

2. Let $F = \mathbb{Z}_3$ and $V = \mathbb{Z}_3(i) = \{a + bi \mid a, b \in \mathbb{Z}_3\}$. Then $\dim_F V = 2 > 1$. So by Corollary 3.2.15, $R = F \ltimes V$ is a finite Manis valuation ring which is not a chained ring.

Next, we give examples of non-arithmetical Prüfer rings.

Example 3.2.17. 1. Since $\dim_{\mathbb{R}} \mathbb{R}^2 = 2 \neq 1$, so by Corollary 3.2.15, $\mathbb{R} \ltimes \mathbb{R}^2$ is a non-arithmetical Prüfer ring.

2. Let $R = \mathbb{R} \ltimes \mathbb{R}[x]$. Because $\dim_{\mathbb{R}} \mathbb{R}[x] = \infty$, then by Corollary 3.2.15, R is a non-arithmetical Prüfer ring.

3.3 Atomic rings and the ACCP via trivial ring extension

In this section we determine some irreducible elements in $R \ltimes M$ and also we determine when $R \ltimes M$ has the ascending chain condition for principal ideals, and we determine a sufficient condition for $R \ltimes M$ to be atomic. We start by recalling some definitions concerning factorization in commutative rings with zero divisors and in modules.

Definition 3.3.1 ([6])**.** Let R be a ring and M an R-module.

1. Two elements $m, n \in M$ are associates ($m \sim n$) if $Rm = Rn$. Taking $M = R$ gives the notion of "associates" in R.

2. A nonunit $a \in R$ is irreducible if $a = bc$ implies $a \sim b$ or $a \sim c$. And a is m-irreducible if Ra is a maximal element of the set of proper principal ideals of R. Also a is prime if $a \mid bc \Rightarrow a \mid b$ or $a \mid c$.

3. An element $m \in M$ is R-primitive if for $a \in R$ and $n \in M$, $m = an \Rightarrow m \sim n$.

4. R is called atomic if every (nonzero) nonunit of R is a product of irreducibles.

5. We say that R satisfies the ascending chain condition on principal ideals (ACCP) if every ascending chain of principal ideals of R terminates.

Remark 3.3.2 ([6])**.** Let R be a ring, M an R-module, and $a \in R$.

1. If a is m-irreducible, then a is irreducible.

2. If a is prime, then a is irreducible.

3. An element $m \in M$ is R-primitive if and only if Rm is a maximal cyclic R-submodule of M.

Proof.

1. Suppose that a is m-irreducible. If $a = bc$ with $Ra \neq Rb$ and $Ra \neq Rc$, then $Ra \subsetneq Rb$ and $Ra \subsetneq Rc$, so Ra is not maximal in the set of all principal ideals of R which is a contradiction since a is m-irreducible. So either $Ra = Rb$ or $Ra = Rc$, that is, either $a \sim b$ or $a \sim c$. Hence a is irreducible. Thus, a is m-irreducible implies a is irreducible.

2. Suppose that a is prime. If $a = bc$, then $a \mid bc$. Since a is prime, so either $a \mid b$ or $a \mid c$. If $a \mid b$, then $Rb \subseteq Ra$, but $Ra \subseteq Rb$ since $a = bc \in Rb$. So $Ra = Rb$, that is, $a \sim b$. Similarly, if $a \mid c$, then $a \sim c$. So a is irreducible. Thus, a is prime implies a is irreducible.

3. Assume that $m \in M$ is R-primitive. Let Rn be a proper cyclic submodule of M such that $Rm \subseteq Rn$. Then $m = rn$ for some $r \in R$. But m is R-primitive, so $m \sim n$ and hence $Rm = Rn$. Thus, Rm is a maximal cyclic R-submodule of M. Conversely, assume that Rm is a maximal cyclic R-submodule of M. Let $a \in R$ and $n \in M$. If $m = an$, then $Rm \subseteq Rn$. By hypothesis, $Rm = Rn$. So $m \sim n$. Thus, m is R-primitive

□

Theorem 3.3.3. *Let R be a ring. If R satisfies the ACCP, then R is atomic.*

Proof. See [4, Theorem 3.2]. □

The following theorem gives some irreducible elements of $R \ltimes M$ where R is an integral domain.

Theorem 3.3.4 ([4, 5])**.** *Let R be a ring, M an R-module, and $E = R \ltimes M$.*

1. *Let $m, n \in M$. Then $m \sim n$ if and only if $(0, m) \sim (0, n)$.*

2. *If R is an integral domain and $0 \neq m \in M$, then m is primitive in M if and only if $(0, m)$ is irreducible in E.*

3. *If R is an integral domain and $0 \neq a \in R$, then a is irreducible in R if and only if $(a, 0)$ is irreducible in E.*

4. *If R is an integral domain and $0 \neq a \in R$, then a is irreducible in R implies (a, m) is irreducible in E for all $m \in M$.*

5. *Suppose that R has a nontrivial idempotent and $M \neq 0$. Then no element $(0, m)$ of $0 \ltimes M$ is irreducible in E.*

Proof. 1. Let $m, n \in M$. Now $m \sim n$ if and only if $Rm = Rn$ if and only if $0 \ltimes Rm = 0 \ltimes Rn$ if and only if $E(0, m) = E(0, n)$ if and only if $(0, m) \sim (0, n)$.

