Faculty of Graduate Studies

Master of Mathematics

Filters and Various Types of Convergence

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Submitted in partial fulfillment of the requirements for the degree of Master of Mathematics, Faculty of Graduate Studies, Hebron University, Hebron, Palestine.

2022

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Dedication

To the soul of Professor of Mathematics M. E. Abd El-Monsef.

To Prof. Abdelmonem Mohamed Kozae and Prof. Ahmed A. Salama from Egypt.

To Prof. Maximilian Ganster and Prof. Saeid Jafari from Austria.

To Prof. Ali Ahmad Ali Fora from Jordan.

Acknowledgment

Thanks to Allah, for his help and for giving me the ability to complete this thesis.

I would like to express my deep gratitude to my supervisor Dr. Bassam A. Manasrah for his valuable suggestions, extensive discussion, encouragement, and continuous guidance and to Mr. Karol Giedroyc from Canada for his collegial assistance. Thanks to the Internal Examinar Dr. Yousef Manasrah and the External Examiner Dr. Mohammad Abu Eideh.

I would like to express the appreciation for my parents and my helpful wife for their fruitful assistance and support to complete this thesis.

I also wish to express my thanks to all the staff of the Department of Mathematics, College of Sciences and Mathematics, Hebron University.

Finally, I must thank all who encouraged me and reach my goal, especially to Dr. Ali Abdel-mohsin Altawaiha, Mr. Abdel-mu'ti Abu Eisheh and my collegue, Mohammad Waleed Adarbeh.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان

المرشحات وأنواع مختلفة من التقارب

FILTERS AND VARIOUS TYPES OF CONVERGENCE

أقر بأن ما إشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص،
باستثناء ما تم الإشارة إليه حيثما ورد، وأن هذه الرسالة ككل لم تقدم
من قبل لنيل أي درجة علمية أو بحث علمي أو بحث لدى أي مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the result of the researcher's work, and has not been submitted elsewhere for any other degree or qualification.

Contents

Abstract

In this thesis, we try to facilitate the concept of convergence of filters by introducing different types of convergence, including the following: δ, θ and rc-convergence. The similarities among δ, θ, and rc-limit points as well as their δ, θ, and rc-cluster points are studied. A number of results, whose statements are parallel to those in the usual sense of convergence, are established.

Some of the topological properties could be characterized by filters, the δ, θ, and rc-cluster (δ, θ, and rc-adherent) points of a subset are introduced in the obvious way. Furthermore, δ, θ, and rc-convergence are studied under various types of topologies, such as, the convergence of filters in the product topology.

Some types of compactness are introduced, including the following: nearly compact, quasi-H-closed, and S-closed spaces, without any axiom of separation to be assumed. These versions of compactness are characterized through filters.

The concepts of almost-strongly closed graph, strongly closed graph, and rc-strongly closed graph are defined. These concepts, which are characterized by filters, are parallel to the closed graph concept in the usual sense of convergence. Also, a filter notion leads to an important theory of convergence structure in topological spaces.

الملخص

في هذه الرسالة، نحاول تسهيل مفهوم تقارب المرشحات من خلال تقديم أنواع مختلفة من التقارب، تشمل تقارب من نوع δ و θ و rc ،كذلك أوجه التشابة بين نقاط النهاية و نقاط التراكم من نوع δ و θ و rc تمت دراستها، تم دراسة عدد من النتائج لتلك الأنواع التي تكون موازية للتقارب العادي.

يمكن تمثيل بعض الخصائص الطوبولوجية عن طريق المرشحات، نقاط التراكم و الإلصاق من نوع δ و θ و rc لمجموعة تقدم بطريقة واضحة. علاوة على ذلك، تتم دراسة التقارب من الأنواع δ و θ و rc تحت أنواع مختلفة من الطبولوجيا ، مثل تقارب المرشحات في طوبولوجيا الضرب.

يتم تقديم بعض أنواع من فضاءات التراص، تشمل nearly compact و quasi-H-closed و S-closed دون افتراض أي من مسلمات الفصل. يتم تمثيل هذه الأنواع من التراص من خلال المرشحات.

يتم تعريف مفاهيم strongly closed graph و almost-strongly closed graph و rc-strongly closed graph . هذه المفاهيم يتم تمثيلها عن طريق المرشحات، موازية لمفهوم closed graph بالمعنى المعتاد للتقارب. كذلك تؤدي فكرة المرشح إلى نظرية مهمة لبنية التقارب في الفضاءات الطوبولوجية.

Introduction and Historical Overview of Filters

The notion of a limit has a significant role in analysis and it has been received frequent and varied treatment by many mathematicians. Of the several theories available, there are two which appear to be most popular in current use. One is that of *net*, which was initiated by E. H. Moore and H. L. Smith [72] and has been discussed and improved by J. L. Kelley [55]; the other is that of *filter*, discovered by Henri Cartan, which finally eliminates countability from topology by replacing the notion of a sequence and introduces the modern notion of compactness [20, 21]. The theory of filters is the convergence theory of choice for many topologists. Filters appear in *order and lattice theory*, but can also be found in topology. It appears that the net notion is predominant in the U.S.A., whereas the filter theory reigns supreme in France. In 1915, a paper by E. H. Moore appeared in the Proceedings of the National Academy of Science U.S.A. titled *Definition of limit in general integral analysis* [71]. This study of unordered summability of sequences led to a theory of convergence by E. H. Moore and H. L. Smith titled *A general theory of limits* which appeared in the American Journal of Mathematics in 1922 [72]. Convergence has been studied via *filter bases* [1] by Vietoris [116]. He was the first to introduce filter bases, directed sets, nets and the related convergence and introduces the modern notion of compactness [1]. Also, convergence has been studied via *filter bases* by G. Birkhoff [13, 14], via *filters* by Cartan [21], and via *ultrafilters* by Cartan [20]. These concepts enabled André Weil [117] and Bourbaki [15] to provide a particularly elegant treatment of compactness.

1. called Kränze by Vietoris and overlapping systems by Birkhoff.

With the developement of topology, this Moore-Smith convergence was applied to topology by G. Birkhoff [14] under a still more general form, using the concept of directed sets and nets. But this is not the only theory of convergence to be found in topology. Independently, Wallman in 1938 developed something very close to Cartan's ultrafilters with his idea of maximal subsets having the "finite intersection property". In fact, the Rudin-Frolík as well as the Rudin-Keisler order of ultrafilters were introduced in a more general setting of filters by Katětov [6].

In 1940, J. W. Tukey made extensive use of the theory in his monograph titled *Convergence and uniformity in topology*, published in the Annals of Mathematics Studies series [114]. Tukey worked with objects that were generalizations of sequences that he referred to as *phalanxes*. They were a special case of the objects that are usually called *nets* today.

After G. Birkhoff reinvented the filter base notion [13], J. Schmidt wrote a paper in 1952 titled *Beiträge zur Filtertheorie II* and he dedicated to Prof. Vietoris [99], but without mentioning that also directed sets and nets can already be found in Vietoris [116]. Topological spaces are presented as a special subclass of convergence spaces of particular interest, but a large part of the material usually developed in a topology textbook is treated in the larger field of convergence spaces. To many, especially students, convergence theory is more natural and intuitive than classical topology. On the other hand, the framework of convergence is easier, more powerful and far-reaching which, highlights a need for a theory of convergence in various branches of analysis.

Chapter 1

Preliminaries

We start with a review of the basic concepts of the theory of filters, which are needed in the later chapters. The notion of modified open and closed sets such as semi-open, regular open and regular closed sets are also introduced. Semi-regularizations and their related topologies are also introduced. Regular, almost regular, Semi-regular, extremally disconnected, and weakly-T_2 spaces are discussed as well. Prime spaces are studied such as completely normal, and fully normal spaces.

1.1 Filters

In topology and analysis, filters are used to define convergence in a manner similar to the role of sequences in a metric space. Filters provide very general contexts to unify the various notions of limit to arbitrary topological spaces. A sequence is usually indexed by the natural numbers, which are a totally ordered set. Thus, limits in first-countable spaces can be described by sequences. However, if the space is not first-countable, nets or filters must be used. Nets generalize the notion of a sequence by requiring the index set simply be a directed set. Filters can be thought of as sets built from multiple nets [90].

1.1.1 Ordered Sets

Definition 1.1.1. [12, 78] Let X be a set and $\leq$ be a relation on X. Then $\leq$ is called

(i) *reflexive* if $x \leq x$ for all $x \in X$.

(ii) *transitive* if $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in X$.

(iii) *antisymmetric* if $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in X$.

Definition 1.1.2. [12, 78] Let X be a set and $\leq$ be a relation on X. The relation $\leq$ is called a *partial ordering* on X if it is reflexive, transitive and antisymmetric. The pair $(X, \leq)$ is a *partially ordered set* and it is abbreviated by poset.

Definition 1.1.3. [12] Let $(X, \leq)$ be a poset and $A \subseteq X$. An element $x \in X$ is called an *upper bound* of A if $y \leq x$ for every $y \in A$.

Definition 1.1.4. [12] Let $(X, \leq)$ be a poset and $A \subseteq X$. An element $x \in X$ is called a *supremum* (or *the least upper bound*) of A if

(i) x is an upper bound of A.

(ii) $x \leq y$ for every upper bound y of A.

1.1.2 Filters, Filter Bases and Filter Subbases

A filter is like an algebraic road function with directions to points or places in a topological space. More precisely, it is a certain subset of a poset. In this subsection, the poset we focus on is the power set of a set X. That is, we consider filters of subsets of a given set [17].

Definition 1.1.5. [9] A *topology* on a nonempty set X is a collection τ of subsets of X satsifying the following axioms:

(i) $\emptyset$ and $X \in \tau$;

(ii) If $\mathcal{A} \subseteq \tau$, then $\bigcup\limits_{A \in \mathcal{A}} A \in \tau$;

(iii) If $A_1, A_2, \cdots, A_n \in \tau$, then $\bigcap\limits_{i=1}^{n} A_n \in \tau$.

In words, axiom (ii) states that: an arbitrary union of elements of τ is an element of τ, while axiom (iii) states that: a finite intersection of elements of τ is an element of τ. In fact, axiom (iii) can be replaced by the equivalent form: If A_1 and $A_2 \in \tau$, then $A_1 \cap A_2 \in \tau$.

Definition 1.1.6. [9] A nonempty set X equipped with a topology τ is called a *topological space*, denoted by the pair (X, τ).

For simplicity, when no confusion occurs concerning τ, we say that X is a topological space, or simply a space.

We give the definition of a filter on a set (not necessarily a topological space) because it is of independent interest.

Definition 1.1.7. [9] A *filter* on a set X is a nonempty family $\mathfrak{F}$ of subsets of X such that

(i) $\emptyset \notin \mathfrak{F}$;

(ii) If $A, B \in \mathfrak{F}$ then $A \cap B \in \mathfrak{F}$;

(iii) If $A \in \mathfrak{F}$ and $A \subseteq B \subseteq X$ then $B \in \mathfrak{F}$.

Now, let us examine a few elementary consequences of this definition. By an induction argument it easily follows from (ii) that a filter is closed under finite intersections. Also, from (iii) it follows that X is a member of any filter on X.

Definition 1.1.8. [16, 79] Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two filters on X. If $\mathfrak{F}_2 \subseteq \mathfrak{F}_1$, then we say that $\mathfrak{F}_1$ is *finer* than $\mathfrak{F}_2$ and $\mathfrak{F}_2$ is *coarser* than $\mathfrak{F}_1$.

Definition 1.1.9. [64] Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two filters on X. Then we say that $\mathfrak{F}_2$ is a *subfilter* of $\mathfrak{F}_1$ if $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$.

Definition 1.1.10. [60, 79] Given a set X, the set of all filters on X is denoted by $\mathbf{F}(X)$.

Remark 1.1. *[30] The ordered pair $(\boldsymbol{F}(X), \subseteq)$ is a poset.*

Definition 1.1.11. [101] Let $\Phi(X)$ be any nonempty class of filters on a nonempty set X. That is, $\Phi(X) \subseteq \mathbf{F}(X)$. Then a filter $\mathcal{F}$ on X is said to be a *supremum* (*infimum*) of $\Phi(X)$ with respect to the inclusion $\subseteq$ if $\mathcal{F}$ is finer (coarser) than each member of $\Phi(X)$ and if $\mathcal{F}'$ is any filter on X which is finer (coarser) than each member of $\Phi(X)$, then $\mathcal{F}$ is coarser (finer) than $\mathcal{F}'$ and it is denoted by $\bigvee_{\mathcal{G}\in\Phi(X)} \mathcal{G}$ $(\bigwedge_{\mathcal{G}\in\Phi(X)} \mathcal{G})$.

Remark 1.2. *[30] The supremum of a family of filters on a set X doesn't always exist while the infimum always exists.*

Example 1.1.1. *[9] Given $X \neq \emptyset$ and $x \in X$. Then $\mathcal{F} = \{A \subseteq X : x \in A\}$ is a filter on X, called the principal filter generated by x. It is denoted by $\langle x \rangle$.*

Now, we discuss a useful way of describing filters using a base or subbase. The treatment here parallels the corresponding ideas from topology.

Definition 1.1.12. [98] Let X be a set and $\emptyset \neq \mathcal{B} \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is a *filter base* in X if it satisfies the following conditions:

(i) $\emptyset \notin \mathcal{B}$.

(ii) If $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Example 1.1.2. *Let (X, τ) be a topological space and $x \in X$. The collection $\tau(x)$ of all open neighborhoods of x is a filter base in X.*

Remark 1.3. *Every filter on a set X is a filter base in X but the converse is not true.*

Definition 1.1.13. [98] If $\mathcal{B}$ is a filter base in X, then $\mathcal{F} = \{F \subseteq X : F \supseteq B, B \in \mathcal{B}\}$ is a filter on X called the *filter generated* by $\mathcal{B}$ and it is denoted by $\langle \mathcal{B} \rangle_X$. In this case we say that $\mathcal{B}$ is a filter base for $\mathcal{F}$. The filter $\langle \mathcal{B} \rangle_X$ is uniquely determined by $\mathcal{B}$ and it is the smallest filter on X containing $\mathcal{B}$.

Notation 1. *For a filter base $\mathcal{B}$ in a set X, when there is no confusion, we will just write the filter "$\langle \mathcal{B} \rangle$" instead of "$\langle \mathcal{B} \rangle_X$".*

Example 1.1.3. *Let (X,τ) be a topological space. For each $x \in X$, the filter generated by $\tau(x)$ is the τ-neighborhood filter of x, $\mathcal{U}_\tau(x)$.*

Notation 2. *For a topological space (X,τ) and $x \in X$, when there is no confusion, we will just write the τ-neighborhood filter of x, "$\mathcal{U}(x)$" instead of "$\mathcal{U}_\tau(x)$".*

Remark 1.4. *[26, 48] Let $\mathcal{B}$ and $\mathcal{C}$ be two filter bases in a set X. If $B \cap C \neq \emptyset$ for all $B \in \mathcal{B}$ and all $C \in \mathcal{C}$, then we write $\mathcal{B}(\cap)\mathcal{C}$. Otherwise, we write $\mathcal{B} \perp \mathcal{C}$.*

Proposition 1.1.1. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be filter bases in a set X and $\mathcal{A} \subseteq \mathcal{B}$.

(i) If $\mathcal{B}(\cap)\mathcal{C}$, then $\mathcal{A}(\cap)\mathcal{C}$.

(ii) If $\mathcal{A} \perp \mathcal{C}$, then $\mathcal{B} \perp \mathcal{C}$.

Proof. (i) Suppose that $\mathcal{B}(\cap)\mathcal{C}$. Let $A \in \mathcal{A}$ and $C \in \mathcal{C}$, then $A \in \mathcal{B}$ since $\mathcal{A} \subseteq \mathcal{B}$. So, $A \in \mathcal{B}$ and $C \in \mathcal{C}$ but $\mathcal{B}(\cap)\mathcal{C}$, then $A \cap C \neq \emptyset$. Hence, $\mathcal{A}(\cap)\mathcal{C}$.

(ii) Suppose that $\mathcal{A} \perp \mathcal{C}$, then there exist $A \in \mathcal{A}$ and $C \in \mathcal{C}$ such that $A \cap C = \emptyset$ but $\mathcal{A} \subseteq \mathcal{B}$, so $A \in \mathcal{B}$. Hence, there exist $A \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $A \cap C = \emptyset$. Therefore, $\mathcal{B} \perp \mathcal{C}$.

□

Definition 1.1.14. [30, 56] Let $\mathcal{F}$ and $\mathcal{G}$ be two filters on a set X. If $\mathcal{F}(\cap)\mathcal{G}$, then the family $\mathcal{B} = \{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}$ is a filter base in X. The filter generated by $\mathcal{B}$ is the *supremum* filter of $\{\mathcal{F}, \mathcal{G}\}$ and it is denoted by $\mathcal{F} \vee \mathcal{G}$. We say in this case that $\mathcal{F} \vee \mathcal{G}$ exists. If $\mathcal{F} \perp \mathcal{G}$, we say that $\mathcal{F} \vee \mathcal{G}$ fails to exist and the pair $\{\mathcal{F}, \mathcal{G}\}$ hasn't an upper bound.

Proposition 1.1.2. [30] If $\{\mathcal{F}_\alpha : \alpha \in \Delta\}$ is a family of filters on a set X, then $\bigcap\limits_{\alpha\in\Delta} \mathcal{F}_\alpha$ is a filter on X and $\{\bigcup\limits_{\alpha\in\Delta} F_\alpha : F_\alpha \in \mathcal{F}_\alpha\}$ is a filter base for $\bigcap\limits_{\alpha\in\Delta} \mathcal{F}_\alpha$.

Proposition 1.1.3. [30] Let $\Phi(X)$ be any nonempty class of filters on a nonempty set X, then the filter $\bigcap\limits_{\mathcal{F}\in\Phi(X)} \mathcal{F}$ is the infimum of the set $\Phi(X)$ in the poset $(\mathbf{F}(X), \subseteq)$.

As we use $\wedge$ to denote infimum, we will write $\bigwedge\limits_{\mathcal{F}\in\Phi(X)} \mathcal{F}$ and $\bigcap\limits_{\mathcal{F}\in\Phi(X)} \mathcal{F}$ interchangeably [30, 60].

Proposition 1.1.4. [49] Let $\mathcal{F}$ and $\mathcal{G}$ be two filters on a set X. Then

(i) $\mathcal{F}(\cap)\mathcal{G}$ if and only if $\mathcal{F} \vee \mathcal{G} \neq \mathcal{P}(X)$.

(ii) $\mathcal{F} \perp \mathcal{G}$ if and only if $\mathcal{F} \vee \mathcal{G} = \mathcal{P}(X)$.

Proof. (i) Suppose that $\mathcal{F}(\cap)\mathcal{G}$. Then $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and all $G \in \mathcal{G}$. So, $\emptyset \notin \mathcal{F} \vee \mathcal{G}$, and hence $\mathcal{F} \vee \mathcal{G} \neq \mathcal{P}(X)$. Conversely, suppose that $\mathcal{F} \vee \mathcal{G} \neq \mathcal{P}(X)$, then $\emptyset \notin \mathcal{F} \vee \mathcal{G}$ since $\mathcal{F} \vee \mathcal{G}$ is closed under the superset operation. So, $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and all $G \in \mathcal{G}$. Thus, $\mathcal{F}(\cap)\mathcal{G}$.

(ii) $\mathcal{F} \perp \mathcal{G}$ if and only if $F \cap G = \emptyset$ for some $F \in \mathcal{F}$ and $G \in \mathcal{G}$ if and only if $\emptyset \in \mathcal{F} \vee \mathcal{G}$ if and only if $\mathcal{F} \vee \mathcal{G} = \mathcal{P}(X)$.

□

Definition 1.1.15. [95] Let $\mathcal{F}$ be a filter on a set X and let $A \subseteq X$. Then the *trace* of $\mathcal{F}$ on A is denoted by $\mathcal{F}\big|_A$ and is defined by $\mathcal{F}\big|_A = \{F \cap A : F \in \mathcal{F}\}$.

Definition 1.1.16. [23] Let (X, τ) be a topological space, $\mathcal{B}$ be a filter base in X and $A \subseteq X$. We say that $\mathcal{B}$ *meets* A if and only if for every $B \in \mathcal{B}$, $B \cap A \neq \emptyset$.

Theorem 1.1.1. [95] If $\mathcal{F}$ is a filter on a set X and $A \subseteq X$, then $\mathcal{F}\big|_A$ is a filter on A if and only if $\mathcal{F}$ meets A.

Proof. Assume that $\mathcal{F}\big|_A$ is a filter on A, then $\emptyset \notin \mathcal{F}\big|_A$. So, $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$. That is, $\mathcal{F}$ meets A. Conversely, suppose that $\mathcal{F}$ meets A. Then $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$. So, $\emptyset \notin \mathcal{F}\big|_A$. Let $F_1 \cap A$ and $F_2 \cap A \in \mathcal{F}\big|_A$. Then $F_1 \cap F_2 \in \mathcal{F}$, and so $(F_1 \cap A) \cap (F_2 \cap A) = (F_1 \cap F_2) \cap A \in \mathcal{F}\big|_A$. Next, let $F \cap A \subseteq P \subseteq A$. Then $(F \cup P) \cap A = (F \cap A) \cup (P \cap A) = (F \cap A) \cup P = P$. Since $F \subseteq F \cup P$ and $F \in \mathcal{F}$, then $F \cup P \in \mathcal{F}$. So, $P = (F \cup P) \cap A \in \mathcal{F}\big|_A$. □

Proposition 1.1.5. [101] If $\mathcal{F}$ is a filter on Y and $Y \subseteq X$, then $\mathcal{F}$ is a filter base in X.

Proposition 1.1.6. [95] Let $\mathcal{F}$ be a filter on a set X and $A \subseteq X$. If $\mathcal{F}$ meets A, then $\mathcal{F}\big|_A$ is a filter base in X and the filter $\mathcal{G} = \langle \mathcal{F}\big|_A \rangle_X$ containing A and finer than $\mathcal{F}$.

Proof. First, since $\mathcal{F}$ meets A, then $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$, hence by Theorem 1.1.1, $\mathcal{F}\big|_A$ is a filter on A, and so by Proposition 1.1.5 part (ii), $\mathcal{F}\big|_A$ is a filter base in X. Next, since $A \supseteq F \cap A$ for all $F \in \mathcal{F}$, then $A \in \mathcal{G}$. Also, since for all $F \in \mathcal{F}$, $F \supseteq F \cap A$, then for all $F \in \mathcal{F}$, $F \in \mathcal{G}$. That is, $\mathcal{F} \subseteq \mathcal{G}$. □

Definition 1.1.17. [79] A collection $\mathcal{A}$ of sets in a topological space, is said to have the *finite intersection property*, written **F.I.P** iff whenever $A_1, \dots, A_n \in \mathcal{A}$, then $\bigcap\limits_{i=1}^{n} A_i \neq \emptyset$.

Definition 1.1.18. [98] Let X be a set and $\emptyset \neq \mathcal{S} \subseteq \mathcal{P}(X)$. then we say that $\mathcal{S}$ is a *filter subbase* on X if $\mathcal{S}$ has the **F.I.P**.

Definition 1.1.19. [98] If $\mathcal{S}$ is a filter subbase on X, then

$$\mathcal{F} = \left\{F \subseteq X : \ \exists\ S_1, \dots, S_n \in \mathcal{S} \text{ such that } F \supseteq \bigcap_{i=1}^{n} S_i\right\}$$

is a filter on X called the *filter generated* by $\mathcal{S}$ and is denoted by $\langle \mathcal{S} \rangle$. In this case, we say that $\mathcal{S}$ is a filter subbase for $\mathcal{F}$. The filter $\langle \mathcal{S} \rangle$ is uniquely determined by $\mathcal{S}$ and it is the smallest filter on X containing $\mathcal{S}$.

Definition 1.1.20. [48, 120] Let $\mathcal{S}$ be a filter subbase on X. Then

$$\mathcal{B} = \left\{\bigcap_{i=1}^{n} S_i : S_1, \dots, S_n \in \mathcal{S}, n \in \mathbb{N}\right\}$$

is a filter base in X called the *filter base generated* by $\mathcal{S}$ and is denoted by $[\mathcal{S}]_X$.

Notation 3. *For a filter subbase $\mathcal{S}$ on a set X, when there is no confusion, we will just write the filter base "$[\mathcal{S}]$" instead of "$[\mathcal{S}]_X$".*

Proposition 1.1.7. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be filter subbases on X. If $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $[\mathcal{S}_1] \subseteq [\mathcal{S}_2]$.

Proof. Let $B \in [\mathcal{S}_1]$, then $B = \bigcap\limits_{i=1}^{n} S_i$, where $S_i \in \mathcal{S}_1$ for each $i = 1, \dots, n$ but $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $S_i \in \mathcal{S}_2$ for each $i = 1, \dots, n$. Hence, $B \in [\mathcal{S}_2]$. Thus, $[\mathcal{S}_1] \subseteq [\mathcal{S}_2]$. □

Proposition 1.1.8. [98] Let $\mathcal{S}$ be a filter subbase on X. Then $\mathcal{S}$ and $[\mathcal{S}]$ generate the same filter on X.

Proof. We have $\mathcal{S} \subseteq [\mathcal{S}]$, so that $\langle \mathcal{S} \rangle \subseteq \langle [\mathcal{S}] \rangle$. But since $\langle \mathcal{S} \rangle$ is closed under finite intersections, then $[\mathcal{S}] \subseteq \langle \mathcal{S} \rangle$, and hence $\langle [\mathcal{S}] \rangle \subseteq \langle \mathcal{S} \rangle$. Therefore, $\langle \mathcal{S} \rangle = \langle [\mathcal{S}] \rangle$. $\square$

1.1.3 Ultrafilters

Definition 1.1.21. [9] Let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is called an *ultrafilter* if for any $A \subseteq X$, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.

Example 1.1.4. *[9] Let X be a nonempty set. For any $x \in X$, the principal filter $\langle x \rangle$ is an ultrafilter on X.*

Definition 1.1.22. [79] A filter $\mathcal{F}$ on X is a *maximal* filter iff whenever $\mathcal{G}$ is a filter on X with $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G} = \mathcal{F}$.

Theorem 1.1.2. [9] Let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}$ is a maximal filter.

Theorem 1.1.3. [79] For every filter $\mathcal{F}$ on a set X, there is an ultrafilter $\mathcal{G}$ that is finer than $\mathcal{F}$.

Theorem 1.1.4. [27] If A is a nonempty subset of X, then every filter on X which meets A is contained in an ultrafilter on X which also meets A.

Theorem 1.1.5. [2, 95] Let A be a nonempty subset of X. If $\mathcal{F}$ is an ultrafilter on X which meets A, then $\mathcal{F}\big|_A$ is an ultrafilter on A and $\mathcal{F}\big|_A \subseteq \mathcal{F}$.

Proof. Let $B \subseteq A$. Then $B \subseteq X$, so $B \in \mathcal{F}$ or $X - B \in \mathcal{F}$ since $\mathcal{F}$ is an ultrafilter on X. So, $A \cap B \in \mathcal{F}\big|_A$ or $A \cap (X - B) \in \mathcal{F}\big|_A$, and hence $B \in \mathcal{F}\big|_A$ or $A - B \in \mathcal{F}\big|_A$. Therefore, $\mathcal{F}\big|_A$ is an ultrafilter on A. Next, since $\mathcal{F}$ meets A, then $X - A \notin \mathcal{F}$ but $\mathcal{F}$ is an ultrafilter on X, so $A \in \mathcal{F}$. Thus, $F \cap A \in \mathcal{F}$ for all $F \in \mathcal{F}$. That is, $\mathcal{F}\big|_A \subseteq \mathcal{F}$. $\square$

Definition 1.1.23. [17] Let X and Y be two nonempty sets and $f : X \to Y$ be any function. If $\mathcal{F}$ is a filter on X, then $\{f(F) : F \in \mathcal{F}\}$ is a filter base in Y, which generates a filter $f(\mathcal{F})$ called the *image* of $\mathcal{F}$ under f. That is, $f(\mathcal{F}) = \langle\{f(F) : F \in \mathcal{F}\}\rangle_Y$.

Definition 1.1.24. [107] Let X and Y be two nonempty sets and $f : X \to Y$ be any function. If $\mathcal{G}$ is a filter on Y and $\mathcal{G}$ meets $f(X)$, then $\{f^{-1}(G) : G \in \mathcal{G}\}$ is a filter base in X, which generates a filter $f^{-1}(\mathcal{G})$ called the *inverse image* of $\mathcal{G}$ under f. That is, $\langle\{f^{-1}(G) : G \in \mathcal{G}\}\rangle_X$.

Theorem 1.1.6. [119] Let X, Y be two sets and $f : X \to Y$ be an onto function. If $\mathcal{F}$ is an ultrafilter on X, then $f(\mathcal{F})$ is an ultrafilter on Y.

1.2 Modified Open and Closed Sets

1.2.1 Semi-open Sets

The notion of semi-open sets in a topological space, which is one of the generalizations of open sets, plays an important role in several of the recent research in General Topology. This notion was originally given in 1963 by Levine [62], who demonstrated that the family of semi-open sets is closed under arbitrary unions and the family of interiors of semi-open sets coincides with the topology of space. Semi-open sets have been used to define and study new versions of interior, closure, separation axioms and continuity. It can be seen by the following definition.

Definition 1.2.1. [62] Let (X, τ) be a topological space and $S \subseteq X$. Then S is called *semi-open* if there exists an open set U in X such that $U \subseteq S \subseteq \overline{U}$. The set of all semi-open sets in (X, τ) is denoted by $\mathrm{SO}(X, \tau)$. As usual, we simply write $\mathrm{SO}(X)$ if no confusion would arise.

Notation 4. *For a topological space* (X, τ) *and* $x \in X$, *we simply write* $\mathrm{SO}_\tau(x) = \{S \subseteq X : S \in \mathrm{SO}(X, \tau)$ *and* $x \in S\}$ *by* $\mathrm{SO}(x)$ *if no confusion would arise.*

Example 1.2.1. *In $\mathbb{R}$ with the usual topology, the intervals (a,b), $[a,b)$, $(a,b]$ and $[a,b]$ are semi-open sets.*

The following theorem gives the characterizations of the semi-open sets.

Theorem 1.2.1. [5, 25] Let X be a topological space and $S \subseteq X$. Then the following are equivalent:

(i) S is semi-open in X.

(ii) $S \subseteq \overline{S^\circ}$.

(iii) $\overline{S} = \overline{S^\circ}$.

Proof.

(i) $\Longrightarrow$ (ii) Assume that S is semi-open in X, then there exists an open set U in X such that $U \subseteq S \subseteq \overline{U}$. Then $U = U^\circ \subseteq S^\circ$. So, $\overline{U} \subseteq \overline{S^\circ}$ but $S \subseteq \overline{U}$, then $S \subseteq \overline{S^\circ}$.

(ii) $\Longrightarrow$ (iii) Assume that $S \subseteq \overline{S^\circ}$, then $\overline{S} \subseteq \overline{\overline{S^\circ}} = \overline{S^\circ}$ but $\overline{S^\circ} \subseteq \overline{S}$. Hence, $\overline{S} = \overline{S^\circ}$.

(iii) $\Longrightarrow$ (i) Assume that $\overline{S} = \overline{S^\circ}$. Let $U = S^\circ$, then U is open in X and $U = S^\circ \subseteq S \subseteq \overline{S} = \overline{S^\circ} = \overline{U}$. Hence, there exists an open set U in X such that $U \subseteq S \subseteq \overline{U}$. Therefore, S is semi-open in X.

$\square$

Proposition 1.2.1. [25, 62] Let X be a topological space.

(i) All open sets in X are semi-open in X.

(ii) Any union of semi-open sets in X is semi-open in X.

The intersection of two semi-open sets is not semi-open in general. The next example can be shown.

Example 1.2.2. *In $\mathbb{R}$ with the usual topology, $[1,2]$ and $[2,3]$ are semi-open but $\{2\} = [1,2] \cap [2,3]$ is not semi-open since $\{2\} \nsubseteq \overline{\{2\}^\circ} = \emptyset$.*

Now, we recall some results which will be useful in the sequel. We start with a theorem which will be used frequently.

Theorem 1.2.2. [5, 42] Let X be a topological space and $A \subseteq X$. If U is an open set in X, then $U \cap \overline{A} \subseteq \overline{U \cap A}$.

Proof. Let $x \in U \cap \overline{A}$ and let V be any open set in X containing x. Then $V \cap U$ is open containing x. But $x \in \overline{A}$, so $(V \cap U) \cap A \neq \emptyset$, and thus $V \cap (U \cap A) \neq \emptyset$. Hence, $x \in \overline{U \cap A}$. Therefore, $U \cap \overline{A} \subseteq \overline{U \cap A}$. $\square$

Corollary 1.2.1. *Let X be a topological space, $A \subseteq X$ and U be an open set in X. Then $U \cap A = \emptyset$ if and only if $U \cap \overline{A} = \emptyset$.*

Proof. If $U \cap \overline{A} = \emptyset$, then $U \cap A \subseteq U \cap \overline{A} = \emptyset$, and so $U \cap A = \emptyset$. Conversely, suppose that $U \cap A = \emptyset$. Since U is open, then by Theorem 1.2.2, $U \cap \overline{A} \subseteq \overline{U \cap A} = \overline{\emptyset} = \emptyset$. Hence, $U \cap \overline{A} = \emptyset$. $\square$

The following two propositions are easy consequences of the definitions of open sets and semi-open sets.

Proposition 1.2.2. [80] Let X be a topological space. If U is open in X and A is semi-open in X, then $U \cap A$ is semi-open in X.

Proof. Assume that U is open in X and A is semi-open in X, then there is an open set G in X such that $G \subseteq A \subseteq \overline{G}$. So, $U \cap G \subseteq U \cap A \subseteq U \cap \overline{G}$. But since U is open in X, then by Theorem 1.2.2, $U \cap \overline{G} \subseteq \overline{U \cap G}$. Hence, we have $U \cap G \subseteq U \cap A \subseteq \overline{U \cap G}$ and $U \cap G$ is open in X. Thus, $U \cap A$ is semi-open in X. $\square$

Proposition 1.2.3. [62] Let X be a topological space and $A \subseteq B \subseteq X$. If A is semi-open in X, then A is semi-open in B.

Proof. Since A is semi-open in X, then there is an open set U in X such that $U \subseteq A \subseteq \overline{U}$. Now, since $U \subseteq A \subseteq B$, then $U = U \cap B \subseteq A \cap B \subseteq \overline{U} \cap B$. This implies that, $U \subseteq A \subseteq \mathrm{Cl}_B(U)$. Since U is open in X, then $U \cap B$ is open in B but $U = U \cap B$, so U is open in B. Therefore, A is semi-open in B. $\square$

The converse of proposition 1.2.3 need not be true in general which will be shown by the following example.

Example 1.2.3. *Let $X = \mathbb{R}$ with the usual topology. Let $A = \{0\} = B$. Since A is open in B, then A is semi-open in B but A is not semi-open in X.*

Theorem 1.2.3. [80] Let X be a topological space and $A \subseteq B \subseteq X$. If B is semi-open in X. Then A is semi-open in X if and only if A is semi-open in B.

Proof. Assume that A is semi-open in X, then by Proposition 1.2.3, A is semi-open in B. Conversely, assume that A is semi-open in B, then there exists an open set V in B such that $V \subseteq A \subseteq \mathrm{Cl}_B(V)$. Since V is open in B, then there exists an open set U in X such that $V = U \cap B$. Therefore, we have $U \cap B \subseteq A \subseteq \mathrm{Cl}_B(U \cap B) = \overline{U \cap B} \cap B \subseteq \overline{U \cap B}$. Since U is open in X and B is semi-open in X, then by Proposition 1.2.2, $U \cap B$ is semi-open in X. So, there exists an open set O in X such that $O \subseteq U \cap B \subseteq \overline{O}$. Then $O \subseteq A$. But $A \subseteq \overline{U \cap B} \subseteq \overline{\overline{O}} = \overline{O}$. Hence, there exists an open set O in X such that $O \subseteq A \subseteq \overline{O}$. Therefore, A is semi-open in X. $\square$

1.2.2 Regular Open and Regular Closed Sets

Definition 1.2.2. [110] A subset S of a topological space (X, τ) is said to be *regular open* if $\overline{S}^{\circ} = S$. The set of all regular open sets in (X, τ) is denoted by $\mathrm{RO}(X, \tau)$. As usual, we simply write $\mathrm{RO}(X)$ if no confusion would arise.

Definition 1.2.3. [110] A subset S of a topological space (X, τ) is said to be *regular closed* if $\overline{S^{\circ}} = S$. The set of all regular closed sets in (X, τ) is denoted by $\mathrm{RC}(X, \tau)$. As usual, we simply write $\mathrm{RC}(X)$ if no confusion would arise.

Notation 5. *For a topological space (X, τ) and $x \in X$.*

(i) $\mathrm{RO}_\tau(x) = \{S \subseteq X : S \in \mathrm{RO}(X, \tau)$ *and* $x \in S\}$. *When no confusion arises, we write "*$\mathrm{RO}(x)$*" instead of "*$\mathrm{RO}_\tau(x)$*".*

(ii) $\mathrm{RC}_\tau(x) = \{S \subseteq X : S \in \mathrm{RC}(X, \tau)$ *and* $x \in S\}$. *When no confusion arises, we write "*$\mathrm{RC}(x)$*" instead of "*$\mathrm{RC}_\tau(x)$*".*

Proposition 1.2.4. [25] Let (X, τ) be a topological space and $\mathcal{C}$ be the family of all closed sets in (X, τ).

(i) $\mathrm{RO}(X, \tau) \subseteq \tau \subseteq \mathrm{SO}(X, \tau)$.

(ii) $\mathrm{RC}(X, \tau) \subseteq \mathcal{C}$.

(iii) $\mathrm{RC}(X, \tau) \subseteq \mathrm{SO}(X, \tau)$.

Proof.

(i) Suppose that U is regular open, then $U = \overline{U}^\circ$ is open. So, $\mathrm{RO}(X, \tau) \subseteq \tau$. Next, $\tau \subseteq \mathrm{SO}(X, \tau)$ by Proposition 1.2.1.