2. Let R be an integral domain and $0 \neq m \in M$. Assume that m is primitive in M. Suppose $(0, m) = (a, n)(b, l)$ where $a, b \in R$ and $n, l \in M$. Then $ab = 0$ and $m = al + bn$. Since R is an integral domain, either $a = 0$ or $b = 0$. If $a = 0$, then $m = bn$. So $m \sim n$ since m is primitive. Hence by (1), $(0, m) \sim (0, n) = (a, n)$. Similarly if $b = 0$, then $(0, m) \sim (0, l) = (b, l)$. So $(0, m)$ is irreducible. Conversely, assume that $(0, m)$ is irreducible in E. Let $a \in R$ and $n \in M$. If $m = an$, then $(0, m) = (a, 0)(0, n)$. Since $(0, m)$ is irreducible, either $(0, m) \sim (a, 0)$

or $(0,m) \sim (0,n)$. Now $E(0,m) = 0 \ltimes Rm \neq Ra \ltimes aM = E(a,0)$ since $a \neq 0$ (for if $a = 0$, then $m = 0$, a contradiction). So $(0,m) \nsim (a,0)$. Hence $(0,m) \sim (0,n)$. By (1), $m \sim n$. Thus m is primitive.

3. Let R be an integral domain and $0 \neq a \in R$. Suppose that $(a,0)$ is irreducible and $a = bc$. Then $(a,0) = (b,0)(c,0)$. So $E(a,0) = E(b,0)$ or $E(a,0) = E(c,0)$. Hence $Ra = Rb$ or $Ra = Rc$. That is, $a \sim b$ or $a \sim c$. So a is irreducible. Conversely, suppose that a is irreducible and $(a,0) = (b,m)(c,n)$. Then $a = bc$. Since a is irreducible and R is an integral domain, b or c is a unit of R. If b is a unit of R, then (b,m) is a unit of E and so $E(a,0) = E(c,n)$, that is, $(a,0) \sim (c,n)$. Similarly if c is a unit of R, then $(a,0) \sim (b,m)$. Hence $(a,0)$ is irreducible.

4. Let R be an integral domain, and let $0 \neq a \in R$. Assume that a is irreducible. Let $m \in M$ be an arbitrary, and suppose $(a,m) = (b,n)(c,l)$. So $a = bc$. Since a is irreducible, either b or c is a unit of R. If b is a unit of R, (b,n) is a unit of E, so $(c,l) = (b,n)^{-1}(a,m) \in E(a,m)$, hence $E(a,m) = E(c,l)$, that is, $(a,m) \sim (c,l)$. Similarly, if c is a unit of R, then $(a,m) \sim (b,n)$. Thus, (a,m) is irreducible in E.

5. Let e be an idempotent of R such that $e \neq 0,1$. So $e = e^2$. Let $m \in M$ be an arbitrary. Then $(0,m) = (e-e^2, em+(m-em)) = (e(1-e), em+(1-e)m) = (e,m)(1-e,m)$. If $(0,m) \sim (e,m)$ or $(0,m) \sim (1-e,m)$, then $(e,m) \in E(0,m) = 0 \ltimes Rm$ or $(1-e,m) \in E(0,m) = 0 \ltimes Rm$. But then $e = 0$ or $e = 1$ which is a contradiction since $e \neq 0,1$. Hence $(0,m) \nsim (e,m)$ and $(0,m) \nsim (1-e,m)$. So $(0,m)$ is not irreducible. Therefore, there is no $m \in M$ such that $(0,m)$ is irreducible in $R \ltimes M$.

□

Next, we use Theorem 3.3.4 to give an example of an irreducible element that is neither prime nor m-irreducible.

Example 3.3.5 ([4, Example 5.7]). Let $R = \mathbb{Z} \ltimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, and let $a = (0,(0,1)) \in R$. Since $\mathbb{Z}(0,1) = 0 \oplus \mathbb{Z}_2$ is a maximal cyclic $\mathbb{Z}$-submodule of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, then $(0,1)$ is primitive in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. So by Theorem 3.3.4 (2), a is irreducible. Note that $(0,(1,0))(2,(1,0)) = (0,(0,0)) \in Ra$, but $(0,(1,0)),(2,(1,0)) \notin Ra$. This means that Ra is not a prime ideal of R. So a is not prime. Next, we show that a is not m-irreducible. Let $b = (3,(0,0)) \in R$. Then $a = ab \in Rb$ but $b \notin Ra$. So $Ra \subsetneq Rb$. Hence Ra is not a maximal element of the set of proper principal ideals of R. Thus a is not m-irreducible.

Definition 3.3.6. Let R be a ring. An R-module M is said to satisfy the ascending chain condition on cyclic submodules (ACCC) if every ascending chain of cyclic submodules of M terminates.

The following theorem determines when $R \ltimes M$ satisfies the ACCP where R is an integral domain.

Theorem 3.3.7 ([4]). *Let R be an integral domain, M an R-module, and $E = R \ltimes M$.*

1. *If R satisfies the ACCP, then every ascending chain of principal ideals of E containing a principal ideal of the form $E(a,m)$ where $0 \neq a \in R$ terminates.*

2. *E satisfies the ACCP if and only if R satisfies the ACCP and M satisfies the ACCC.*

Proof. 1. Suppose $0 \neq a \in R$ and $E(a,m) \subsetneq E(b,n)$. Then $(a,m) = (b,n)(c,l)$ for some $(c,l) \in R \ltimes M$. So $a = bc$. If c is a unit of R, then (c,l) is a unit of E and so $(b,n) \in E(a,m)$, a contradiction. So c is not a unit of R and hence $Ra \subsetneq Rb$. Thus, if R satisfies the ACCP, then every ascending chain of principal ideals of E containing a principal ideal of the form $E(a,m)$ where $0 \neq a \in R$ terminates.

2. ($\Rightarrow$). Suppose that E satisfies the ACCP. We know by Theorem 2.1.10, that $E(a,0) = Ra \ltimes aM$ and $E(0,m) = 0 \ltimes Rm$ for all $a \in R$ and all $m \in M$. Now since E satisfies the ACCP, then E satisfies the ACCP on ideals of the form $E(a_1,0) \subseteq E(a_2,0) \subseteq \cdots$ and $E(0,n_1) \subseteq E(0,n_2) \subseteq \cdots$. Hence, R satisfies the ACCP and M satisfies the ACCC.

   ($\Leftarrow$). Suppose that R satisfies the ACCP and M satisfies the ACCC. Let $E(a_1,n_1) \subseteq E(a_2,n_2) \subseteq \cdots$ be an ascending chain of principal ideals of E. If $a_i = 0$ for each i, then $E(0,n_1) \subseteq E(0,n_2) \subseteq \cdots$. So $Rn_1 \subseteq Rn_2 \subseteq \cdots$ which stops since M satisfies the ACCC. Hence the original chain in E terminates. If $a_i \neq 0$ for some i, then by (1), the chain terminates. It follows that E satisfies the ACCP.

□

Definition 3.3.8 ([6]). Let R be a ring. An R-module M is said to satisfy the MCC if every cyclic submodule of M is contained in a maximal (not necessarily proper) cyclic submodule.

The following theorem gives a sufficient condition for $R \ltimes M$ to be atomic.

Theorem 3.3.9 ([4]). *Let R be an integral domain and M an R-module. If R satisfies the ACCP and M satisfies the MCC, then $R \ltimes M$ is atomic.*

Proof. Assume that R satisfies the ACCP and M satisfies the MCC. Let $(0,0) \neq (a,n) \in R \ltimes M$ be a nonunit. Either $a \neq 0$ or $a = 0$. If $a \neq 0$, then by Theorem 3.3.7 (1), (a,n) is a product of irreducibles. If $a = 0$, then $n \neq 0$. Since M satisfies the MCC, then $Rn \subseteq Rm$ where Rm is a maximal cyclic submodule of M. So m is primitive and hence by Theorem 3.3.4 (2), $(0,m)$ is irreducible. Now $n = cm$ for some $0 \neq c \in R$. Since R satisfies the ACCP, R is atomic. So $(c,0)$ is either a unit or a product of irreducible. Hence $(0,n) = (c,0)(0,m)$ is a product of irreducibles. Therefore, $R \ltimes M$ is atomic. □

Theorem 3.3.10. *Let R be an integral domain and M an R-module. If $R \ltimes M$ is atomic, then M satisfies the MCC.*

Proof. Suppose that $R \ltimes M$ is atomic. Let Rn be a cyclic submodule of M. Then $(0,n) = (a_1,n_1)\cdots(a_s,n_s)$ where each (a_i,n_i) is an irreducible of $R \ltimes M$. Now $a_1 \cdots a_s = 0$, so say $a_s = 0$. Then $n = a_1 \cdots a_{s-1} n_s$. So $Rn \subset Rn_s$. Since $(0,n_s) = (a_s,n_s)$ is an irreducible, then by Theorem 3.3.4 (2), n_s is primitive. Hence Rn_s is a maximal cyclic submodule of M. □

Next, we consider a particular construction using trivial ring extension, $R = D \ltimes D/\mathrm{m}$ where D is a local integral domain and m is a maximal ideal of D. The next result shows that R is an atomic ring and that R satisfies the ACCP if and only if D satisfies the ACCP.