(ii) Suppose that F is regular closed, then $F = \overline{F^\circ}$ is closed. So, $\mathrm{RC}(X, \tau) \subseteq \mathcal{C}$.

(iii) Let F be regular closed, then $F = \overline{F^\circ}$, and so $F \subseteq \overline{F^\circ}$. Hence, by Theorem 1.2.1, F is semi-open. Therefore, $\mathrm{RC}(X, \tau) \subseteq \mathrm{SO}(X, \tau)$.

$\square$

Theorem 1.2.4. [25, 33, 120] Let (X, τ) be a topological space and $S \subseteq X$. Then S is regular closed in X if and only if $X - S$ is regular open in X.

Proof. $S \in \mathrm{RC}(X)$ if and only if $\overline{S^\circ} = S$ if and only if $X - \overline{S^\circ} = X - S$ if and only if $\overline{X - S}^\circ = X - S$ if and only if $X - S \in \mathrm{RO}(X)$. $\square$

Theorem 1.2.5. [42] Let (X, τ) be a topological space. If A or B is open in X, then $\overline{A \cap B}^\circ = \overline{A}^\circ \cap \overline{B}^\circ$.

Proof. Suppose that B is open in X. Since $\overline{A}^\circ$ and B are open, then by Theorem 1.2.2, $\overline{A}^\circ \cap \overline{B} \subseteq \overline{\overline{A}^\circ \cap B}$ and $\overline{A} \cap B \subseteq \overline{A \cap B}$, respectively. Thus,

$$\overline{A}^\circ \cap \overline{B}^\circ \subseteq \overline{A}^\circ \cap \overline{B} \subseteq \overline{\overline{A}^\circ \cap B} \subseteq \overline{\overline{A} \cap B} \subseteq \overline{\overline{A \cap B}} = \overline{A \cap B}.$$

Hence, $(\overline{A}^\circ \cap \overline{B}^\circ)^\circ \subseteq \overline{A \cap B}^\circ$, and so $\overline{A}^\circ \cap \overline{B}^\circ \subseteq \overline{A \cap B}^\circ$. The other case is similar. Next, since $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, then $\overline{A \cap B}^\circ \subseteq (\overline{A} \cap \overline{B})^\circ = \overline{A}^\circ \cap \overline{B}^\circ$. So, $\overline{A \cap B}^\circ \subseteq \overline{A}^\circ \cap \overline{B}^\circ$. Therefore, $\overline{A \cap B}^\circ = \overline{A}^\circ \cap \overline{B}^\circ$. $\square$

Proposition 1.2.5. [110] Let (X, τ) be a topological space. If A and $B \in \mathrm{RO}(X)$, then $A \cap B \in \mathrm{RO}(X)$.

Proof. Let $A, B \in \mathrm{RO}(X)$, then $A = \overline{A}^{\circ}$ and $B = \overline{B}^{\circ}$. But since A and B are open, then by Lemma 1.2.5, we have $\overline{A \cap B}^{\circ} = \overline{A}^{\circ} \cap \overline{B}^{\circ} = A \cap B$. Hence, $A \cap B \in \mathrm{RO}(X)$. $\square$

Note that by the usual induction argument, any finite intersection of regular open sets is a regular open set but a finite union of regular open sets need not be regular open. It can be easily shown by the following example.

Example 1.2.4. *Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{a\}$ and $B = \{b\}$. Then $\overline{A}^{\circ} = A$ and $\overline{B}^{\circ} = B$. That is, A and B are regular open sets of X. But $A \cup B = \{a, b\}$ is not regular open since $\overline{\{a, b\}}^{\circ} = X \neq \{a, b\}$.*

Proposition 1.2.6. [110] Let (X, τ) be a topological space.

(i) Any finite union of regular closed sets is a regular closed set.

(ii) A finite intersection of regular closed sets need not be regular closed.

Proof.

(i) This follows from Theorem 1.2.4 and Proposition 1.2.5.

(ii) Let $X = \mathbb{R}$ with the usual topology. Let $A = [0, 1]$ and $B = [1, 2]$. Then A and B are regular closed in X. But $A \cap B = \{1\}$ is not regular closed since $\overline{\{1\}^{\circ}} = \emptyset \neq \{1\}$.

$\square$

Proposition 1.2.7. [33, 53] Let (X, τ) be a topological space. Then

(i) $\mathrm{RO}(X) = \{\overline{U}^{\circ} : U \in \tau\}$.

(ii) $\mathrm{RC}(X) = \{\overline{U} : U \in \tau\}$.

(iii) $\mathrm{RC}(X) = \{\overline{S} : S \in \mathrm{SO}(X)\}$.

Proof.

(i) Let $G \in \mathrm{RO}(X)$, then $G = \overline{G}^{\circ}$ and $G \in \tau$. So, $G \in \{\overline{U}^{\circ} : U \in \tau\}$. Next, let $U \in \tau$, then we show that $G = \overline{U}^{\circ} \in \mathrm{RO}(X)$. Since $U \in \tau$, then $U = U^{\circ} \subseteq \overline{U}^{\circ} = G$ but then $\overline{U}^{\circ} \subseteq \overline{G}^{\circ}$, so $G \subseteq \overline{G}^{\circ}$. But $\overline{G}^{\circ} = \overline{\overline{U}^{\circ}}^{\circ} \subseteq \overline{U}^{\circ} = G$. Thus, $G = \overline{G}^{\circ}$. Hence, $G \in \mathrm{RO}(X)$.

(ii) Let $F \in \mathrm{RC}(X)$. Then $F = \overline{F^\circ}$ but since $F^\circ \in \tau$, then $F \in \{\overline{U} : U \in \tau\}$. Next, let $U \in \tau$, then we show that $\overline{U} \in \mathrm{RC}(X)$. Since $\overline{U} = \overline{U^\circ} \subseteq \overline{\overline{U}^\circ} \subseteq \overline{U}$, then $\overline{\overline{U}^\circ} = \overline{U}$. Thus, $\overline{U} \in \mathrm{RC}(X)$.

(iii) Let $S \in \mathrm{SO}(X)$. Then by Theorem1.2.1, $\overline{S} = \overline{S^\circ}$. Since $S^\circ \in \tau$, then by part (ii), $\overline{S} = \overline{S^\circ} \in \mathrm{RC}(X)$. Next, let $R \in \mathrm{RC}(X)$, then $R = \overline{R^\circ}$. Since $R^\circ \in \tau$, then $R^\circ \in \mathrm{SO}(X)$. Thus, $R \in \{\overline{S} : S \in \mathrm{SO}(X)\}$.

□

1.3 Semi-Regularizations and Their Related Topologies

Recall that a topological space X is a *T_1-space* if for each $x \neq y$ in X, there exist open sets U and V in X such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$ [64]. A topological space X is *Hausdorff* (or T_2) if for each $x \neq y$ in X, there exist open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ [64]. A topological space X is said to be *regular* if for each $x \in X$ and each open U in X containing x, there exists an open set V such that $x \in V \subseteq \overline{V} \subseteq U$ [33].

1.3.1 Semi-regular Spaces

The semi-regularization method was first defined and studied by M. H. Stone [110]. It can be easily seen by the following definition.

Definition 1.3.1. [110] If (X, τ) is a topological space, then the topology τ_s generated by the regular open sets of (X, τ) is called the *semi-regularization* topology of τ and is coarser than τ. (X, τ) is said to be a *semi-regular* space if $\tau_s = \tau$.

Example 1.3.1. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\mathrm{RO}(X, \tau) = \{\emptyset, X, \{a\}, \{b\}\}$ is a base for τ. Hence, $\tau_s = \tau$. Therefore, (X, τ) is a semi-regular space.*

The regular open sets of a topological space (X, τ) need not be a base for τ as shown in the following example.

Example 1.3.2. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then $\mathrm{RO}(X, \tau) = \{\emptyset, X\}$ is not a base for τ. In fact, $\mathrm{RO}(X, \tau)$ is a base for $\tau_s = \{\emptyset, X\}$.*

Theorem 1.3.1. [76] A topological space (X, τ) is semi-regular if for each $x \in X$ and each open U in (X, τ) containing x, there exists an open set V in (X, τ) such that $x \in V \subseteq \overline{V}^{\circ} \subseteq U$.

Proposition 1.3.1. [73] Every regular space is semi-regular. The converse is not always true. As an example the simplified Arens square [109, Example 81].

Proof. Let (X, τ) be a regular space. Let $U \in \tau$ and $x \in U$, then by regularity of (X, τ), there exists an open set V in (X, τ) such that $x \in V \subseteq \overline{V} \subseteq U$. Then, $x \in V = V^{\circ} \subseteq \overline{V}^{\circ} \subseteq U^{\circ} = U$. Hence, $x \in V \subseteq \overline{V}^{\circ} \subseteq U$. So, by Definition 1.3.1, (X, τ) is semi-regular. □

Recall that if τ_1 and τ_2 are two topologies on a set X such that $\tau_1 \subseteq \tau_2$, then $\mathrm{Cl}_{\tau_2}(A) \subseteq \mathrm{Cl}_{\tau_1}(A)$ and $\mathrm{Int}_{\tau_1}(A) \subseteq \mathrm{Int}_{\tau_2}(A)$ for any subset A of X [58].

Lemma 1.3.1. *[4, 73, 113] Let (X, τ_s) be the semi-regularization space of a topological space (X, τ). Then*

(i) $\mathrm{Cl}_{\tau}(U) = \mathrm{Cl}_{\tau_s}(U)$ *for each* $U \in \tau$.

(ii) $\mathrm{Int}_{\tau}(F) = \mathrm{Int}_{\tau_s}(F)$ *for each* $F \in \mathrm{RC}(X, \tau)$.

(iii) $\mathrm{RO}(X, \tau) = \mathrm{RO}(X, \tau_s)$.

(iv) $\mathrm{RC}(X, \tau) = \mathrm{RC}(X, \tau_s)$.

Proof. (i) Let $U \in \tau$. Then $\mathrm{Cl}_{\tau}(U) \subseteq \mathrm{Cl}_{\tau_s}(U)$ since $\tau_s \subseteq \tau$. Let $x \in \mathrm{Cl}_{\tau_s}(U)$ and let $V \in \tau$ with $x \in V$. Then $x \in V \subseteq \mathrm{Int}_{\tau}\mathrm{Cl}_{\tau}(V)$ and $\mathrm{Int}_{\tau}\mathrm{Cl}_{\tau}(V) \in \tau_s$. So, $\mathrm{Int}_{\tau}\mathrm{Cl}_{\tau}(V) \cap U \neq \emptyset$, and hence $\mathrm{Cl}_{\tau}(V) \cap U \neq \emptyset$, but since $U \in \tau$, so by Corollary 1.2.1, we have $V \cap U \neq \emptyset$. Therefore, $x \in \mathrm{Cl}_{\tau}(U)$.

(ii) Let $F \in \mathrm{RC}(X, \tau)$. Then $\mathrm{Int}_{\tau_s}(F) \subseteq \mathrm{Int}_{\tau}(F)$ since $\tau_s \subseteq \tau$. Let $x \in \mathrm{Int}_{\tau}(F)$. Since $F \in \mathrm{RC}(X, \tau)$, then $F = \mathrm{Cl}_{\tau}\mathrm{Int}_{\tau}(F)$. Now, $\mathrm{Int}_{\tau}(F) = \mathrm{Int}_{\tau}\mathrm{Cl}_{\tau}\mathrm{Int}_{\tau}(F) \in \tau_s$. So, $x \in \mathrm{Int}_{\tau}(F) \subseteq F$ and $\mathrm{Int}_{\tau}(F) \in \tau_s$. Thus, $x \in \mathrm{Int}_{\tau_s}(F)$.

(iii) Since $\mathrm{Cl}_\tau(U) \in \mathrm{RC}(X,\tau)$ for all $U \in \tau$. Then by parts (i) and (ii), $\mathrm{Int}_\tau\mathrm{Cl}_\tau(U) = \mathrm{Int}_{\tau_s}\mathrm{Cl}_{\tau_s}(U)$ for all $U \in \tau$. Since $\mathrm{RO}(X,\tau) \subseteq \tau$ and $\mathrm{RO}(X,\tau_s) \subseteq \tau$, then we have $U \in \mathrm{RO}(X,\tau)$ if and only if $U = \mathrm{Int}_\tau\mathrm{Cl}_\tau(U)$ if and only if $U = \mathrm{Int}_{\tau_s}\mathrm{Cl}_{\tau_s}(U)$ if and only if $U \in \mathrm{RO}(X,\tau_s)$.

(iv) This follows from Theorem 1.2.4 and part (iii).

□

Proposition 1.3.2. [73] For any topological space (X,τ), $(\tau_s)_s = \tau_s$, that is, (X,τ_s) is semi-regular.

Proof. Clearly, $(\tau_s)_s \subseteq \tau_s$. Let $U \in \tau_s$. Then U is a union of sets in $\mathrm{RO}(X,\tau)$ but by Lemma 1.3.1, $\mathrm{RO}(X,\tau) = \mathrm{RO}(X,\tau_s)$, so U is a union of sets in $\mathrm{RO}(X,\tau_s)$. Thus, $U \in (\tau_s)_s$. Hence, $\tau_s \subseteq (\tau_s)_s$. Therefore, $(\tau_s)_s = \tau_s$. □

Definition 1.3.2. [73] A topological property P is called *semi-regular* if (X,τ) has property P if and only if (X,τ_s) has property P.

Lemma 1.3.2. *[73] If A and B are disjoint open sets in (X,τ), then $\overline{A}^\circ$ and $\overline{B}^\circ$ are disjoint open sets in (X,τ_s) containing A and B, respectively.*

Proof. Suppose that A and B are disjoint open sets in (X,τ). Since $\overline{A}^\circ$ and $\overline{B}^\circ$ are regular open sets in (X,τ), then $\overline{A}^\circ$ and $\overline{B}^\circ$ are open sets in (X,τ_s). Now, $A = A^\circ \subseteq \overline{A}^\circ$. Similarly, $B \subseteq \overline{B}^\circ$. Also, by Theorem 1.2.5, $\overline{A}^\circ \cap \overline{B}^\circ = \overline{A \cap B}^\circ = \overline{\emptyset}^\circ = \emptyset$. □

Recall that if (X,τ_1) is a T_j-space and τ_2 is a topology on X such that $\tau_1 \subseteq \tau_2$, then (X,τ_2) is a T_j-space for $j = 1,2$ [64].

Proposition 1.3.3. [73] A topological space (X,τ) is Hausdorff if and only if (X,τ_s) is Hausdorff. That is, the Hausdorff property is semi-regular.

Proof. Clearly, if (X,τ_s) is Hausdorff, then (X,τ) is Hausdorff since $\tau_s \subseteq \tau$. Conversely, suppose that (X,τ) is Hausdorff. Let $x \neq y$ in X, then there exist $U,V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. By Lemma 1.3.2, $\overline{U}^\circ$ and $\overline{V}^\circ$ are disjoint open sets in (X, τ_s) containing U and V, respectively. So, $\overline{U}^\circ$ and $\overline{V}^\circ$ are open sets in (X, τ_s) such that $x \in \overline{U}^\circ$, $y \in \overline{V}^\circ$ and $\overline{U}^\circ \cap \overline{V}^\circ = \emptyset$. Therefore, (X, τ_s) is Hausdorff. $\square$

1.3.2 Almost-Regular Spaces

Almost-regular topological spaces was introduced and studied in 1969 by Singal and Arya [102]. It can be easily seen by the following definition.

Definition 1.3.3. [102] A topological space (X, τ) is said to be *almost-regular* if for each regular closed subset A of (X, τ) and each point $x \notin A$, there exist open sets U and V in (X, τ) such that $A \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.

Proposition 1.3.4. Every regular space is almost-regular.

Proof. Assume that (X, τ) is regular. Let A be a regular closed subset of (X, τ) and $x \notin A$. Then A is closed in (X, τ) and $x \notin A$. But X is regular, so there are disjoint open sets U and V in (X, τ) such that $A \subseteq U$ and $x \in V$. Thus, (X, τ) is almost-regular. $\square$

Theorem 1.3.2. [22] A topological space (X, τ) is almost-regular if and only if for each $x \in X$ and each regular open set U containing x, there exists a regular open set V such that $x \in V \subseteq \overline{V} \subseteq U$.

Theorem 1.3.3. [73] A topological space (X, τ) is almost-regular if and only if (X, τ_s) is regular.

Proof. Let (X, τ) be almost-regular, C be a closed set in (X, τ_s) and $x \in X - C$. Now, $C = \bigcap\limits_{i \in I} C_i$, where C_i is regular closed in (X, τ) for each $i \in I$. Since $x \in X - C$, then there is some $j \in I$ such that $x \in X - C_j$. So, by almost-regularity of (X, τ), there are disjoint open sets U and V in (X, τ) such that $C \subseteq C_j \subseteq U$ and $x \in V$. By Lemma 1.3.2, there are disjoint open sets U' and V' in (X, τ_s) such that $C \subseteq U \subseteq U'$ and $x \in V \subseteq V'$. Hence, (X, τ_s) is regular.

Conversely, let C be a regular closed set in (X, τ) and $x \in X - C$. Then C is a closed set in (X, τ_s) and $x \in X - C$. But since (X, τ_s) is regular, then there are disjoint open sets U and V in (X, τ_s) such that $C \subseteq U$ and $x \in V$. Since $\tau_s \subseteq \tau$, then $U, V \in \tau$. Therefore, (X, τ) is almost-regular. □

Corollary 1.3.1. *[73] Almost-regularity is a semi-regular property.*

Proof.

$$
\begin{aligned}
(X, \tau_s) \text{ is almost-regular iff } (X, (\tau_s)_s) &\text{ is regular by Theorem 1.3.3} \\
&\text{iff } (X, \tau_s) \text{ is regular by Proposition 1.3.2} \\
&\text{iff } (X, \tau) \text{ is almost-regular by Theorem 1.3.3.}
\end{aligned}
$$

□

Corollary 1.3.2. *[105]. A topological space (X, τ) is semi-regular and almost-regular if and only if it is regular.*

Proof. If (X, τ) is regular, then by Propositions 1.3.1 and 1.3.4, (X, τ) is semi-regular and almost-regular. Conversely, if (X, τ) is almost-regular, then by Theorem 1.3.3, (X, τ_s) is regular but (X, τ) is semi-regular, then $\tau_s = \tau$. So, (X, τ) is regular. □

1.3.3 Extremally Disconnected Spaces

Definition 1.3.4. [120] A topological space (X, τ) is said to be *extremally disconnected* if for each $U \in \tau$, $\overline{U} \in \tau$.

Example 1.3.3.

(i) The co-finite topological space $(\mathbb{R}, \tau_{\text{cof.}})$ is extremally disconnected.

(ii) The left ray topological space $(\mathbb{R}, \tau_{\text{left.}})$ is extremally disconnected.

Proposition 1.3.5. [18] A topological space (X, τ) is extremally disconnected if and only if (X, τ_s) is extremally disconnected.

Proof. Assume that (X, τ) is extremally disconnected. Let $U \in \tau_s$, then $U \in \tau$ since $\tau_s \subseteq \tau$. So, by Lemma 1.3.1 part (i), $\mathrm{Cl}_{\tau_s}(U) = \mathrm{Cl}_\tau(U)$. But $U \in \tau$ implies $\mathrm{Cl}_\tau(U) \in \tau$. Thus, $\mathrm{Cl}_\tau(U) = \mathrm{Int}_\tau(\mathrm{Cl}_\tau(U)) \in \tau_s$. So, $\mathrm{Cl}_{\tau_s}(U) \in \tau_s$. Therefore, (X, τ_s) is extremally disconnected.