Theorem 3.3.11 ([3]). *Let (D,m) be a local integral domain with maximal ideal* m *and let $R = D \ltimes D/\mathrm{m}$. Then*

1. *R is always atomic.*
2. *R satisfies the ACCP if and only if D satisfies the ACCP.*

Proof. 1. Let $(r,\bar{n})$ be a nonzero nonunit of R. Then $r \notin U(D)$. So $r \in \mathrm{m}$ since $D - U(D) = \mathrm{m}$. Either $r = 0$ or $r \neq 0$. First, assume $r = 0$, then $\bar{n} \neq \bar{0}$ since $(r,\bar{n})$ is nonzero. We claim that $(r,\bar{n}) = (0,\bar{n})$ is irreducible. Suppose $(0,\bar{n}) = (a,\bar{x})(b,\bar{y})$. Then $0 = ab$. Since D is an integral domain, either $a = 0$ or $b = 0$. If $a = b = 0$, then $(0,\bar{n}) = (0,\bar{0})$, a contradiction. So only one of a and b is 0, say a is 0. Then $(0,\bar{n}) = (0,\bar{x})(b,\bar{y}) = (0,b\bar{x})$, so $\bar{n} = b\bar{x}$, hence $n - bx \in \mathrm{m}$. Since $\bar{n} \neq \bar{0}$, then $n \notin \mathrm{m}$, so $b \notin \mathrm{m}$ for if $b \in \mathrm{m}$, then $bx \in \mathrm{m}$, but then $n = bx + (n - bx) \in \mathrm{m}$, a contradiction. Hence $b \in U(D)$, thus $(b,\bar{y})$ is a unit of R. Similarly, if $b = 0$, then $(a,\bar{x})$ is a unit of R. So $(r,\bar{n}) = (0,\bar{n})$ is an irreducible. Next, assume $r \neq 0$. If $(r,\bar{n}) = (a,\bar{x})(b,\bar{y})$, then $r = ab$. But $r \in \mathrm{m}$, so either $a \in \mathrm{m}$ or $b \in \mathrm{m}$. If $a \in \mathrm{m}$ but $b \notin \mathrm{m}$, then $b \in U(D)$, so

$(b, \bar{y})$ is a unit of R . Similarly, if $a \notin \mathrm{m}$ but $b \in \mathrm{m}$, then $(a, \bar{x})$ is a unit of R. It follows that $(r, \bar{n})$ is an irreducible. If both $a, b \in \mathrm{m}$, then $a\bar{y} = b\bar{x} = \bar{0}$. So

$$(r, \bar{n}) = (a, \bar{x})(b, \bar{y}) = (ab, \bar{0}) = (ab, a\bar{1} + b\bar{1}) = (a, \bar{1})(b, \bar{1}).$$

But $(s, \bar{1})$ is an irreducible for all $0 \neq s \in \mathrm{m}$. Indeed, if $(s, \bar{1}) = (c, \bar{k})(d, \bar{l})$, then $(s, \bar{1}) = (cd, c\bar{l} + d\bar{k})$. If both $c, d \in \mathrm{m}$, then $(s, \bar{1}) = (cd, \bar{0})$, a contradiction. So either $c \notin \mathrm{m} = D - U(D)$ or $d \notin \mathrm{m} = D - U(D)$, hence either $(c, \bar{k})$ or $(d, \bar{l})$ is a unit of R. Therefore, $R = D \ltimes D/\mathrm{m}$ is an atomic ring.

2. Since the only cyclic submodules of D/m are $D\bar{0}$ and $D\bar{n} = D/\mathrm{m}$ where $\bar{0} \neq \bar{n} \in D/\mathrm{m}$. So D/m satisfies the ACCC. It follows by Theorem 3.3.7, that R satisfies the ACCP if and only if D satisfies the ACCP.

□

The last theorem shows that an atomic ring need not satisfy the ACCP.

Definition 3.3.12 ([6]). Let R be a ring. We say that R is r-atomic if every regular nonunit of R is a product of irreducibles and that R satisfies the r-ACCP if every ascending chain of regular principal ideals terminates.

Remark 3.3.13. Let R be a ring.

1. If R is atomic, then R is r-atomic. Where the converse is not true in general.

2. If a and b are two regular elements of R, then $a \sim b$ if and only if $a = ub$ for some unit u of R.

The following theorem shows how a trivial ring extension can be used to give examples of rings satisfying the factorization properties for the regular elements.

Theorem 3.3.14 ([6]). *Let R be an integral domain and M an R-module.*

1. *If R satisfies the ACCP, then $R \ltimes M$ satisfies the r-ACCP.*

   Suppose further that $M = M_S$ where $S = R - (Z(R) \cup Z_R(M)) = R - Z_R(M)$.