Conversely, suppose that (X, τ_s) is extremally disconnected. Let $U \in \tau$, then by Lemma 1.3.1 part (i), $\mathrm{Cl}_\tau(U) = \mathrm{Cl}_{\tau_s}(U)$. Now, since $\mathrm{Int}_\tau \mathrm{Cl}_\tau(U) \in \tau_s$, then $\mathrm{Cl}_{\tau_s}(\mathrm{Int}_\tau \mathrm{Cl}_\tau(U)) \in \tau_s \subseteq \tau$. Again, by Lemma 1.3.1 part (i), $\mathrm{Cl}_{\tau_s}(\mathrm{Int}_\tau \mathrm{Cl}_\tau(U)) = \mathrm{Cl}_\tau(\mathrm{Int}_\tau \mathrm{Cl}_\tau(U))$ since $\mathrm{Int}_\tau \mathrm{Cl}_\tau(U) \in \tau$. Hence, $\mathrm{Cl}_\tau(\mathrm{Int}_\tau \mathrm{Cl}_\tau(U)) \in \tau$. Since $U \in \tau$, then by Proposition 1.2.7 part (ii), $\mathrm{Cl}_\tau(U) \in \mathrm{RC}(X, \tau)$. So, $\mathrm{Cl}_\tau(\mathrm{Int}_\tau \mathrm{Cl}_\tau(U)) = \mathrm{Cl}_\tau \mathrm{Int}_\tau(\mathrm{Cl}_\tau(U)) = \mathrm{Cl}_\tau(U)$. Thus, $\mathrm{Cl}_\tau(U) \in \tau$. Therefore, (X, τ) is extremally disconnected. □

Proposition 1.3.6. Let (X, τ) be a topological space. Then (X, τ) is extremally disconnected if and only if $\mathrm{RO}(X, \tau) = \mathrm{RC}(X, \tau)$.

Proof. Suppose that $\mathrm{RO}(X, \tau) = \mathrm{RC}(X, \tau)$. Let $U \in \tau$. Then by Proposition 1.2.7, $\overline{U} \in \mathrm{RC}(X, \tau)$, and hence $\overline{U} \in \mathrm{RO}(X, \tau)$. Thus, $\overline{U} \in \tau$. Therefore, (X, τ) is extremally disconnected. Conversely, suppose that (X, τ) is extremally disconnected. Now,

$$
\begin{aligned}
\mathrm{RO}(X, \tau) &= \{\overline{U}^{\circ} : U \in \tau\} && \text{by Proposition 1.2.7 part (i)} \\
&= \{\overline{U} : U \in \tau\} && \text{since } X \text{ is extremally disconnected} \\
&= \mathrm{RC}(X, \tau) && \text{by Proposition 1.2.7 part (ii).}
\end{aligned}
$$

□

1.3.4 Weakly-T_2 Spaces

Definition 1.3.5. [108] A topological space (X, τ) is called *weakly-T_2* if each point of X is an intersection of regular closed sets.

Proposition 1.3.7. [3] A topological space (X, τ) is weakly-T_2 if and only if for each $x \neq y$ in X, there exists a regular closed set F such that $x \in F$ but $y \notin F$.

Proof. Suppose that X is weakly-T_2. Let $x \neq y$ in X. Then $y \notin \{x\}$ but $\{x\} = \bigcap\{F \in \mathrm{RC}(X) : x \in F\}$ since X is weakly-T_2. So, there exists $F \in \mathrm{RC}(X)$ with $x \in F$ such that $y \notin F$.

Conversely, let $x \in X$. Suppose, by the way of a contradiction, that $\{x\} \neq \bigcap\{F \in \mathrm{RC}(X) : x \in F\}$. Then there exists $y \in \bigcap\{F \in \mathrm{RC}(X) : x \in F\}$ such that $x \neq y$. By hypothesis, there exists $F \in \mathrm{RC}(X)$ such that $x \in F$ but $y \notin F$. This implies $y \notin \bigcap\{F \in \mathrm{RC}(X) : x \in F\}$, which is a contradiction. Hence, $\{x\} = \bigcap\{F \in \mathrm{RC}(X) : x \in F\}$. Therefore, (X, τ) is weakly-T_2. □

Theorem 1.3.4. [29] A topological space (X, τ) is weakly-T_2 if and only if (X, τ_s) is a T_1-space.

Proof. Assume that (X, τ) is weakly-T_2. Let $x \in X$. We show that $\{x\}$ is closed in (X, τ_s). Now, $\{x\} = \bigcap\{F \in \mathrm{RC}(X) : x \in F\}$. Since every regular open set in (X, τ) is open in (X, τ_s), then by Theorem 1.2.4, every regular closed set in (X, τ) is closed in (X, τ_s). So, $\{x\}$ is an intersection of closed sets in (X, τ_s). Hence, $\{x\}$ is closed in (X, τ_s). Therefore, (X, τ_s) is T_1.

Conversely, let $x \in X$. Suppose, by the way of contradiction, that $\{x\} \neq \bigcap\{F \in \mathrm{RC}(X) : x \in F\}$. Then there exists $y \in \bigcap\{F \in \mathrm{RC}(X) : x \in F\}$ such that $y \neq x$ but (X, τ_s) is T_1, so there exists $V \in \tau_s$ such that $y \in V$ but $x \notin V$. Since $y \in V \in \tau_s$, then $y \in U \subseteq V$ for some $U \in \mathrm{RO}(X)$. As $x \notin V$, then $x \notin U$. Let $F = X - U$, then $F \in \mathrm{RC}(X)$ and $x \in F$. So, $y \in F = X - U$, and hence $y \notin U$, which is a contradiction. Therefore, (X, τ) is weakly-T_2. □

The following implications hold.

Proposition 1.3.8. [3] Hausdorff $\Longrightarrow$ weakly-T_2 $\Longrightarrow$ T_1.

Proof. Suppose that (X, τ) is Hausdorff. Then by Theorem 1.3.3, (X, τ_s) is Hausdorff. So, (X, τ_s) is T_1. Hence, by Theorem 1.3.4, (X, τ) is weakly-T_2. Next, suppose that (X, τ) is weakly-T_2. Then again by Theorem 1.3.4, (X, τ_s) is T_1 but $\tau_s \subseteq \tau$, so (X, τ) is T_1. □

1.4 Prime Topological Spaces and Examples

The notion of prime spaces was defined in [38] where the sequential convergence was studied. Some authors do not use the name prime spaces, they speak simply about topological spaces having unique nonisolated point instead as in [69].

1.4.1 Prime Spaces

Definition 1.4.1. [100] Let A be a subset of a topological space (X, τ) and a point $x \in X$, then x is said to be an *isolated* point of A if there exists $U \in \mathcal{U}(x)$ such that $U \cap A = \{x\}$. The set of all isolated points of A is denoted by $\mathrm{I}(A)$.

Definition 1.4.2. [100] Let A be a subset of a topological space (X, τ) and a point $x \in X$, then x is said to be a *cluster* (or an *accumulation*) point of A if for every $U \in \mathcal{U}(x)$, $U \cap (A - \{x\}) \neq \emptyset$. The set of all cluster points of A is denoted by A'.

Theorem 1.4.1. [100] Let (X, τ) be a topological space and $x \in X$. Then x is an isolated point of X if and only if $\{x\}$ is open in X.

Definition 1.4.3. [38] A topological space is called a *prime space*, if it has precisely one nonisolated (or cluster) point.

Proposition 1.4.1. Let $\mathcal{F}$ be a filter on a nonempty set Y and a point $p \notin Y$. If $X = Y \cup \{p\}$, then the family $\tau = \mathcal{P}(Y) \cup \{F \cup \{p\} : F \in \mathcal{F}\}$, where $\mathcal{P}(Y)$ is the power set of Y, is a topology on X.

Proof. First, $\emptyset \in \mathcal{P}(Y) \subseteq \tau$ and $X = Y \cup \{p\} \in \tau$ since $Y \in \mathcal{F}$. Next, let $A, B \in \tau$. If $A, B \in \mathcal{P}(Y)$, then $A \cap B \in \mathcal{P}(Y) \subseteq \tau$. If $A \in \mathcal{P}(Y)$ and $B = F \cup \{p\}$ for some $F \in \mathcal{F}$, then $A \cap B = A \cap (F \cup \{p\}) = A \cap F \in \mathcal{P}(Y) \subseteq \tau$. If $A = F_1 \cup \{p\}$ and $B = F_2 \cup \{p\}$ for some $F_1, F_2 \in \mathcal{F}$, then $A \cap B = (F_1 \cap F_2) \cup \{p\} \in \tau$ since $F_1 \cap F_2 \in \mathcal{F}$. Finally, let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\} \subseteq \tau$. If $U_\alpha \subseteq Y$ for any $\alpha \in \Delta$, then $\bigcup\limits_{\alpha \in \Delta} U_\alpha \subseteq Y$, and hence $\bigcup\limits_{\alpha \in \Delta} U_\alpha \in \mathcal{P}(Y) \subseteq \tau$. If for any $\alpha \in \Delta$,

$U_\alpha = F_\alpha \cup \{p\}$, $F_\alpha \in \mathfrak{F}$, then $\bigcup\limits_{\alpha\in\Delta} U_\alpha = \bigcup\limits_{\alpha\in\Delta} (F_\alpha \cup \{p\}) = \Big(\bigcup\limits_{\alpha\in\Delta} F_\alpha\Big) \cup \{p\}$ but $\bigcup\limits_{\alpha\in\Delta} F_\alpha \in \mathfrak{F}$ since $\bigcup\limits_{\alpha\in\Delta} F_\alpha \supseteq F_\alpha$ and $F_\alpha \in \mathfrak{F}$. So, $\bigcup\limits_{\alpha\in\Delta} U_\alpha \in \tau$. If $\mathfrak{U} = \{U_\alpha \subseteq Y : \alpha \in \Delta_1\} \cup \{F_\alpha \cup \{p\} : \alpha \in \Delta_2\}$ where $\Delta = \Delta_1 \cup \Delta_2$, then $\bigcup\limits_{\alpha\in\Delta} U_\alpha = \bigcup\limits_{\alpha\in\Delta_1} U_\alpha \cup \bigcup\limits_{\alpha\in\Delta_2} U_\alpha$. So, $\bigcup\limits_{\alpha\in\Delta} U_\alpha = \bigcup\limits_{\alpha\in\Delta_1} U_\alpha \cup \bigcup\limits_{\alpha\in\Delta_2} (F_\alpha \cup \{p\}) = \Big(\bigcup\limits_{\alpha\in\Delta_1} U_\alpha \cup \bigcup\limits_{\alpha\in\Delta_2} F_\alpha\Big) \cup \{p\}$. Since for any $\alpha \in \Delta_2$, $F_\alpha \subseteq \bigcup\limits_{\alpha\in\Delta_2} U_\alpha \cup \bigcup\limits_{\alpha\in\Delta_2} F_\alpha \subseteq Y$, then $\bigcup\limits_{\alpha\in\Delta_2} U_\alpha \cup \bigcup\limits_{\alpha\in\Delta_2} F_\alpha \in \mathfrak{F}$. Hence, $\bigcup\limits_{\alpha\in\Delta} U_\alpha \in \tau$. Therefore, τ is a topology on X. □

Proposition 1.4.2. [24, 77] The topological space (X, τ) defined in Proposition 1.4.1, is a prime space and is denoted by (X, τ_p).

Proof. If $p \in \mathrm{I}(X)$, then $\{p\} \in \tau_p$. This implies $\{p\} = F \cup \{p\}$ for some $F \in \mathfrak{F}$, so $F \subseteq \{p\}$ but $F \neq \emptyset$. Thus, $p \in F \subseteq Y$ and thus, $p \in Y$, which is a contradiction. Hence, p is not an isolated point of X. Next, let $y \in Y$, then $y \in Y$, and so $\{y\}$ is open in X. Thus, $y \in \mathrm{I}(X)$. Hence, (X, τ_p) has one nonisolated point, namely, p. Therefore, (X, τ_p) is a prime space. □

1.4.2 Completely Normal Spaces

Recall that a topological space (X, τ) is called *normal* if for every two disjoint closed sets A and B in X, there exist open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$ [101]. A family $\{A_\alpha\}_{\alpha\in\Delta}$ of subsets of a set X is called a *cover* of X if $\bigcup\limits_{\alpha\in\Delta} A_\alpha = X$. If X is a topological space and all sets A_α are open (closed), we say that the cover $\{A_\alpha\}$ is *open* (*closed*) [35].

Definition 1.4.4. [101] Let (X, τ) be a topological space and $A, B \subseteq X$. Then A and B are said to be *separated* if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

Definition 1.4.5. [101] A topological space (X, τ) is called *completely normal* if for every two separated sets A and B in X, there exist open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Theorem 1.4.2. The prime space (X, τ_p) is completely normal.

Proof. Let A and B be two separated sets in X. That is, $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. If $p \notin \overline{A}$, then $\overline{A} \subseteq Y$, and so $\overline{A}$ is open in X. Since $\overline{A} \cap B = \emptyset$, then $B \subseteq X - \overline{A}$. Hence, $\overline{A}$ and $X - \overline{A}$ are disjoint open sets in X such that $A \subseteq \overline{A}$ and $B \subseteq X - \overline{A}$. If $p \in \overline{A}$, then $p \notin B$ since $\overline{A} \cap B = \emptyset$. So, $X - \overline{B}$ and B are disjoint open sets in X such that $A \subseteq X - \overline{B}$ and $B \subseteq B$. Therefore, (X, τ_p) is completely normal. $\square$

1.4.3 Fully Normal Spaces

Definition 1.4.6. [45, 94] Let X be a set, $\mathcal{U}$ a cover of X and $x \in X$. Then $\mathrm{St}(x, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : x \in U\}$ is said to be a *star* of x with respect to $\mathcal{U}$.

Definition 1.4.7. [45, 94] Let X be a set, $\mathcal{U}$ and $\mathcal{V}$ covers of X. Then $\mathcal{V}$ is said to be a *star-refinement* of $\mathcal{U}$ if for every $x \in X$, there exists $U \in \mathcal{U}$ such that $\mathrm{St}(x, \mathcal{V}) \subseteq U$.

Example 1.4.1. *Let $X = \{1, 2, 3\}$. Consider the cover $\mathcal{U} = \{\{1, 2\}, \{2\}, \{2, 3\}\}$ of X. Then the cover $\mathcal{V} = \{\{1\}, \{2\}, \{3\}\}$ is a star-refinement of $\mathcal{U}$ since*

$$\begin{aligned} \{1, 2\} \in \mathcal{U} &\text{ such that } \mathrm{St}(1, \mathcal{V}) = \{1\} \subseteq \{1, 2\}, \\ \{2\} \in \mathcal{U} &\text{ such that } \mathrm{St}(2, \mathcal{V}) = \{2\} \subseteq \{2\}, \text{ and} \\ \{2, 3\} \in \mathcal{U} &\text{ such that } \mathrm{St}(3, \mathcal{V}) = \{3\} \subseteq \{2, 3\}. \end{aligned}$$

While the cover $\mathcal{U}' = \{\{1, 2\}, \{2, 3\}\}$ is not a star-refinement of $\mathcal{U}$ since $\mathrm{St}(2, \mathcal{U}') = \{1, 2\} \cup \{2, 3\} = \{1, 2, 3\} = X$ and $X \nsubseteq U$ for all $U \in \mathcal{U}$.

Definition 1.4.8. [45] A topological space (X, τ) is called *fully normal* if every open cover of X has an open star-refinement.

Theorem 1.4.3. Any discrete space is fully normal.

Proof. Let X be a discrete space and $\mathcal{U}$ be any open cover of X. Consider $\mathcal{V} = \{\{x\} : x \in X\}$, then $\mathcal{V}$ is an open cover of X and $\mathrm{St}(x, \mathcal{U}) = \{x\}$ for all $x \in X$. Now, for all $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Hence, for all $x \in X$, $U_x \in \mathcal{U}$ is such that $\mathrm{St}(x, \mathcal{U}) = \{x\} \subseteq U_x$. Thus, $\mathcal{V}$ is an open star-refinement of $\mathcal{U}$. Therefore, X is fully normal. $\square$

Theorem 1.4.4. The prime space (X, τ_p) is fully normal.

Proof. Let $\mathcal{U}$ be an open cover of X. Then $X = \bigcup_{U \in \mathcal{U}} U$. So, for all $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since U_p is open in X and $p \in U_p$, then there exists $F_\circ \in \mathcal{F}$ such that $U_p = F_\circ \cup \{p\}$. Let $\mathcal{V} = \{\{y\} : y \in Y\} \cup \{F_\circ \cup \{p\}\}$, then we claim that $\mathcal{V}$ is an open star-refinement of $\mathcal{U}$. Clearly, $\mathcal{V}$ is an open cover of X. Let $x \in X$, then $x \in Y$ or $x = p$. Now, suppose that $x \in Y$, then either $x \in F_\circ$ or $x \notin F_\circ$. If $x \in F_\circ$, then $\mathrm{St}(x, \mathcal{V}) = \{x\} \cup (F_\circ \cup \{p\}) = F_\circ \cup \{p\} = U_p$ and $U_p \in \mathcal{U}$. If $x \notin F_\circ$, then $\mathrm{St}(x, \mathcal{V}) = \{x\} \subseteq U_x$ and $U_x \in \mathcal{U}$. Next, suppose that $x = p$, then $\mathrm{St}(x, \mathcal{V}) = F_\circ \cup \{p\} \subseteq U_p$ and $U_p \in \mathcal{U}$. Therefore, (X, τ_p) is fully normal. $\square$

Chapter 2

Convergence of Filters

We study convergence of filters. We will start by introducing the definition of a limit of a filter and define a cluster point of a filter. In addition, the connections between a continuous function and limits of filters as well as cluster points of filters are investigated. Hausdorff and compact spaces are characterized by our present structure. A closed graph concept is defined and characterized by filters.

2.1 Limit and Cluster Points of Filters

Definition 2.1.1. [33] Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F}$ *τ-converges* to x, written $\mathcal{F} \longrightarrow x$, iff $\mathcal{U}_\tau(x) \subseteq \mathcal{F}$. In such a case, x is called the *limit* of $\mathcal{F}$.

Notation 6. *When referring to a topological space (X, τ), when no confusion may arise, we will simply say that "converges" instead of "τ-converges".*

Definition 2.1.2. [33] Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F}$ *accumulates* at x, written $\mathcal{F} \propto x$, iff $\mathcal{F}(\cap)\mathcal{U}_\tau(x)$ iff for each $F \in \mathcal{F}$ and for each $U \in \mathcal{U}_\tau(x)$, $F \cap U \neq \emptyset$. In such a case, x is called the *cluster* point of $\mathcal{F}$.

Proposition 2.1.1. [120] Let $\mathfrak{F}$ be a filter on a topological space (X, τ) and $x \in X$. If $\mathfrak{F} \longrightarrow x$, then $\mathfrak{F} \propto x$.