2. *If R is atomic, then $R \ltimes M$ is r-atomic.*

Proof. 1. Assume R satisfies the ACCP. Let $(R \ltimes M)(a, m)$ be a regular principal ideal of $R \ltimes M$. Then there is $(b, n) \in R \ltimes M$ such that $(b, n)(a, m)$ is regular. So $ba \neq 0$ and hence $a \neq 0$. By Theorem 3.3.7 (1), $R \ltimes M$ satisfies the r-ACCP.

2. Suppose that $M = M_S$ where $S = R - Z(M)$. Let (r, m) be a regular nonunit of $R \ltimes M$. Then $r \in S$ and r is a nonunit of R. Since $M = M_S$, $m = rm'$ for some $m' \in M$. So $(r, m) = (r, 0)(1, m')$. Hence $(r, 0) \sim (r, m)$. By the remark above, $(r, m) = u(r, 0)$ for some unit u of $R \ltimes M$. Now, R is atomic implies that $r = r_1 \cdots r_n$ where each r_i is an irreducible of R. So each $(r_i, 0)$ is a regular irreducible of $R \ltimes M$. Hence, $(r, m) = u(r, 0) = u(r_1, 0) \cdots (r_n, 0)$ is a product of irreducibles. Therefore, $R \ltimes M$ is r-atomic.

□

The following example gives a ring R that is r-atomic but not atomic.

Example 3.3.15 ([4, Example 5.5])**.** Let $R = \mathbb{Z} \ltimes (\mathbb{Z}_2 \oplus \mathbb{Q})$. Now since $\mathbb{Z}$ is atomic, then by Theorem 3.3.14(2), R is r-atomic. Next, note that $\mathbb{Z}(1, 0)$ is a maximal cyclic submodule of $\mathbb{Z}_2 \oplus \mathbb{Q}$, but no other nonzero cyclic submodule is contained in a maximal cyclic submodule since $\mathbb{Z}(1, a) \subset \mathbb{Z}(1, a/3)$ and $\mathbb{Z}(0, a) \subset \mathbb{Z}(0, a/3)$ for any $a \in \mathbb{Q} - \{0\}$. So $\mathbb{Z}_2 \oplus \mathbb{Q}$ does not satisfy the MCC. Hence by Theorem 3.3.10, R is not atomic.

Chapter 4

Applications

In this chapter, we determine the structure of Boolean-like rings using the trivial ring extension, also we determine clean and nil-clean rings of the form $R \ltimes M$.

4.1 Structure of Boolean-like rings

This section is devoted to study the structure of Boolean-like rings using trivial ring extension. We start by recalling the following definition.

Definition 4.1.1 ([2])**.** Let R be a ring.

1. R is called a Boolean-like ring if char $R = 2$ and $xy(1+x)(1+y) = 0$ for all $x, y \in R$.

2. R is called an n-Boolean ring if char $R = 2$ and $x_1 \cdots x_n(1+x_1)\cdots(1+x_n) = 0$ for all $x_1, ..., x_n \in R$.

Remark 4.1.2. Boolean rings are 1-Boolean rings, and Boolean-like rings are 2-Boolean rings.

The following proposition gives an equivalent condition for a ring R to be n-Boolean.

Proposition 4.1.3 ([6])**.** Let R be a ring. Then R is n-Boolean if and only if char $R = 2$, $R/nil(R)$ is Boolean, and $nil(R)^n = 0$

Proof. Suppose that R is n-Boolean. Then by Definition 4.1.1, char $R = 2$. Now, let $x \in R$. Then $x^n(1+x)^n = 0$ since R is n-Boolean. So $(x(1+x))^n = 0$. But then $x(1+x) \in nil(R)$. So $\overline{x(1+x)} = \bar{0}$ or $\bar{x}(\bar{1}+\bar{x}) = \bar{0}$. Hence $R/nil(R)$ is 1-Boolean or equivalently, $R/nil(R)$ is Boolean. Next, let $x_1, ..., x_n \in nil(R)$. We have $x_1 \cdots x_n(1+x_1)\cdots(1+x_n) = 0$ since R is n-Boolean. As

$x_i \in nil(R)$ for all $i = 1, ..., n$, $1+x_i \in U(R)$ for all $i = 1, ..., n$. So $x_1 \cdots x_n(1+x_1)\cdots(1+x_n) = 0$ gives that $x_1 \cdots x_n = 0$. Hence $nil(R)^n = 0$. Conversely, let $x_1, ..., x_n \in R$. Then $\bar{x_1}, ..., \bar{x_n} \in R/nil(R)$. But $R/nil(R)$ is Boolean, so $\bar{x_i}(\bar{1} + \bar{x_i}) = \bar{0}$ for all $i = 1, ..., n$. But then $x_i(1 + x_i) \in nil(R)$ for all $i = 1, ..., n$. So $x_1 \cdots x_n(1 + x_1) \cdots (1 + x_n) = x_1(1 + x_1) \cdots x_n(1 + x_n) \in nil(R)^n = 0$. Thus, R is n-Boolean. □

Next, we determine a sufficient condition for $R \ltimes M$ to be n-Boolean ring. We start by the following lemma.