Proof. Assume that $\mathfrak{F} \longrightarrow x$ and $U \in \mathfrak{U}_\tau(x)$, then $U \in \mathfrak{F}$. Hence, $U \cap F \neq \emptyset$ for all $F \in \mathfrak{F}$. Therefore, $\mathfrak{F} \propto x$. $\square$

The converse of proposition 2.1.1 need not be true as the following example shows.

Example 2.1.1. *Let* $X = \{1, 2, 3\}$ *and* $\tau = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$. *Let* $\mathfrak{F} = \{X, \{1, 3\}\}$. *Then* $\mathfrak{U}_\tau(1) = \{\{1\}, \{1, 2\}, \{1, 3\}, X\}$. *Clearly,* $\mathfrak{F} \propto 1$ *but* $\mathfrak{F} \not\longrightarrow 1$.

Definition 2.1.3. [101] Let (X, τ) be a topological space, $E \subseteq X$ and $x \in X$. We say that x is an *adherent* point of E iff for all $U \in \tau(x)$, $U \cap E \neq \emptyset$.

Remark 2.1. *[101] Let* (X, τ) *be a topological space,* $E \subseteq X$ *and* $x \in X$. *Then* $x \in \overline{E}$ *if and only if* x *is an adherent point of* E.

Definition 2.1.4. [101] Let $\mathfrak{F}$ be a filter on a topological space (X, τ). A point $x \in X$ is said to be an *adherent* point of $\mathfrak{F}$ if x is an adherent point of every set in $\mathfrak{F}$. The *adherence* of $\mathfrak{F}$, $\mathrm{Adh}_\tau(\mathfrak{F})$, is the set of all adherent points of $\mathfrak{F}$.

Remark 2.2. *[2] Let* (X, τ) *be a topological space. If* $\mathfrak{F}$ *is a filter on* X, *then* $\mathrm{Adh}_\tau(\mathfrak{F}) = \bigcap_{F \in \mathfrak{F}} \overline{F}$.

Theorem 2.1.1. [2] Let $\mathfrak{F}$ be a filter on a topological space (X, τ) and $x \in X$. Then $x \in \mathrm{Adh}_\tau(\mathfrak{F})$ if and only if $\mathfrak{F} \propto x$.

Proof.

$$\begin{aligned}
x \in \mathrm{Adh}_\tau(\mathfrak{F}) &\text{ iff } x \in \bigcap_{F \in \mathfrak{F}} \overline{F} \\
&\text{ iff } x \in \overline{F} \text{ for all } F \in \mathfrak{F} \\
&\text{ iff } V \cap F \neq \emptyset \text{ for all } V \in \mathfrak{U}_\tau(x) \text{ and for all } F \in \mathfrak{F} \\
&\text{ iff } \mathfrak{F}(\cap)\mathfrak{U}_\tau(x) \\
&\text{ iff } \mathfrak{F} \propto x.
\end{aligned}$$

$\square$

Theorem 2.1.2. [9] Let (X, τ) be a topological space, $E \subseteq X$ and $x \in X$. Then $x \in \overline{E}$ if and only if there exists a filter $\mathcal{F}$ on X such that $E \in \mathcal{F}$ and $\mathcal{F} \longrightarrow x$.

Proof. Suppose that there exists a filter $\mathcal{F}$ on X such that $E \in \mathcal{F}$ and $\mathcal{F} \longrightarrow x$. Let $U \in \mathcal{U}_\tau(x)$. But $\mathcal{U}_\tau(x) \subseteq \mathcal{F}$ since $\mathcal{F} \longrightarrow x$. So, $U \in \mathcal{F}$. Hence, $U \cap E \neq \emptyset$. Thus, $x \in \overline{E}$.

Conversely, suppose that $x \in \overline{E}$, then $U \cap E \neq \emptyset$ for all $U \in \mathcal{U}_\tau(x)$. Consider the filter $\mathcal{F} = \langle \mathcal{U}_\tau(x)\big|_E \rangle$. Then by Proposition 1.1.6, $E \in \mathcal{F}$ and $\mathcal{U}_\tau(x) \subseteq \mathcal{F}$. Therefore, $E \in \mathcal{F}$ and $\mathcal{F} \longrightarrow x$. $\square$

Theorem 2.1.3. [2, 95] Let (X, τ) be a topological space, $E \subseteq X$ and $x \in X$. Then $x \in \overline{E}$ if and only if there exists a filter $\mathcal{F}$ on X such that $\mathcal{F}$ meets E and $\mathcal{F} \longrightarrow x$.

Proof. Suppose that $x \in \overline{E}$. Then by Theorem 2.1.2, there exists a filter $\mathcal{F}$ on X such that $E \in \mathcal{F}$ and $\mathcal{F} \longrightarrow x$. Since $E \in \mathcal{F}$, then for all $F \in \mathcal{F}$, $F \cap E \neq \emptyset$. That is, $\mathcal{F}$ meets E.

Conversely, suppose that $\mathcal{F}$ is a filter on X such that $\mathcal{F} \longrightarrow x$ and $F \cap E \neq \emptyset$ for all $F \in \mathcal{F}$. Since $\mathcal{F} \longrightarrow x$, then $U \in \mathcal{F}$ for all $U \in \mathcal{U}_\tau(x)$. So, by hypothesis, $U \cap E \neq \emptyset$ for all $U \in \mathcal{U}_\tau(x)$. Therefore, $x \in \overline{E}$. $\square$

Theorem 2.1.4. [9] Let (X, τ) be a topological space and $A \subseteq X$. Then A is closed if and only if whenever a filter $\mathcal{F} \longrightarrow x$ with $A \in \mathcal{F}$, then $x \in A$.

Proof. Assume that a filter $\mathcal{F} \longrightarrow x$ and $A \in \mathcal{F}$. Then by Theorem 2.1.2, $x \in \overline{A}$. But $\overline{A} = A$ since A is closed. So, $x \in A$. Conversely, let $x \in \overline{A}$. Then by Theorem 2.1.2, there is a filter $\mathcal{F}$ on X such that $\mathcal{F} \longrightarrow x$ and $A \in \mathcal{F}$. So by hypothesis, $x \in A$. Thus, $\overline{A} \subseteq A$. But $A \subseteq \overline{A}$, and hence $\overline{A} = A$. Therefore, A is closed. $\square$

Theorem 2.1.5. [9] Let (X, τ) be a topological space and $A \subseteq X$. Then A is open in X if and only if whenever a filter $\mathcal{F} \longrightarrow x \in A$, then $A \in \mathcal{F}$.

Proof. Suppose that A is open in X. If a filter $\mathcal{F} \longrightarrow x \in A$, then $A \in \mathcal{F}$ since $A \in \mathcal{U}_\tau(x)$. Conversely, suppose, by the way of contradiction, that there exists $x \in A$ such that $x \notin A^\circ$. Then $x \in X - A^\circ = \overline{X - A}$, so by Theorem 2.1.2, there exists a filter $\mathcal{F}$ on X such that $X - A \in \mathcal{F}$ and $\mathcal{F} \longrightarrow x$. Since $\mathcal{F} \longrightarrow x \in A$, then by hypothesis, $A \in \mathcal{F}$. But then $\emptyset = A \cap (X - A) \in \mathcal{F}$, which is a contradiction. Thus, for all $x \in A$, $x \in A^\circ$. That is, $A \subseteq A^\circ$. Therefore, A is open in X. $\square$

Remark 2.3. *[95] Let $\mathcal{F}$ and $\mathcal{G}$ be filters on a topological space (X, τ) and $x \in X$.*

(i) The principal filter $\langle x \rangle \longrightarrow x$.

(ii) If $\mathcal{F} \longrightarrow x$ and $\mathcal{G} \longrightarrow x$, then $\mathcal{F} \cap \mathcal{G} \longrightarrow x$.

Theorem 2.1.6. [40] Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \longrightarrow x$ if and only if for every subfilter $\mathcal{F}'$ of $\mathcal{F}$, $\mathcal{F}' \longrightarrow x$.

Proof. If every subfilter of $\mathcal{F}$ τ-converges to $x \in X$, then so does $\mathcal{F}$ because it is a subfilter of itself. Conversely, if $\mathcal{F} \longrightarrow x$ and $\mathcal{F}'$ is a subfilter of $\mathcal{F}$, then $\mathcal{U}_\tau(x) \subseteq \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{F}'$. That is, $\mathcal{U}_\tau(x) \subseteq \mathcal{F}'$. So, $\mathcal{F}' \longrightarrow x$. $\square$

Theorem 2.1.7. [2, 95] Let (X, τ) be a topological space and $\mathcal{F}$ be a filter on X. Then $\mathcal{F} \longrightarrow x$ if and only if every subfilter $\mathcal{G}$ of $\mathcal{F}$ has a subfilter $\mathcal{H}$ such that $\mathcal{H} \longrightarrow x$.

Proof. Suppose, by the way of contradiction, that $\mathcal{F} \not\longrightarrow x$, then there is an open set U containing x such that $U \notin \mathcal{F}$. Then for all $F \in \mathcal{F}$, $F \cap (X - U) \neq \emptyset$ (for if, $F \cap (X - U) = \emptyset$ for some $F \in \mathcal{F}$, then $F \subseteq U$, and so $U \in \mathcal{F}$, which is a contradiction). So, $\mathcal{G} = \langle \mathcal{F}\big|_{X-U} \rangle$ is a subfilter of $\mathcal{F}$ containing $X - U$ by Proposition 1.1.6. By hypothesis, $\mathcal{G}$ has a subfilter $\mathcal{H}$ which τ-converges to x. Since U is an open set containing x, then $U \in \mathcal{H}$ but $X - U \in \mathcal{G} \subseteq \mathcal{H}$. So, $\emptyset \in \mathcal{H}$, which is a contradiction. The converse follows from Theorem 2.1.6. $\square$

Theorem 2.1.8. [9] Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \propto x$ if and only if there exists a subfilter $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \longrightarrow x$.

Proof. Suppose that there exists a subfilter $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \longrightarrow x$. Then $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{U}_\tau(x) \subseteq \mathcal{F}'$. So, if $U \in \mathcal{U}_\tau(x)$ and $F \in \mathcal{F}$, then $U \in \mathcal{F}'$ and $F \in \mathcal{F}'$. So, $U \cap F \in \mathcal{F}'$, and hence $U \cap F \neq \emptyset$. Therefore, $\mathcal{F} \propto x$.

Conversely, assume that $\mathcal{F} \propto x$. We will construct a subfilter $\mathcal{F}'$ of $\mathcal{F}$ that τ-converges to x. Since $\mathcal{F} \propto x$, then $F \cap U \neq \emptyset$ for all $F \in \mathcal{F}$ and all $U \in \mathcal{U}_\tau(x)$. Let $\mathcal{F}' = \mathcal{F} \vee \mathcal{U}_\tau(x)$. Then $\mathcal{F}'$ is a filter on X such that $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{U}_\tau(x) \subseteq \mathcal{F}'$. Thus, $\mathcal{F}'$ is a subfilter of $\mathcal{F}$ and $\mathcal{F}' \longrightarrow x$. □

Theorem 2.1.9. [95] Let (X, τ) be a topological space, $\mathcal{F}'$ be a subfilter of $\mathcal{F}$ on X and $x \in X$. If $\mathcal{F}' \propto x$, then $\mathcal{F} \propto x$.

Proof. Suppose that $\mathcal{F}' \propto x$. Let $U \in \mathcal{U}_\tau(x)$ and $F \in \mathcal{F}$, but $\mathcal{F}'$ is a subfilter of $\mathcal{F}$, so $F \in \mathcal{F}'$. Thus, $U \cap F \neq \emptyset$. Hence, $\mathcal{F} \propto x$. □

Theorem 2.1.10. [120] Let $\mathcal{F}$ be an ultrafilter on a topological space (X, τ) and $x \in X$. Then $\mathcal{F} \longrightarrow x$ if and only if $\mathcal{F} \propto x$.

Proof. If $\mathcal{F} \longrightarrow x$, then by Proposition 2.1.1, $\mathcal{F} \propto x$. Conversely, suppose that $\mathcal{F} \propto x$. Let $U \in \mathcal{U}_\tau(x)$. Then $U \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. So, $\mathcal{F}$ meets U. But $\mathcal{F}$ is an ultrafilter on X, then by the proof of Theorem 1.1.5, $U \in \mathcal{F}$. Therefore, $\mathcal{F} \longrightarrow x$. □

2.2 Convergence in Hausdorff Spaces

Theorem 2.2.1. [120] A topological space (X, τ) is Hausdorff if and only if each filter $\mathcal{F}$ on X τ-converges to at most one point in X.

Proof. Suppose that X is a Hausdorff space and $\mathcal{F}$ is a filter on X such that $\mathcal{F} \longrightarrow x$ and $\mathcal{F} \longrightarrow y$. Assume that $x \neq y$. But X is Hausdorff, so there exist $U \in \mathcal{U}_\tau(x)$ and $V \in \mathcal{U}_\tau(y)$ such that $U \cap V = \emptyset$. Now, since $\mathcal{F} \longrightarrow x$, then $\mathcal{U}_\tau(x) \subseteq \mathcal{F}$, and so $U \in \mathcal{F}$. Also, since $\mathcal{F} \longrightarrow y$, then $\mathcal{U}_\tau(y) \subseteq \mathcal{F}$, and so $V \in \mathcal{F}$. Thus, $U \cap V \neq \emptyset$, which is a contradiction. So, we must have $x = y$.

Conversely, suppose that X is not a Hausdorff space. Then there exists $x \neq y$ in X such that $U \cap V \neq \emptyset$ for all $U \in \mathcal{U}_\tau(x)$ and all $V \in \mathcal{U}_\tau(y)$. Let $\mathcal{F} = \mathcal{U}_\tau(x) \vee \mathcal{U}_\tau(y)$, then $\mathcal{F}$ is a filter on X such that $\mathcal{U}_\tau(x) \subseteq \mathcal{F}$ and $\mathcal{U}_\tau(y) \subseteq \mathcal{F}$. Thus, the filter $\mathcal{F}$ τ-converges to both x and y. But then by hypothesis, $x = y$, which is a contradiction. Therefore, X must be Hausdorff. $\square$

Theorem 2.2.2. [60] Let (X, τ) be a Hausdorff space, $\mathcal{F}$ be a filter on X and $x \in X$. If $\mathcal{F} \longrightarrow x$, then x is the unique cluster point of $\mathcal{F}$.

Proof. If $\mathcal{F} \longrightarrow x$, then by Proposition 2.1.1, $\mathcal{F} \propto x$. Now, suppose that $y \in X$ is a cluster point of $\mathcal{F}$ with $x \neq y$. But since X is Hausdorff, then there exist $U \in \mathcal{U}_\tau(x)$ and $V \in \mathcal{U}_\tau(y)$ such that $U \cap V = \emptyset$. Now, since $\mathcal{F} \longrightarrow x$, then $\mathcal{U}_\tau(x) \subseteq \mathcal{F}$, and so $U \in \mathcal{F}$. But then, $U \cap V \neq \emptyset$ since $\mathcal{F} \propto y$, which is a contradiction. Therefore, $x = y$. $\square$

2.3 Convergence and Functions

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is *continuous* at $x \in X$ if for every open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq V$. If this condition is satisfied at each $x \in X$, then f is said to be continuous on X. It is possible to define the continuous function f in terms of neighborhoods. A function f is *continuous* at a point $x \in X$ if for every $V \in \mathcal{U}_\sigma(f(x))$, there exists $U \in \mathcal{U}_\tau(x)$ such that $f(U) \subseteq V$ [92, 120].

Theorem 2.3.1. [120] Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \longrightarrow x$, then $f(\mathcal{F}) \longrightarrow f(x)$ in Y.

Proof. Suppose that $\mathcal{F} \longrightarrow x$ in X. Let $V \in \mathcal{U}_\sigma(f(x))$. Since f is continuous at $x \in X$, then there exists $U \in \mathcal{U}_\tau(x)$ such that $f(U) \subseteq V$. But since $\mathcal{F} \longrightarrow x$, then $U \in \mathcal{F}$. Hence, $V \in f(\mathcal{F})$. Thus, $\mathcal{U}_\sigma(f(x)) \subseteq f(\mathcal{F})$. That is, $f(\mathcal{F}) \longrightarrow f(x)$.

Conversely, let $V \in \mathcal{U}_\sigma(f(x))$. Since $\mathcal{U}_\tau(x) \subseteq \mathcal{U}_\tau(x)$, then $\mathcal{U}_\tau(x) \longrightarrow x$. By hypothesis, $f(\mathcal{U}_\tau(x)) \longrightarrow f(x)$. That is, $\mathcal{U}_\sigma(f(x)) \subseteq f(\mathcal{U}_\tau(x))$. Therefore, there exists $U \in \mathcal{U}_\tau(x)$ such that $f(U) \subseteq V$. Therefore, f is continuous at $x \in X$. $\square$

Theorem 2.3.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \propto x$, then $f(\mathcal{F}) \propto f(x)$ in Y.

Proof. Suppose that $\mathcal{F} \propto x$ in X and $V \in \mathcal{U}_\sigma(f(x))$, then there exists $U \in \mathcal{U}_\tau(x)$ such that $f(U) \subseteq V$ since f is continuous at $x \in X$. Since $U \in \mathcal{U}_\tau(x)$ and $\mathcal{F} \propto x$, then $U \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, and so $f(U \cap F) \neq \emptyset$ for all $F \in \mathcal{F}$. But $V \cap f(F) \supseteq f(U) \cap f(F) \supseteq f(U \cap F)$ for all $F \in \mathcal{F}$. Hence, $V \cap f(F) \neq \emptyset$ for all $F \in \mathcal{F}$. Therefore, $f(\mathcal{F}) \propto f(x)$ in Y.

Conversely, suppose, by the way of contradiction, that f is not continuous at $x \in X$, then there exists $V \in \mathcal{U}_\sigma(f(x))$ such that $f(U) \nsubseteq V$ for any $V \in \mathcal{U}_\tau(x)$. So, $U \nsubseteq f^{-1}(V)$ for any $U \in \mathcal{U}_\tau(x)$. Let $\mathcal{B} = \{U - f^{-1}(V) : U \in \mathcal{U}_\tau(x)\}$, then $\mathcal{B}$ is a filter base in X. Let $\mathcal{F} = \langle \mathcal{B} \rangle_X$, then we claim that $\mathcal{F} \propto x$ but $f(\mathcal{F}) \not\propto f(x)$. Let $U \in \mathcal{U}_\tau(x)$ and $F \in \mathcal{F}$, then $F \supseteq B$ for some $B \in \mathcal{B}$. This implies that $F \supseteq W - f^{-1}(V)$ for some $W \in \mathcal{U}_\tau(x)$. Since $U \cap W \in \mathcal{U}_\tau(x)$, then $(U \cap W) - f^{-1}(V) \neq \emptyset$ but $U \cap F \supseteq U \cap (W - f^{-1}(V)) \neq \emptyset$. Hence, $\mathcal{F} \propto x$. Next, since $X \in \tau(x)$, then $B = X - f^{-1}(V) \in \mathcal{B} \subseteq \mathcal{F}$, so $f(B) \in f(\mathcal{F})$. We claim that $V \cap f(B) = \emptyset$. For if $f(b) \in V$ for some $b \in B$, then $b \in f^{-1}(V)$ and $b \in X - f^{-1}(V)$, so $b \in (X - f^{-1}(V)) \cap f^{-1}(V) = \emptyset$, which is a contradiction. Since $V \in \mathcal{U}_\sigma(f(x))$, $f(B) \in f(\mathcal{F})$ and $V \cap f(B) = \emptyset$, then $f(\mathcal{F}) \not\propto f(x)$ in Y. □

Lemma 2.3.1. *[64] Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of topological spaces and consider the product space $\prod_{\alpha \in \Delta} X_\alpha$. Let $A_\alpha \subseteq X_\alpha$ for each $\alpha \in \Delta$ and π_α be the α^{th} projection function. Then $\prod_{\alpha \in \Delta} A_\alpha = \bigcap_{\alpha \in \Delta} \pi_\alpha^{-1}(A_\alpha)$.*

Theorem 2.3.3. [9] Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of topological spaces and let $\mathcal{F}$ be a filter on $X = \prod_{\alpha \in \Delta} X_\alpha$. Then $\mathcal{F} \longrightarrow x$ in X if and only if $\pi_\alpha(\mathcal{F}) \longrightarrow \pi_\alpha(x)$ in X_α for all $\alpha \in \Delta$.

Proof. Assume that $\mathcal{F} \longrightarrow x$ in X. Since π_α is continuous for all $\alpha \in \Delta$, then by Theorem 2.3.1, $\pi_\alpha(\mathcal{F}) \longrightarrow \pi_\alpha(x)$ for all $\alpha \in \Delta$.

Conversely, suppose that $\pi_\alpha(\mathcal{F}) \longrightarrow \pi_\alpha(x)$ for all $\alpha \in \Delta$. Let U be any neighborhood of x in X. Then $x \in \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i) \subseteq U$, where $U_i \in \mathcal{U}(\pi_{\alpha_i}(x))$ for all

$i = 1, \ldots, n$. But $\pi_{\alpha_i}(\mathcal{F}) \longrightarrow \pi_{\alpha_i}(x)$ for all $i = 1, \ldots, n$, and so $U_i \in \pi_{\alpha_i}(\mathcal{F})$ for all $i = 1, \ldots, n$, and hence for all $i = 1, \ldots, n$, there exists $F_i \in \mathcal{F}$ such that $\pi_{\alpha_i}(F_i) \subseteq U_i$. Then $F_i \subseteq \pi_{\alpha_i}^{-1}(U_i)$ for all $i = 1, \ldots, n$. So, $\bigcap\limits_{i=1}^{n} F_i \subseteq \bigcap\limits_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i)$. Since $\bigcap\limits_{i=1}^{n} F_i \in \mathcal{F}$, then $\bigcap\limits_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i) \in \mathcal{F}$. So, $U \in \mathcal{F}$ and thus, $\mathcal{F} \longrightarrow x$ in X. $\square$

Theorem 2.3.4. Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of topological spaces and let $\mathcal{F}$ be a filter on $X = \prod\limits_{\alpha \in \Delta} X_\alpha$. If $\mathcal{F} \propto x$ in X, then $\pi_\alpha(\mathcal{F}) \propto \pi_\alpha(x)$ in X_α for all $\alpha \in \Delta$.

Proof. Assume that $\mathcal{F} \propto x \in X$. Since π_α is continuous for every $\alpha \in \Delta$, then by Theorem 2.3.2, $\pi_\alpha(\mathcal{F}) \propto \pi_\alpha(x)$ for every $\alpha \in \Delta$. $\square$

2.4 Compactness

2.4.1 Characterizations of Compactness

Definition 2.4.1. [9] A topological space (X, τ) is called *compact* iff each open cover of X has a finite subcover.

Definition 2.4.2. [92] A subset A of a topological space (X, τ) is said to be

(i) a *compact subspace* if the space (A, τ_A) is compact.

(ii) a *compact relative* to X if for every cover $\{V_\alpha : \alpha \in \Delta\}$ of A by open sets in X, there exists a finite subset Ω of Δ such that $A \subseteq \bigcup\limits_{\alpha \in \Omega} V_\alpha$.

Theorem 2.4.1. [92, 120] A set A of a topological space (X, τ) is a compact subspace if and only if A is a compact relative to X.

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a cover of A by open sets of X. Since $U_\alpha \in \tau$ for each $\alpha \in \Delta$, then $A \cap U_\alpha \in \tau_A$ for each $\alpha \in \Delta$. Hence, $A \cap U_\alpha$ is open in A for each $\alpha \in \Delta$. Now, since $A \subseteq \bigcup\limits_{\alpha \in \Delta} U_\alpha$, then $A = A \cap \left(\bigcup\limits_{\alpha \in \Delta} U_\alpha \right) = \bigcup\limits_{\alpha \in \Delta} (A \cap U_\alpha)$. So, $\{A \cap U_\alpha : \alpha \in \Delta\} \subseteq \tau_A$ is a cover of A. But A is compact, then there exists a finite subset Ω of Δ such that $A = \bigcup\limits_{\alpha \in \Omega} (A \cap U_\alpha)$. Hence, $A \subseteq \bigcup\limits_{\alpha \in \Omega} U_\alpha$. Therefore, A is compact relative to X.

Conversely, suppose that A is compact relative to X. Let $\{U_\alpha : \alpha \in \Delta\} \subseteq \tau_A$ be a cover of A. Then for each $\alpha \in \Delta$, $U_\alpha = V_\alpha \cap A$, where $V_\alpha \in \tau$. Thus, $\{V_\alpha : \alpha \in \Delta\} \subseteq \tau$ is a cover of A. Since A is a compact relative to X. Then there exists a finite subset Ω of Δ such that $A \subseteq \bigcup\limits_{\alpha\in\Omega} V_\alpha$. Thus, $A = A \cap (\bigcup\limits_{\alpha\in\Omega} V_\alpha) = \bigcup\limits_{\alpha\in\Omega}(A \cap V_\alpha) = \bigcup\limits_{\alpha\in\Omega} V_\alpha$. Therefore, A is a compact subspace of X. □

The property of a topological space being compact is not a semi-regular property, as the following example shows.

Example 2.4.1. *Consider the topological space $(\mathbb{R}, \tau)$, where τ is the left ray topology. Then $(\mathbb{R}, \tau)$ is not compact. But $\tau_s = \{\emptyset, \mathbb{R}\}$, so $(\mathbb{R}, \tau_s)$ is compact.*

We are now ready to make characterizations of compact spaces using the convergence of filters.

Theorem 2.4.2. [120] For a topological space (X, τ), the following are equivalent:

(i) X is compact.

(ii) Each filter on X has a cluster point.

(iii) Each ultrafilter on X τ-converges.

(iv) For each family $\mathcal{C}$ of closed sets of X such that $\bigcap\limits_{C\in\mathcal{C}} C = \emptyset$, there exists a finite subfamily $\mathcal{C}'$ of $\mathcal{C}$ such that $\bigcap\limits_{C\in\mathcal{C}'} C = \emptyset$.

Proof.

(i) $\Longrightarrow$ (ii) Assume that there is a filter $\mathcal{F}$ on X such that $\mathcal{F} \not\propto x$ for all $x \in X$. This means that for all $x \in X$, there exist $G_x \in \tau(x)$ and $F_x \in \mathcal{F}$ such that $F_x \cap G_x = \emptyset$. Consider $\mathcal{U} = \{G_x : x \in X\}$. Clearly, $\mathcal{U}$ is open cover of X. But X is compact, so there exist $x_1, \ldots, x_n \in X$ such that $X = \bigcup\limits_{i=1}^{n} G_{x_i}$. Now, for every $i = 1, \ldots, n$, choose F_{x_i} such that $F_{x_i} \cap G_{x_i} = \emptyset$ and let $F_\circ = \bigcap\limits_{i=1}^{n} F_{x_i}$. Then $F_\circ \in \mathcal{F}$ and $F_\circ \subseteq F_{x_i}$ for all $i = 1, \ldots, n$. So,

$$F_\circ = F_\circ \cap X = F_\circ \cap (\bigcup_{i=1}^{n} G_{x_i}) = \bigcup_{i=1}^{n}(F_\circ \cap G_{x_i}) \subseteq \bigcup_{i=1}^{n}(F_{x_i} \cap G_{x_i}) = \emptyset.$$

This implies $F_\circ = \emptyset$, which is a contradiction. Therefore, $\mathcal{F}$ must have a cluster point in X.

(ii) $\Longrightarrow$ (iii) Let $\mathcal{F}$ be an ultrafilter on X. Then by hypothesis, $\mathcal{F} \propto x \in X$. But then, $\mathcal{F} \longrightarrow x$ by Theorem 2.1.10 and and since $\mathcal{F}$ is an ultrafilter on X.

(iii) $\Longrightarrow$ (iv) Let $\mathcal{C} = \left\{F_\alpha : \alpha \in \Delta\right\}$ be a family of closed subsets of X such that $\bigcap\limits_{\alpha\in\Delta} F_\alpha = \emptyset$. Suppose, by the way of contradiction, that for each finite subset Ω of Δ, $\bigcap\limits_{\alpha\in\Omega} F_\alpha \neq \emptyset$. Let $\mathcal{B} = \{\bigcap\limits_{\alpha\in\Omega} F_\alpha : \Omega$ is a finite subset of $\Delta\}$, then $\mathcal{B}$ is a filter base in X. Hence, by Theorem 1.1.4, the filter $\mathcal{F} = \langle\mathcal{B}\rangle_X$ is contained in some ultrafilter $\mathcal{F}'$ on X. So, by hypothesis, $\mathcal{F}' \longrightarrow x \in X$. But then $\mathcal{F}' \propto x$. Hence, $\mathcal{F} \propto x$ by Theorem 2.1.9. Therefore, we have constructed a filter $\mathcal{F}$ on X which has $x \in X$ as a cluster point. Now, $x \notin \emptyset = \bigcap\limits_{\alpha\in\Delta} F_\alpha$. So, $x \notin F_{\alpha_\circ}$ for some $\alpha_\circ \in \Delta$. This implies that $X - F_{\alpha_\circ} \in \tau(x)$. But $F_{\alpha_\circ} \in \mathcal{F}$ by the construction of $\mathcal{F}$ and $\mathcal{F} \propto x$, so $(X - F_{\alpha_\circ}) \cap F_{\alpha_\circ} \neq \emptyset$, which is a contradiction. Therefore, there exists a finite subset Ω of Δ such that $\bigcap\limits_{\alpha\in\Omega} F_\alpha \neq \emptyset$.

(iv) $\Longrightarrow$ (i) Let $\mathcal{U}$ be an open cover of X. Then $\mathcal{C} = \{X - U : U \in \mathcal{U}\}$ is a family of closed subsets of X with $\bigcap\limits_{U\in\mathcal{U}} (X - U) = X - \bigcup\limits_{U\in\mathcal{U}} U = X - X = \emptyset$. So, by hypothesis, there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $\bigcap\limits_{i=1}^{n} (X - U_i) = \emptyset$. So, $\emptyset = \bigcap\limits_{i=1}^{n} (X - U_i) = X - \bigcup\limits_{i=1}^{n} U_i$. Hence, $X = \bigcup\limits_{i=1}^{n} U_i$. Therefore, X is compact.

□

Theorem 2.4.3. [92, 101] Let X be a topological space and $A \subseteq X$. Then the following are equivalent:

(i) A is compact.

(ii) Every filter on X which meets A accumulates at some point of A.

(iii) Every ultrafilter on X which meets A τ-converges to some point of A.

(iv) For every family $\mathcal{C}$ of closed sets of X such that $\Big(\bigcap\limits_{C\in\mathcal{C}} C\Big) \cap A = \emptyset$, there exists a finite subfamily $\mathcal{C}'$ of $\mathcal{C}$ such that $\Big(\bigcap\limits_{C\in\mathcal{C}'} C\Big) \cap A = \emptyset$.

Proof. Similar to the proof of Theorem 2.4.2. □

Theorem 2.4.4. [120] A closed subset of a compact space X is compact.

Proof. Let X be compact and $A \subseteq X$ be closed. Let $\mathcal{F}$ be an ultrafilter on X which meets A. Then by Theorem 2.4.2, $\mathcal{F} \longrightarrow x$ for some $x \in X$ since X is compact. Now, since $\mathcal{F}$ is an ultrafilter on X and $\mathcal{F}$ meets A, then $A \in \mathcal{F}$. So, we have $\mathcal{F} \longrightarrow x$ and $A \in \mathcal{F}$ but A is closed, so by Theorem 2.1.4, $x \in A$. Hence, every ultarfilter $\mathcal{F}$ on X which meets A τ-converges to some point of A. Therefore, A is compact relative to X by Theorem 2.4.3. □

Theorem 2.4.5. [120] Let X be a Hausdorff space and $A \subseteq X$. If A is compact, then A is closed.

Proof. Let A be a compact subset of X and X be a Hausdorff space. Let $x \in \overline{A}$. Then by Theorem 2.1.3, there exists a filter on X which meets A such that $\mathcal{F} \longrightarrow x$. But since A is compact, then by Theorem 2.4.3, $\mathcal{F} \propto a$ for some $a \in A$. But also by Theorem 2.1.8, $\mathcal{F}$ has a subfilter $\mathcal{F}'$ such that $\mathcal{F}' \longrightarrow a$. Also, by Theorem 2.1.6, $\mathcal{F}' \longrightarrow x$ since $\mathcal{F}'$ is a subfilter of $\mathcal{F}$ and $\mathcal{F} \longrightarrow x$. Now, X is Hausdorff implies $x = a$ by Theorem 2.2.1. Therefore, $x \in A$. So, $\overline{A} \subseteq A$. Hence, $\overline{A} = A$. Therefore, A is closed. □

Theorem 2.4.6. [120] Let $f : (X, \tau) \to (Y, \sigma)$ be an onto continuous function. If X is compact, then Y is compact.

Proof. Let $f : X \to Y$ be continuous, $A \subseteq X$ be compact and $\mathcal{G}$ be a filter on Y. Since f is onto, then $f^{-1}(\mathcal{G})$ is a filter on X but X is compact, then $f^{-1}(\mathcal{G}) \propto x \in X$. As f is continuous and by Theorem 2.3.2, then $ff^{-1}(\mathcal{G}) \propto f(x) \in Y$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$, so by Theorem 2.1.9, $\mathcal{G} \propto f(x)$. Therefore, Y is compact. □

Corollary 2.4.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous function. If $A \subseteq X$ is compact, then $f(A) \subseteq Y$ is compact.*

Proof. Since $f : (X, \tau) \to (Y, \sigma)$ is continuous, then $f\big|_A : A \to f(A)$ is continuous and onto, so by Theorem 2.4.6, $f(A)$ is compact. □

Theorem 2.4.7. [120] The product $X = \prod\limits_{\alpha \in \Delta} X_\alpha$ is compact if and only if each space X_α, $\alpha \in \Delta$ is compact.

Proof. Assume that X is compact. Since the projection π_α is continuous and onto for all $\alpha \in \Delta$, then by Corollary 2.4.1, $X_\alpha = \pi_\alpha(X)$ is compact for all $\alpha \in \Delta$.

Conversely, let $\mathcal{F}$ be an ultrafilter on X. Since π_α is onto for all $\alpha \in \Delta$, then by Theorem 1.1.6, $\pi_\alpha(\mathcal{F})$ is an ultrafilter on X_α for all $\alpha \in \Delta$. But X_α is compact for all $\alpha \in \Delta$. Thus, by Theorem 2.4.2, $\pi_\alpha(\mathcal{F}) \longrightarrow x_\alpha \in X_\alpha$ for all $\alpha \in \Delta$. Let $x = (x_\alpha)_{\alpha\in\Delta}$, then $x \in X$ and $\pi_\alpha(x) = x_\alpha$ for all $\alpha \in \Delta$. So, $\pi_\alpha(\mathcal{F}) \longrightarrow \pi_\alpha(x)$ for all $\alpha \in \Delta$. Hence, $\mathcal{F} \longrightarrow x$ by Theorem 2.3.3. Therefore, $X = \prod_{\alpha\in\Delta} X_\alpha$ is compact by Theorem 2.4.2. $\square$

2.4.2 Closed Graphs

For two sets X and Y and any function $f : X \to Y$, the subset $\Gamma_f = \{(x, f(x)) : x \in X\}$ of the product $X \times Y$ is called the *graph* of f [63].

Proposition 2.4.1. Let $f : X \to Y$ be a function. If $A \subseteq X$ and $B \subseteq Y$, then $(A \times B) \cap \Gamma_f = \emptyset$ if and only if $f(A) \cap B = \emptyset$.

Proof. Suppose that $(A\times B)\cap\Gamma_f = \emptyset$. Suppose on the contrary that $f(A)\cap B \neq \emptyset$, then there exists $b \in f(A) \cap B$. So, there exists $a \in A$ such that $b = f(a)$. But then, $(a, b) \in (A \times B) \cap \Gamma_f$, which is a contradiction. Thus, $f(A) \cap B = \emptyset$.

Conversely, assume that $f(A) \cap B = \emptyset$. Suppose on the contrary that $(A \times B) \cap \Gamma_f \neq \emptyset$, then there exists $(a, b) \in (A\times B)\cap\Gamma_f$, so $a \in A$, $b \in B$ and $f(a) = b$, hence $b \in f(A) \cap B$ and this implies $f(A) \cap B \neq \emptyset$, which is a contradiction. Therefore, $(A \times B) \cap \Gamma_f = \emptyset$ $\square$

Definition 2.4.3. [63] A function $f : (X, \tau) \to (Y, \sigma)$ is said to have a *closed graph* if Γ_f is a closed subset of $X \times Y$.

Proposition 2.4.2. A function $f : (X, \tau) \to (Y, \sigma)$ has a closed graph if and only if for all $(x, y) \in X \times Y$, with $(x, y) \notin \Gamma_f$, there exist $U \in \tau(x)$ and $V \in \sigma(y)$ such that $(U \times V) \cap \Gamma_f = \emptyset$.

P.E. Long [63] has obtained the following characterization of functions with closed graph.

Theorem 2.4.8. [63] Let $f : (X, \tau) \to (Y, \sigma)$ be a function, then f has a closed graph if and only if for each $x \in X$ and each $y \in Y$, with $(x, y) \notin \Gamma_f$, there exist $U \in \tau(x)$ and $V \in \sigma(y)$ such that $f(U) \cap V = \emptyset$.

Proof. This follows from Propositions 2.4.2 and 2.4.1. □

A useful characterization of functions with closed graphs in terms of filters is given in the following theorem.

Theorem 2.4.9. [119] A function $f : (X, \tau) \to (Y, \sigma)$ has a closed graph if and only if whenever $\mathfrak{F}$ is a filter on X with $\mathfrak{F} \longrightarrow x$ and $f(\mathfrak{F}) \longrightarrow y$ in Y, then $(x, y) \in \Gamma_f$.

Proof. Assume that f has a closed graph. Let $\mathfrak{F} \longrightarrow x$ and $f(\mathfrak{F}) \longrightarrow y$. Suppose on the contrary that $(x, y) \notin \Gamma_f$. Since f has a closed graph, then by Theorem 2.4.8, there exist $U \in \tau(x)$ and $V \in \sigma(y)$ such that $f(U) \cap V = \emptyset$. But since $\mathfrak{F} \longrightarrow x$ and $U \in \tau(x)$, then $U \in \mathfrak{F}$, and hence $f(U) \in f(\mathfrak{F})$. On the other hand, $f(\mathfrak{F}) \longrightarrow y$ and $V \in \sigma(y)$, so $V \in f(\mathfrak{F})$. Thus, $f(U) \cap V \neq \emptyset$, which is a contradiction. Therefore, $(x, y) \in \Gamma_f$.