Lemma 4.1.4. *Let R be a ring and M an R-module. Then for each $k \in \mathbb{N}$, $nil(R \ltimes M)^k = nil(R)^k \ltimes nil(R)^{k-1}M$.*

Proof. To prove this lemma, we will use induction. By Theorem 2.1.19 (1), we have $nil(R \ltimes M) = nil(R) \ltimes M$. For $k = 2$,

$$(nil(R) \ltimes M)^2 = nil(R)^2 \ltimes 2nil(R)M = nil(R)^2 \ltimes nil(R)^1M.$$

So its true for $k = 2$. If $(nil(R) \ltimes M)^k = nil(R)^k \ltimes nil(R)^{k-1}M$, then

$$\begin{aligned}
(nil(R) \ltimes M)^{k+1} &= (nil(R)^k \ltimes nil(R)^{k-1}M)(nil(R) \ltimes M) \\
&= nil(R)^{k+1} \ltimes 2nil(R)^kM \\
&= nil(R)^{k+1} \ltimes nil(R)^kM \\
&= nil(R)^{k+1} \ltimes nil(R)^kM.
\end{aligned}$$

Hence for each $k \in \mathbb{N}$, $nil(R \ltimes M)^k = (nil(R) \ltimes M)^k = nil(R)^k \ltimes nil(R)^{k-1}M$. □

Theorem 4.1.5 ([2]). *Let R be a ring and M an R-module. If R is n-Boolean, then $R \ltimes M$ is (n+1)-Boolean. Moreover, $R \ltimes M$ is n-Boolean if and only if $nil(R)^{n-1}M = 0$*

Proof. Suppose that R is n-Boolean. Then char $R = 2$. So for any $x \in M$, $2x = 2(1.x) = (2.1)x = 0x = 0$. Hence char $(R \ltimes M) = 2$. Now $(R \ltimes M)/nil(R \ltimes M) = (R \ltimes M)/nil(R) \ltimes M \cong R/nil(R)$ is a Boolean ring. Next, since R is n-Boolean, $nil(R)^n = 0$. Hence by Lemma 4.1.4, $nil(R \ltimes M)^{n+1} = nil(R)^{n+1} \ltimes nil(R)^nM = 0$. So R is n-Boolean implies that $R \ltimes M$ is an (n+1)-Boolean ring and $R \ltimes M$ is n-Boolean $\Leftrightarrow nil(R \ltimes M)^n = 0 \Leftrightarrow nil(R)^{n-1}M = 0$. □

The following theorem gives a structure theory for Boolean-like rings using trivial ring extension.

Theorem 4.1.6 ([2]). *(Structure theory for Boolean-like rings) If B is a Boolean ring and M is a B-module, then $B \ltimes M$ is a Boolean-like ring. Conversely, suppose that R is a Boolean-like ring. Then $\overline{R} = R/nil(R)$ is a Boolean ring and $R \cong \overline{R} \ltimes nil(R)$ where $nil(R)$ is considered as an $\overline{R}$-module (since $nil(R)^2 = 0$). Equivalently, if $B = \{b \in R \mid b = b^2\}$, then B is a Boolean subring of R (with $B \cong \overline{R}$) and $R \cong B \ltimes nil(R)$ where $nil(R)$ is considered as a B-module.*

Proof. If B is a Boolean ring, then B is a 1-Boolean ring and by Theorem 4.1.5, B is a 2-Boolean ring or equivalently, B is a Boolean-like ring.

Conversely, suppose that R is a Boolean-like ring. Then char $R = 2$, $\overline{R} = R/nil(R)$ is a Boolean ring, and $nil(R)^2 = 0$. Since char $R = 2$, then $B = \{b \in R \mid b = b^2\}$ is a Boolean subring of R. Let $r \in R$. Then $\bar{r} = \bar{r}^2$ in $\overline{R}$. So $n = r - r^2 \in nil(R)$ and $r = r^2 + n$ where $n \in nil(R)$. We claim that $r^2 \in B$. Since R is a Boolean-like ring and $r \in R$, then $r^2(1+r)^2 = 0$. But char $R = 2$, so $(x+y)^2 = x^2 + y^2$ and $x = -x$ for all $x, y \in R$. Now, $r^2(1+r)^2 = 0$ $\Rightarrow r^2(1+r^2) = 0 \Rightarrow r^2 + r^4 = 0 \Rightarrow r^2 = -r^4 = r^4$. So $r^2 \in B$. Hence $r = b + n$ where $b = r^2 \in B$ and $n \in nil(R)$. This shows that $R = B + nil(R)$. Moreover, $B \cap nil(R) = 0$. So $R = B \oplus nil(R)$. Now $B/(B \cap nil(R)) \cong (B + nil(R))/nil(R)$ gives $B \cong R/nil(R) = \overline{R}$. Because $nil(R)$ is an R-module, $nil(R)/nil(R)^2$ is an $R/nil(R)$-module. But $nil(R)^2 = 0$, so $nil(R)$ is an $\overline{R}$-module and hence $nil(R)$ is a B-module. Finally, define $\varphi : R \to B \ltimes nil(R)$ by $r = b + n \mapsto (b, n)$. Then clearly φ is well-defined (as $R = B \oplus nil(R)$), bijection, and group homomorphism. We show that φ preserve multiplication. Let $r_1 = b_1 + n_1, r_2 = b_2 + n_2 \in R$ where $b_1, b_2 \in B$ and $n_1, n_2 \in nil(R)$. Since $nil(R)^2 = 0$, $n_1 n_2 = 0$. Also, since $nil(R)$ is an ideal of R, $b_1 n_2 + b_2 n_1 \in nil(R)$. Now $\varphi(r_1 r_2) = \varphi(b_1 b_2 + b_1 n_2 + b_2 n_1 + n_1 n_2) = \varphi(b_1 b_2 + b_1 n_2 + b_2 n_1) = (b_1 b_2, b_1 n_2 + b_2 n_1) = (b_1, n_1)(b_2, n_2) = \varphi(r_1)\varphi(r_2)$. $\square$