Conversely, suppose on the contrary that f does not have a closed graph. Then there exists $(x, y) \in X \times Y$ with $(x, y) \notin \Gamma_f$ such that $f(U) \cap V \neq \emptyset$ for all $U \in \tau(x)$ and all $V \in \sigma(y)$. This implies that $U \cap f^{-1}(V) \neq \emptyset$ for all $U \in \tau(x)$ and all $V \in \sigma(y)$. Let $\mathfrak{F} = \{F \subseteq X : F \supseteq U \cap f^{-1}(V), U \in \tau(x), V \in \sigma(y)\}$, then $\mathfrak{F}$ is a filter on X. We claim that $\mathfrak{F} \longrightarrow x$ and $f(\mathfrak{F}) \longrightarrow y$. First, let $U_\circ \in \tau(x)$. Then $U_\circ \supseteq U_\circ \cap f^{-1}(V)$ for each $V \in \sigma(y)$. Hence, $U_\circ \in \mathfrak{F}$. Next, let $V \in \sigma(y)$. Then $V \supseteq f(f^{-1}(V)) \supseteq f(U \cap f^{-1}(V))$ for each $U \in \tau(x)$, but $U \cap f^{-1}(V) \in \mathfrak{F}$ for each $U \in \tau(x)$. So, $V \in f(\mathfrak{F})$. Therefore, we have constructed a filter $\mathfrak{F} \longrightarrow x$ on X for which $f(\mathfrak{F}) \longrightarrow y$ in Y. By hypothesis, $(x, y) \in \Gamma_f$, which is a contradiction. Thus, f must be of closed graph. □

The graph of a continuous function need not be closed as it is shown in the next example.

Example 2.4.2. *Consider the function* $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$*, where* τ *and* σ *are the left ray and right ray topologies on* $\mathbb{R}$*, respectively, given by* $f(x) = -x$*. Then* f *is continuous. But the graph* Γ_f *is not closed since* $(1, 1) \notin \Gamma_f$ *but for any* $a > 1$ *and for any* $b < 1$*, we have* $(x, -x) \in ((-\infty, a) \times (b, \infty)) \cap \Gamma_f$ *where* $x < \min\{a, -b\}$*.*

We are now ready to give a sufficient condition on the codomain of a continuous function f to insure that it has a closed graph.

Theorem 2.4.10. [39] Let $f : (X, \tau) \to (Y, \sigma)$ be continuous and (Y, σ) be Hausdorff. Then f has a closed graph.

Proof. Suppose that $\mathcal{F}$ is a filter on X with $\mathcal{F} \longrightarrow x \in X$ and $f(\mathcal{F}) \longrightarrow y \in Y$. Since f is continuous, then by Theorem 2.3.1, $f(\mathcal{F}) \longrightarrow f(x)$ in Y. But Y is Hausdorff implies $f(x) = y$ by Theorem 2.2.1. So, $(x, y) \in \Gamma_f$. Hence, by Theorem 2.4.9, f has a closed graph. $\square$

Example 2.4.3. *Consider the identity function* $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$*, where* τ *and* σ *are the usual and discrete topologies on* $\mathbb{R}$*, respectively. Then* f *has a closed graph but* f *is not continuous.*

We are now ready to give a sufficient condition on the codomain of a function f has a closed graph to insure that it is continuous.

Theorem 2.4.11. [54] Let (Y, σ) be a compact space. For every topological space (X, τ), each function $f : (X, \tau) \to (Y, \sigma)$ with a closed graph is continuous.

Proof. Assume that (Y, σ) is a compact space. Let (X, τ) be any topological space and $f : (X, \tau) \to (Y, \sigma)$ be a function which has a closed graph. We show that f is continuous. Let $x \in X$ and $V \in \sigma(f(x))$. For each $y \in Y - V$, we have $y \neq f(x)$, this means for each $y \in Y - V$, $(x, y) \notin \Gamma_f$. But Γ_f is closed, then by Theorem 2.4.8, for any $y \in Y - V$, there exist $U_y \in \tau(x)$ and $V_y \in \sigma(y)$ such that $f(U_y) \cap V_y = \emptyset$. Let $\mathcal{V} = \{V\} \cup \{V_y : y \in Y - V\}$, then $\mathcal{V}$ is an open cover for Y. But Y is compact, then $\mathcal{V}$ has a finite subcover, say $\mathcal{V}' = \left\{V, V_{y_1}, \ldots, V_{y_n}\right\}$. So, $Y = V \cup \bigcup\limits_{i=1}^{n} V_{y_i}$. Let $U_x = \bigcap\limits_{i=1}^{n} U_{y_i}$. Then $U_x \in \tau(x)$ and $U_x \subseteq U_{y_i}$ for each

$i = 1, \ldots, n$. So, $f(U_x) \cap (\bigcup_{i=1}^{n} V_{y_i}) = \bigcup_{i=1}^{n} (f(U_x) \cap V_{y_i}) \subseteq \bigcup_{i=1}^{n} \left(f(U_{y_i}) \cap V_{y_i}\right) = \emptyset$. This implies that, $f(U_x) \subseteq Y - \bigcup_{i=1}^{n} V_{y_i} = (V \cup \bigcup_{i=1}^{n} V_{y_i}) - \bigcup_{i=1}^{n} V_{y_i} = V - \bigcup_{i=1}^{n} V_{y_i} \subseteq V$. Thus, f is continuous at the arbitrary point $x \in X$, and so f is continuous on X. $\square$

Theorem 2.4.12. Let $\mathcal{F}$ be a filter on a topological space (Y, σ) and $X = Y \cup \{p\}$ with $p \notin Y$. If $\mathrm{Adh}_\sigma(\mathcal{F}) = \emptyset$, then the prime space (X, τ_p) is Hausdorff.

Proof. Let $x_1 \neq x_2$ in $X = Y \cup \{p\}$. If $x_1, x_2 \in Y$, then $\{x_1\}$ and $\{x_2\}$ are disjoint open sets in X containing x_1 and x_2, respectively. So assume that one of x_1 and x_2 is p, say $x_1 = p$ and $x_2 \in Y$. Since $x_2 \in Y$ and $\mathrm{Adh}_\sigma(\mathcal{F}) = \emptyset$, then $x_2 \notin \mathrm{Adh}_\sigma(\mathcal{F})$, and so by Theorem 2.1.1, $\mathcal{F} \not\propto x_2$, then there exist $U \in \sigma(x_2)$ and $F \in \mathcal{F}$ such that $U \cap F = \emptyset$, so $x_2 \notin F$. Hence, $F \cup \{p\} \in \tau_p(x_1)$ and $\{x_2\} \in \tau_p(x_2)$. Moreover, $(F \cup \{p\}) \cap \{x_2\} = F \cap \{x_2\} = \emptyset$. Therefore, (X, τ_p) is a Hausdorff space. $\square$

Lemma 2.4.1. *Let $X = Z \cup \{p\}$ where Z is a set with $p \notin Z$, (Y, σ) be a topological space and $y \in Y$. Let $g : Z \to (Y, \sigma)$ be a function and $\mathcal{F}$ be a filter on Z. Define a function $\widetilde{g} : (X, \tau_p) \to (Y, \sigma)$ by $\widetilde{g}(z) = g(z)$ for any $z \in Z$ and $\widetilde{g}(p) = y$. Then $g(\mathcal{F}) \longrightarrow y$ in (Y, σ) if and only if $\widetilde{g}$ is continuous on X.*

Proof. Suppose that $g(\mathcal{F}) \longrightarrow y$ in Y and $U \in \sigma$. If $p \notin \widetilde{g}^{-1}(U)$, then $\widetilde{g}^{-1}(U) \subseteq Z$, and so $\widetilde{g}^{-1}(U) \in \tau_p$. If $p \in \widetilde{g}^{-1}(U)$, then $y = \widetilde{g}(p) \in U$. So, $U \in \sigma(y)$ but since $f(\mathcal{F}) \longrightarrow y$, then there exists $F \in \mathcal{F}$ such that $U \supseteq g(F)$. But $g(F) = \widetilde{g}(F)$. So, $\widetilde{g}^{-1}(U) \supseteq \widetilde{g}^{-1}\widetilde{g}(F) \supseteq F$ but $p \notin F$, so $F \subseteq \widetilde{g}^{-1}(U) - \{p\}$. Let $F_\circ = \widetilde{g}^{-1}(U) - \{p\}$, then $F_\circ \in \mathcal{F}$ since $F \subseteq F_\circ \subseteq Z$, $F \in \mathcal{F}$ and $\mathcal{F}$ is a filter on Z. Hence, $\widetilde{g}^{-1}(U) = F_\circ \cup \{p\} \in \tau_p$. Therefore, $\widetilde{g}$ is continuous on X.

Conversely, suppose that $\widetilde{g}$ is continuous on X. Since $\mathcal{F}$ is a filter on Y and $Y \subset X$, then $\mathcal{F}$ is a filter base in X. Let $\mathcal{G} = \langle \mathcal{F} \rangle_X$. Then $\mathcal{G} \longrightarrow p$ in (X, τ_p) and $\widetilde{g}(\mathcal{G}) = g(\mathcal{F})$ but $\widetilde{g}$ is continuous on X, then $\widetilde{g}(\mathcal{G}) \longrightarrow \widetilde{g}(p) = y$ in (Y, σ), so $g(\mathcal{F}) \longrightarrow y$ in (Y, σ). $\square$

Our final result shows that the condition of theorem 2.4.11 characterizes compact spaces if the spaces (X, τ) are chosen from a particular class of topological spaces. In [54], a class of topological spaces containing the class of Hausdorff completely normal and fully normal spaces is denoted by $\mathcal{S}$. Then we use the class $\mathcal{S}$ to obtain the following characterization of compact spaces.

Theorem 2.4.13. [54] A T_1-space (Y,σ) is compact if and only if for every topological space $(X,\tau) \in \mathcal{S}$, each function $f : (X,\tau) \to (Y,\sigma)$ with a closed graph is continuous.

Proof. The first direction follows by Theorem 2.4.11. Conversely, assume that Y is not compact, then there is a filter $\mathcal{F}$ on Y such that $\mathrm{Adh}_\sigma(\mathcal{F}) = \emptyset$. Let $X = Y \cup \{p\}$ where $p \notin Y$. Consider the topological space (X,τ_p). Then by Theorem 2.4.12, (X,τ_p) is Hausdorff. Also, by Theorems 1.4.2 and 1.4.4, (X,τ_p) is completely normal and fully normal. This implies that $(X,\tau_p) \in \mathcal{S}$. Fix a point $b \in Y$ and define $\widetilde{\mathrm{id}}_Y : (X,\tau_p) \to (Y,\sigma)$ by $\widetilde{\mathrm{id}}_Y(x) = \mathrm{id}_Y(x) = x$ for any $x \in Y$ and $\widetilde{\mathrm{id}}_Y(p) = b$. Let $(x,y) \in X \times Y$ and $(x,y) \notin \Gamma_{\widetilde{\mathrm{id}}_Y}$. Consider the case when $x \neq p$. Since $\widetilde{\mathrm{id}}_Y(x) \neq y$ and Y is T_1, then $U_y = Y - \{\widetilde{\mathrm{id}}_Y(x)\} \in \sigma(y)$. Hence, $\{x\} \in \tau_p(x)$, $U_y \in \sigma(y)$ and $\widetilde{\mathrm{id}}_Y(\{x\}) \cap U_y = \{\widetilde{\mathrm{id}}_Y(x)\} \cap (Y - \{\widetilde{\mathrm{id}}_Y(x)\}) = \emptyset$. Consider the case when $x = p$. Then $b = \widetilde{\mathrm{id}}_Y(p) \neq y$. Again, since Y is T_1, then there exists $V_y \in \sigma(y)$ such that $b \notin V_y$. Moreover, since $\mathrm{Adh}_\sigma(\mathcal{F}) = \emptyset$, then by Theorem 2.1.1, we have $\mathcal{F} \not\propto y$, so there exist $W_y \in \sigma(y)$ and $F \in \mathcal{F}$ such that $F \cap W_y = \emptyset$. Let $Z_y = V_y \cap W_y$. Then $Z_y \in \sigma(y)$, $b \notin Z_y$ and $F \cap Z_y = \emptyset$. Hence, $F \cup \{p\} \in \tau_p(x)$, $Z_y \in \sigma(y)$ and $\widetilde{\mathrm{id}}_Y(F \cup \{p\}) \cap Z_y = (\mathrm{id}_Y(F) \cup \{b\}) \cap Z_y = F \cap Z_y = \emptyset$. We have shown, in both cases, that for each $(x,y) \in (X \times Y) - \Gamma_{\widetilde{\mathrm{id}}_Y}$, there exist $U_x \in \tau_p(x)$ and $G_y \in \sigma(y)$ such that $\widetilde{\mathrm{id}}_Y(U_x) \cap G_y = \emptyset$. Thus, by Theorem 2.4.8, $\widetilde{\mathrm{id}}_Y$ has a closed graph. By hypothesis, $\widetilde{\mathrm{id}}_Y$ is continuous, and so by Lemma 2.4.1, $\mathrm{id}_Y(\mathcal{F}) \longrightarrow b$ in (Y,σ) implies $\mathcal{F} \longrightarrow b$ in (Y,σ), thus by Proposition 2.1.1, $\mathcal{F} \propto b$, and hence by Theorem 2.1.1, $\mathrm{Adh}_\sigma(\mathcal{F}) \neq \emptyset$, which is a contradiction. Therefore, (Y,σ) is compact. $\square$

Chapter 3

δ-Convergence of Filters

We study δ-convergence of filters. We will start by introducing the definition of a δ-limit of a filter and define a δ-cluster point of a filter. Some interesting results in semi-regular spaces have been achieved. Various functions: almost-continuous, super-continuous, and δ-continuous are all characterized by filters. As well, the connections between these functions and δ-limits (δ-cluster points) of filters are investigated. Several important notions, such as Hausdorffness, near-compactness, and almost-strongly closed graph can be characterized with the help of filters.

3.1 δ-Limit and δ-Cluster Points of Filters

For a topological space (X, τ), let $\mathcal{U}_{\tau_s}(x)$ be the τ_s-neighborhood filter of x where τ_s is the semi-regularization of τ. We will see immediately that, in semi-regular spaces, δ-convergence of filters is equivalent to convergence of filters and in this case equivalence is also valid for cluster and δ-cluster points.

Definition 3.1.1. [115] Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F}$ *δ-converges* to x, written $\mathcal{F} \xrightarrow{\delta} x$, iff $\mathcal{U}_{\tau_s}(x) \subseteq \mathcal{F}$. In such a case, x is called the *δ-limit* of $\mathcal{F}$.

Notation 7. *For a topological space (X, τ) and $x \in X$, when there is no confusion, we will just write the τ_s-neighborhood filter of x, "$\mathfrak{U}_s(x)$" instead of "$\mathfrak{U}_{\tau_s}(x)$".*

Proposition 3.1.1. Let (X, τ) be a topological space, $\mathfrak{F}$ be a filter on X and $x \in X$. $\mathfrak{F} \xrightarrow{\delta} x$ in (X, τ) if and only if $\mathfrak{F} \longrightarrow x$ in (X, τ_s).

Proof. Since $\mathfrak{U}_s(x)$ is the τ_s-neighborhood filter of x, then clearly, by Definition 3.1.1, $\mathfrak{F} \xrightarrow{\delta} x$ if and only if $\mathfrak{F} \longrightarrow x$ in (X, τ_s). □

Definition 3.1.2. [115] Let (X, τ) be a topological space and $\mathfrak{F}$ be a filter on X and $x \in X$. Then $\mathfrak{F}$ *δ-accumulates* at x, written $\mathfrak{F} \overset{\delta}{\propto} x$, iff $\mathfrak{F}(\cap)\mathfrak{U}_s(x)$ iff for each $F \in \mathfrak{F}$ and for each $G \in \mathfrak{U}_s(x)$, $F \cap G \neq \emptyset$. Equivalently, for each $F \in \mathfrak{F}$ and for each $U \in \tau(x)$, we have $F \cap \overline{U}^{\circ} \neq \emptyset$. In such a case, x is called the *δ-cluster* point of $\mathfrak{F}$.

Proposition 3.1.2. [112] Let $\mathfrak{F}$ be a filter on a topological space (X, τ) and $x \in X$.

(i) If $\mathfrak{F} \xrightarrow{\delta} x$, then $\mathfrak{F} \overset{\delta}{\propto} x$.

(ii) If $\mathfrak{F} \longrightarrow x$, then $\mathfrak{F} \xrightarrow{\delta} x$.

(iii) If $\mathfrak{F} \propto x$, then $\mathfrak{F} \overset{\delta}{\propto} x$.

Proof. (i) Assume that $\mathfrak{F} \xrightarrow{\delta} x$. Let $G \in \mathfrak{U}_s(x)$ and $F \in \mathfrak{F}$. Since $\mathfrak{F} \xrightarrow{\delta} x$ and $G \in \mathfrak{U}_s(x)$, then $G \in \mathfrak{F}$. So, $G \cap F \neq \emptyset$. Therefore, $\mathfrak{F} \overset{\delta}{\propto} x$.

(ii) Assume that $\mathfrak{F} \longrightarrow x$, then $\mathfrak{U}(x) \subseteq \mathfrak{F}$, but $\mathfrak{U}_s(x) \subseteq \mathfrak{U}(x)$ (since $\tau_s \subseteq \tau$), so $\mathfrak{U}_s(x) \subseteq \mathfrak{F}$ and thus $\mathfrak{F} \xrightarrow{\delta} x$.

(iii) Suppose that $\mathfrak{F} \propto x$. Let $G \in \mathfrak{U}_s(x)$ and $F \in \mathfrak{F}$, then $G \in \mathfrak{U}(x)$ since $\mathfrak{U}_s(x) \subseteq \mathfrak{U}(x)$. But $\mathfrak{F} \propto x$, so $G \cap F \neq \emptyset$. Thus, $\mathfrak{F} \overset{\delta}{\propto} x$.

□

The converse of each statement in proposition 3.1.2 need not be true as the following examples show.

Example 3.1.1. *Let $X = \mathbb{R}$ with the usual topology. Let $\mathfrak{F} = \{A \subseteq \mathbb{R} : [0, 1) \subseteq A\}$ be the filter on X generated by $[0, 1)$. Then $\mathfrak{F} \overset{\delta}{\propto} 0$ but $\mathfrak{F} \overset{\delta}{\not\longrightarrow} 0$.*

Example 3.1.2. *Let* $X = \{1,2\}$*,* $\tau_X = \{\emptyset, X, \{1\}\}$ *and* $\mathcal{F} = \{X, \{2\}\}$*. Then* $\mathcal{U}(1) = \{X, \{1\}\}$ *and* $\mathcal{U}_s(1) = \{X\}$*. Clearly,* $\mathcal{F} \xrightarrow{\delta} 1$ *but* $\mathcal{F} \not\to 1$*.*

Example 3.1.3. *In Example 3.1.2,* $\mathcal{F} \overset{\delta}{\propto} 1$ *but* $\mathcal{F} \not\propto 1$*.*

Definition 3.1.3. [115] Let (X,τ) be a topological space, $E \subseteq X$ and $x \in X$. Then x is a *δ-adherent* point of E iff for all $U \in \tau(x)$, $\overline{U}^{\circ} \cap E \neq \emptyset$. Equivalently, for all $G \in \mathrm{RO}(x)$, $G \cap E \neq \emptyset$. The set of all δ-adherent points of a set E is called the *δ-closure* of the set E and is denoted by $\delta\text{-Cl}(E)$.