The following is an illustrative example for Theorem 4.1.6.

Example 4.1.7. Let $A = \mathbb{Z}_2[x]$, $I = Ax^2$, and $R = A/I$. Then $R = \{0+I, 1+I, x+I, 1+x+I\}$. Since char $A = 2$, then char $R = 2$. Now, $nil(R) = \{0+I, x+I\}$. So $nil(R)^2 = 0$ and $R/nil(R) = \{0 + I + nil(R), 1 + I + nil(R)\} \cong \mathbb{Z}_2$ is Boolean. It follows by Proposition 4.1.3, that R is a Boolean-like ring. So by Theorem 4.1.6, $R \cong R/nil(R) \ltimes nil(R)$. In fact, $R \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_2$.

A natural question is whether Theorem 4.1.6 can be extended to n-Boolean rings for $n > 2$. The answer of this question is No. The next example shows that a 3-Boolean ring need not be the trivial ring extension of a 2-Boolean ring.

Example 4.1.8 ([2], pages 74-75)**.** Let $R = \frac{\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]} \times \mathbb{Z}_2$. Then R satisfies the following statements:

1. R is a 3-Boolean ring.

2. R is not a 2-Boolean ring.

3. R is not the trivial ring extension of a 2-Boolean ring.

Proof. 1. Since char $\mathbb{Z}_2[x] = 2$ and char $\mathbb{Z}_2 = 2$, then char $R = 2$. Now

$$nil(R) = nil(\frac{\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}) \times nil(\mathbb{Z}_2) = \frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]} \times 0.$$

So $nil(R)^3 = 0$ and $R/nil(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is a Boolean ring. Thus, by Proposition 4.1.3, R is a 3-Boolean ring.

2. Let $\beta = (x + x^3\mathbb{Z}_2[x], 0)$. Then $\beta \in nil(R)$, so $\beta^2 \in nil(R)^2$. But

$$\beta^2 = (x^2 + x^3\mathbb{Z}_2[x], 0) \neq (0 + x^3\mathbb{Z}_2[x], 0).$$

So $nil(R)^2 \neq 0$ and hence R is not a 2-Boolean ring.

3. By contradiction, suppose that $R \cong A \ltimes M$ where A is a 2-Boolean ring and M is an A-module. Note that $|R| = 8 \cdot 2 = 16$. So $|A| = 1, 2, 4, 8, or 16$. If $|A| \leq 2$ or $|A| = 16$, then A is Boolean or $R \cong A$. If A is Boolean, then by Theorem 4.1.5, $R \cong A \ltimes M$ is a 2-Boolean ring, a contradiction. If $R \cong A$, R is a 2-Boolean ring, a contradiction. So either $|A| > 2$ or $|A| \neq 16$. Hence $|A| = 4$ or 8. Now if $|A| = 4$, then $|M| = 4$. By part (1), $|nil(R)| = 4$. So $|nil(A) \ltimes M| = 4$ and then $|nil(A)| = 1$, that is $nil(A) = 0$. So $A/nil(A) \cong A$. As A is 2-Boolean, then $A \cong A/nil(A)$ is Boolean and hence again by Theorem 4.1.5, $R \cong A \ltimes M$ is 2-Boolean, a contradiction. Next, if $|A| = 8$, then $|M| = 2$. Since A is 2-Boolean, $nil(A)^2 = 0$. Now

$$|nil(R)^2| = |nil(A \ltimes M)^2| = |nil(A)^2 \ltimes nil(A)M| = |nil(A)M|.$$

By part (1), $nil(R)^2 \neq 0$, so $|nil(A)M| = |nil(R)^2| \neq 1$. Since $nil(A)M \subseteq M$ and $|M| = 2$, then $nil(A)M = M$. So $M = nil(A)M = nil(A)nil(A)M = nil(A)^2M = 0$, a contradiction (as $|M| = 2$). Therefore, R is not the trivial ring extension of a 2-Boolean ring.

$\square$

4.2 Clean and nil-clean rings.

In this section we will study the Clean, Weakly clean, Nil-Clean, and Weakly Nil-Clean rings via trivial ring extension. We start with the following definition.

Definition 4.2.1 ([22]). A ring R is called clean if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$.

Example 4.2.2. (Boolean rings are clean rings). Let B be a Boolean ring. Then $U(B) = \{1\}$ and $Id(B) = B$. Since each $b \in B$ can be written as $b = 1 + (b - 1)$, then B is a clean ring.

Definition 4.2.3 ([22]). A ring R is said to be weakly clean if for all $x \in R$, either $x = u + e$ or $x = u - e$ for some unit u and some idempotent e.

The following theorem characterizes when $R \ltimes M$ is a clean ring or weakly clean ring.

Theorem 4.2.4 ([1]). *Let R be a ring and M an R-module. $R \ltimes M$ is clean (weakly clean) if and only if R is clean (weakly clean).*

Proof. We know by Theorem 2.1.19, that $U(R \ltimes M) = U(R) \ltimes M$ and $Id(R \ltimes M) = Id(R) \ltimes 0$. If $R \ltimes M$ is clean and $r \in R$. Then $(r, 0) = (u, m) + (e, 0)$ for some $(u, m) \in U(R \ltimes M)$ and some $(e, 0) \in Id(R \ltimes M)$. So $r = u + e$ for some $u \in U(R)$ and some $e \in Id(R)$. Hence R is clean. A similar argument shows that if R is weakly clean, then $R \ltimes M$ is weakly clean. Conversely, if R is clean and $(r, m) \in R \ltimes M$. Then $r = u + e$ for some $u \in U(R)$ and some $e \in Id(R)$. So $(r, m) = (u + e, m) = (u, m) + (e, 0)$ for some $(u, m) \in U(R \ltimes M)$ and some $(e, 0) \in Id(R \ltimes M)$. Hence $R \ltimes M$ is clean. A similar argument shows that if $R \ltimes M$ is weakly clean, then R is weakly clean. $\square$

Theorem 4.2.4 can be used to give a class of a non-Boolean clean rings.

Example 4.2.5. Let R be a Boolean ring and M a nonzero R-module. Since $M \neq 0$, then $R \ltimes M$ is not a Boolean ring. By Example 4.2.2, R is Boolean gives that R is clean. So by Theorem 4.2.4, $R \ltimes M$ is a clean ring.

Definition 4.2.6 ([11]). A ring R is called nil-clean if each $r \in R$ can be written as $r = q + e$, where $q \in nil(R)$ and $e \in Id(R)$.

Definition 4.2.7 ([10]). A ring R is called weakly nil-clean ring if every $r \in R$ can be presented as either $r = q + e$ or $r = q - e$, where $q \in nil(R)$ and $e \in Id(R)$.

The following theorem characterizes when $R \ltimes M$ is a nil-clean ring or weakly nil-clean ring.

Theorem 4.2.8. *Let R be a ring and M an R-module. $R \ltimes M$ is nil-clean (weakly nil-clean) if and only if R is nil-clean (weakly nil-clean).*

Proof. We know by Theorem 2.1.19, that $nil(R \ltimes M) = nil(R) \ltimes M$ and $Id(R \ltimes M) = Id(R) \ltimes 0$. Now, suppose that $R \ltimes M$ is a nil-clean ring and let $r \in R$. Then $(r, 0) = (q, m) + (e, 0)$ for some $(q, m) \in nil(R \ltimes M)$ and some $(e, 0) \in Id(R \ltimes M)$. So $r = q + e$ for some $q \in nil(R)$ and some $e \in Id(R)$. Hence R is a nil-clean ring. A similar argument shows that if R is weakly nil-clean, then $R \ltimes M$ is weakly nil-clean. Conversely, suppose that R is a nil-clean ring and let $(r, m) \in R \ltimes M$. Then $r = q + e$ for some $q \in nil(R)$ and some $e \in Id(R)$. So $(r, m) = (q + e, m) = (q, m) + (e, 0)$ for some $(q, m) \in nil(R \ltimes M)$ and some $(e, 0) \in Id(R \ltimes M)$. Hence $R \ltimes M$ is a nil-clean ring. A similar argument shows that if $R \ltimes M$ is nil-weakly clean, then R is nil-weakly clean. $\square$

Example 4.2.9. Let $R = \mathbb{Z}_3$, and let M be a nonzero R-module. Then R is a weakly nil-clean ring but not nil-clean. So by Theorem 4.2.8, $R \ltimes M$ is a weakly nil-clean ring but not nil-clean.

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