Proposition 3.1.3. [61] For any subset E of a topological space X, $\overline{E} \subseteq \delta\text{-Cl}(E)$.

Proof. Let $x \in \overline{E}$ and U be open in X containing x. Then $U \cap E \neq \emptyset$. But $U \subseteq \overline{U}^{\circ}$. so $U \cap E \subseteq \overline{U}^{\circ} \cap E$. Thus, $\overline{U}^{\circ} \cap E \neq \emptyset$. Therefore, $x \in \delta\text{-Cl}(E)$. □

Theorem 3.1.1. [115] Let (X,τ) be a topological space and $E \subseteq X$. If E is open in X, then $\overline{E} = \delta\text{-Cl}(E)$.

Proof. $\overline{E} \subseteq \delta\text{-Cl}(E)$ by Proposition 3.1.3. Let $x \in \delta\text{-Cl}(E)$, then $\overline{U}^{\circ} \cap E \neq \emptyset$ for all $U \in \tau(x)$, and so $\overline{U} \cap E \neq \emptyset$ for all $U \in \tau(x)$ but by Corollary 1.2.1, $U \cap E \neq \emptyset$ for all $U \in \tau(x)$. So, $x \in \overline{E}$. Hence, $\delta\text{-Cl}(E) \subseteq \overline{E}$. □

Definition 3.1.4. [115] A subset E of a topological space (X,τ) is called *δ-closed* if $\delta\text{-Cl}(E) = E$. The complement of a δ-closed set is called a *δ-open* set.

Proposition 3.1.4. [115] The family of all δ-open sets in (X,τ) is a new topology on X denoted by τ_δ.

Proposition 3.1.5. Let (X,τ) be a topological space. Then $\tau_\delta = \tau_s$.

Proof. First, we claim that $\delta\text{-Cl}(A) = \mathrm{Cl}_{\tau_s}(A)$ for any $A \subseteq X$. Now, $x \in \delta\text{-Cl}(A)$ if and only if for all $U \in \tau(x)$, $\overline{U}^{\circ} \cap A \neq \emptyset$ if and only if for all $G \in$ RO(x), $G \cap A \neq \emptyset$ if and only if $x \in \mathrm{Cl}_{\tau_s}(A)$. Therefore, for any $A \subseteq X$, $A \in \tau_\delta$ if and only if $\delta\text{-Cl}(X - A) = X - A = \mathrm{Cl}_{\tau_s}(X - A) = X - A$ if and only if $A \in \tau_s$. □

Definition 3.1.5. [115] Let $\mathcal{F}$ be a filter on a topological space (X, τ). A point $x \in X$ is said to be a *δ-adherent* point of $\mathcal{F}$ if x is a δ-adherent point of every set in $\mathcal{F}$. The *δ-adherence* of $\mathcal{F}$, $\delta\text{-Adh}(\mathcal{F})$, is the set of all δ-adherent points of $\mathcal{F}$.

Remark 3.1. *[115] Let X be a topological space. If $\mathcal{F}$ is a filter on X, then* $\delta\text{-Adh}(\mathcal{F}) = \bigcap\limits_{F \in \mathcal{F}} \delta\text{-Cl}(F)$.

Theorem 3.1.2. [115] Let $\mathcal{F}$ be a filter on a topological space (X, τ) and $x \in X$. Then $x \in \delta\text{-Adh}(\mathcal{F})$ if and only if $\mathcal{F} \overset{\delta}{\propto} x$.

Proof.

$$\begin{aligned} x \in \delta\text{-Adh}(\mathcal{F}) &\text{ iff } x \in \bigcap_{F \in \mathcal{F}} \delta\text{-Cl}(F) \\ &\text{ iff } x \in \delta\text{-Cl}(F) \text{ for all } F \in \mathcal{F} \\ &\text{ iff } \overline{U}^{\circ} \cap F \neq \emptyset \text{ for all } U \in \tau(x) \text{ and all } F \in \mathcal{F} \\ &\text{ iff } \mathcal{F}(\cap)\mathcal{U}_s(x) \\ &\text{ iff } \mathcal{F} \overset{\delta}{\propto} x. \end{aligned}$$

□

Theorem 3.1.3. Let (X, τ) be a topological space, $E \subseteq X$ and $x \in X$. Then $x \in \delta\text{-Cl}(E)$ if and only if there exists a filter $\mathcal{F}$ on X such that $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\delta} x$.

Proof. Assume that there exists a filter $\mathcal{F}$ on X such that $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\delta} x$. Let $U \in \tau(x)$, then $\overline{U}^{\circ} \in \mathcal{F}$, so $\overline{U}^{\circ} \cap E \neq \emptyset$. Therefore, $x \in \delta\text{-Cl}(E)$.

Conversely, suppose that $x \in \delta\text{-Cl}(E)$, then $U \cap E \neq \emptyset$ for all $U \in \mathcal{U}_s(x)$. Consider the filter $\mathcal{F} = \langle \mathcal{U}_s(x)\big|_E \rangle$. Then by Proposition 1.1.6, $E \in \mathcal{F}$ and $\mathcal{U}_s(x) \subseteq \mathcal{F}$. Hence, $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\delta} x$. □

Theorem 3.1.4. Let (X, τ) be a topological space, $E \subseteq X$ and $x \in X$. Then $x \in \delta\text{-Cl}(E)$ if and only if there exists a filter $\mathcal{F}$ on X which meets E such that $\mathcal{F} \xrightarrow{\delta} x$.

Proof. Suppose that $x \in \delta\text{-Cl}(E)$. Then by Theorem 3.1.3, there exists a filter $\mathcal{F}$ on X such that $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\delta} x$. Since $E \in \mathcal{F}$, then for all $F \in \mathcal{F}$, $F \cap E \neq \emptyset$. Hence, $\mathcal{F}$ meets E.

Conversely, suppose that $\mathcal{F}$ is a filter on X such that $\mathcal{F} \xrightarrow{\delta} x$ and $F \cap E \neq \emptyset$ for all $F \in \mathcal{F}$. Since $\mathcal{F} \xrightarrow{\delta} x$, then $\overline{U}^{\circ} \in \mathcal{F}$ for all $U \in \tau(x)$. So, by hypothesis, $\overline{U}^{\circ} \cap E \neq \emptyset$ for all $U \in \tau(x)$. Therefore, $x \in \delta\text{-Cl}(E)$. □

Theorem 3.1.5. Let (X, τ) be a topological space and $A \subseteq X$. Then A is δ-closed if and only if whenever a filter $\mathcal{F} \xrightarrow{\delta} x$ with $A \in \mathcal{F}$, then $x \in A$.

Proof. Assume that a filter $\mathcal{F} \xrightarrow{\delta} x$ and $A \in \mathcal{F}$. Then by Theorem 3.1.3, $x \in \delta\text{-Cl}(A)$. But $\delta\text{-Cl}(A) = A$ since A is δ-closed. So, $x \in A$. Conversely, let $x \in \delta\text{-Cl}(A)$. Then by Theorem 3.1.3, there is a filter $\mathcal{F}$ on X such that $\mathcal{F} \xrightarrow{\delta} x$ and $A \in \mathcal{F}$. So by hypothesis, $x \in A$. Thus, $\delta\text{-Cl}(A) \subseteq A$. But $A \subseteq \delta\text{-Cl}(A)$. Therefore, $\delta\text{-Cl}(A) = A$, and hence A is δ-closed. □

Theorem 3.1.6. Let (X, τ) be a topological space and $A \subseteq X$. Then A is δ-open in X if and only if whenever a filter $\mathcal{F} \xrightarrow{\delta} x \in A$, then $A \in \mathcal{F}$.

Proof. Suppose that A is δ-open in X. If a filter $\mathcal{F} \xrightarrow{\delta} x \in A$, then $A \in \mathcal{F}$ since $A \in \mathcal{U}_s(x)$. Conversely, suppose, by the way of contradiction, that A is not δ-open, then $X - A$ is not δ-closed, so there exists $x \in \delta\text{-Cl}(X - A)$ such that $x \notin X - A$. So, $x \in A$. Now, by Theorem 3.1.3, there exists a filter $\mathcal{F}$ on X such that $X - A \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\delta} x$. Since $\mathcal{F} \xrightarrow{\delta} x \in A$, then by hypothesis, $A \in \mathcal{F}$. But then $\emptyset = A \cap (X - A) \in \mathcal{F}$, which is a contradiction. Therefore, A is δ-open. □

Remark 3.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be filters on a topological space (X, τ) and $x \in X$.*

(i) The principal filter $\langle x \rangle \xrightarrow{\delta} x$.

(ii) If $\mathcal{F} \xrightarrow{\delta} x$ and $\mathcal{G} \xrightarrow{\delta} x$, then $\mathcal{F} \cap \mathcal{G} \xrightarrow{\delta} x$.

Theorem 3.1.7. Let X be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \xrightarrow{\delta} x$ if and only if for every subfilter $\mathcal{F}'$ of $\mathcal{F}$, $\mathcal{F}' \xrightarrow{\delta} x$.

Proof. If every subfilter of $\mathcal{F}$ δ-converges to $x \in X$, then so does $\mathcal{F}$ because it is a subfilter of itself. Conversely, suppose that $\mathcal{F} \xrightarrow{\delta} x$ and $\mathcal{F}'$ is a subfilter of $\mathcal{F}$, then $\mathcal{U}_s(x) \subseteq \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{F}'$. So, $\mathcal{U}_s(x) \subseteq \mathcal{F}'$. Therefore, $\mathcal{F}' \xrightarrow{\delta} x$. $\square$

Theorem 3.1.8. Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \xrightarrow{\delta} x$ if and only if every subfilter $\mathcal{G}$ of $\mathcal{F}$ has a subfilter $\mathcal{H}$ such that $\mathcal{H} \xrightarrow{\delta} x$.

Proof. Suppose, by the way of contradiction, that $\mathcal{F} \overset{\delta}{\not\to} x$, then there is a regular open set U in X containing x such that $U \notin \mathcal{F}$. Then $(X - U) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. So, $\mathcal{G} = \langle \mathcal{F}\big|_{X-U} \rangle$ is a subfilter of $\mathcal{F}$ containing $X - U$. By hypothesis, $\mathcal{G}$ has a subfilter $\mathcal{H}$ which δ-converges to x. Since U is a regular open set in X containing x, then $U \in \mathcal{H}$ but $X - U \in \mathcal{G} \subseteq \mathcal{H}$. So, $\emptyset \in \mathcal{H}$, which is a contradiction. The converse follows from Theorem 3.1.7. $\square$

Theorem 3.1.9. [115] Let $\mathcal{F}$ be a filter on a topological space (X, τ) and $x \in X$. Then $\mathcal{F} \overset{\delta}{\propto} x$ if and only if there exists a subfilter $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \xrightarrow{\delta} x$.

Proof. Let $\mathcal{F}$ be a filter on X. Suppose that there exists a subfilter $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \xrightarrow{\delta} x$. Then $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{U}_s(x) \subseteq \mathcal{F}'$. Let $F \in \mathcal{F}$ and $U \in \mathcal{U}_s(x)$, then $F \in \mathcal{F}'$ and $U \in \mathcal{F}'$. So, $F \cap U \neq \emptyset$. Hence, $\mathcal{F} \overset{\delta}{\propto} x$.

Conversely, assume that $\mathcal{F} \overset{\delta}{\propto} x$. We will construct a subfilter $\mathcal{F}'$ of $\mathcal{F}$ that δ-converges to x. Since $\mathcal{F} \overset{\delta}{\propto} x$, then $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and all $G \in \mathcal{U}_s(x)$. Let $\mathcal{F}' = \mathcal{F} \vee \mathcal{U}_s(x)$. Then $\mathcal{F}'$ is a filter on X such that $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{U}_s(x) \subseteq \mathcal{F}'$. Thus, $\mathcal{F}'$ is a subfilter of $\mathcal{F}$ such that $\mathcal{F}' \xrightarrow{\delta} x$. $\square$

Theorem 3.1.10. Let (X, τ) be a topological space, $\mathcal{F}'$ be a subfilter of $\mathcal{F}$ on X and $x \in X$. If $\mathcal{F}' \overset{\delta}{\propto} x$, then $\mathcal{F} \overset{\delta}{\propto} x$.

Proof. Suppose that $\mathcal{F}' \overset{\delta}{\propto} x$. Let $U \in \mathcal{U}_s(x)$ and $F \in \mathcal{F}$ but $\mathcal{F} \subseteq \mathcal{F}'$, then $F \in \mathcal{F}'$. Hence, $U \cap F \neq \emptyset$. Therefore, $\mathcal{F} \overset{\delta}{\propto} x$. $\square$

Theorem 3.1.11. [115] Let $\mathcal{F}$ be an ultrafilter on a topological space X and $x \in X$. Then $\mathcal{F} \xrightarrow{\delta} x$ if and only if $\mathcal{F} \overset{\delta}{\propto} x$.

Proof. If $\mathcal{F} \xrightarrow{\delta} x$, then $\mathcal{F} \overset{\delta}{\propto} x$ by Proposition 3.1.2. Conversely, suppose that $\mathcal{F} \overset{\delta}{\propto} x$. Let $G \in \mathrm{RO}(x)$. Then $G \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. So, $\mathcal{F}$ meets G. But $\mathcal{F}$ is an ultrafilter on X, then by the proof of Theorem 1.1.5, $G \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{\delta} x$. $\square$

3.2 δ-Convergence in Hausdorff Spaces

Theorem 3.2.1. [61] A topological space (X, τ) is Hausdorff if and only if each filter $\mathcal{F}$ on X δ-converges to at most one point in X.

Proof. (X, τ) is Hausdorff if and only if (X, τ_s) is Hausdorff by Proposition 1.3.3 if and only if every filter on X τ_s-converges to at most one point in (X, τ_s) by Theorem 2.2.1 if and only if every filter on X δ-converges to at most one point in (X, τ) by Proposition 3.1.1. $\square$

Theorem 3.2.2. [61] Let (X, τ) be a Hausdorff space, $\mathcal{F}$ be a filter on X and $x \in X$. If $\mathcal{F} \xrightarrow{\delta} x$, then x is the unique δ-cluster point of $\mathcal{F}$.

Proof. If $\mathcal{F} \xrightarrow{\delta} x$, then x is a δ-cluster point of $\mathcal{F}$ by Proposition 3.1.2. Now, suppose that $y \in X$ is a δ-cluster point of $\mathcal{F}$ with $x \neq y$. But since (X, τ) is Hausdorff, then by Proposition 1.3.3, (X, τ_s) is Hausdorff, so there exist $U \in \mathcal{U}_s(x)$ and $V \in \mathcal{U}_s(y)$ such that $U \cap V = \emptyset$. But since $\mathcal{F} \xrightarrow{\delta} x$, then $\mathcal{U}_s(x) \subseteq \mathcal{F}$ and $U \in \mathcal{F}$. But then, $U \cap V \neq \emptyset$ since y is a δ-cluster point of $\mathcal{F}$, which is a contradiction. Therefore, $x = y$. $\square$

3.3 δ-Convergence in Semi-regular Spaces

We will see immediately that, in semi-regular spaces, δ-convergence of filters is equivalent to convergence of filters and in this case equivalence is also valid for cluster and δ-cluster points.

Theorem 3.3.1. Let (X, τ) be a semi-regular space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \xrightarrow{\delta} x$ if and only if $\mathcal{F} \longrightarrow x$.

Proof. If $\mathcal{F} \longrightarrow x$, then $\mathcal{F} \xrightarrow{\delta} x$ by Proposition 3.1.2. Conversely, suppose that $\mathcal{F} \xrightarrow{\delta} x$. Let $U \in \tau(x)$. Since X is semi-regular, then there exists $V \in \tau(x)$ such that $\overline{V}^{\circ} \subseteq U$. Since $V \in \tau(x)$, then $\overline{V}^{\circ} \in \mathrm{RO}(x)$ but $\langle \mathrm{RO}(x) \rangle \subseteq \mathcal{F}$. So, $\overline{V}^{\circ} \in \mathcal{F}$ and thus, $U \in \mathcal{F}$. Therefore, $\mathcal{F} \longrightarrow x$. $\square$

Theorem 3.3.2. [115] Let X be a semi-regular space and $E \subseteq X$. Then $\overline{E} = \delta\text{-Cl}(E)$.

Proof. By Proposition 3.1.3, $\overline{E} \subseteq \delta\text{-Cl}(E)$. Next, let $x \in \delta\text{-Cl}(E)$, then by Theorem 3.1.3, there exists a filter $\mathcal{F}$ on X such that $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\delta} x$. But X is semi-regular, so $\mathcal{F} \longrightarrow x$ by Theorem 3.3.1. Thus, by Theorem 2.1.2, $x \in \overline{E}$. $\square$

Theorem 3.3.3. Let X be a semi-regular space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \overset{\delta}{\propto} x$ if and only if $\mathcal{F} \propto x$.

Proof. If $\mathcal{F} \propto x$, then $\mathcal{F} \overset{\delta}{\propto} x$ by Proposition 3.1.2. Conversely, suppose that $\mathcal{F} \overset{\delta}{\propto} x$, then by Theorem 3.1.9, $\mathcal{F}$ has a subfilter $\mathcal{F}'$ such that $\mathcal{F}' \xrightarrow{\delta} x$. But X is semi-regular, so $\mathcal{F}' \longrightarrow x$ by Theorem 3.3.1. Thus, $\mathcal{F}$ has a subfilter $\mathcal{F}'$ such that $\mathcal{F}' \longrightarrow x$. Hence, $\mathcal{F} \propto x$ by Theorem 2.1.8. $\square$

Theorem 3.3.4. Let (X, τ) be a topological space. Then X is a semi-regular space if and only if for each $x \in X$ and for each filter $\mathcal{F}$ on X, $\mathcal{F} \longrightarrow x$ whenever $\mathcal{F} \xrightarrow{\delta} x$.

Proof. Let $x \in X$. Since $\mathcal{U}_s(x) \xrightarrow{\delta} x$, then by hypothesis, $\mathcal{U}_s(x) \longrightarrow x$. So, $\mathcal{U}(x) \subseteq \mathcal{U}_s(x)$. Now, let $A \in \tau$, then $A \in \mathcal{U}(x)$ for any $x \in A$, and so $A \in \mathcal{U}_s(x)$ for any $x \in A$. Thus, $A \in \tau_s$. Hence, $\tau \subseteq \tau_s$ but $\tau_s \subseteq \tau$, so $\tau = \tau_s$. Therefore, X is semi-regular. The converse follows from Theorem 3.3.1. $\square$

3.4 δ-Convergent Filters and Functions

We will investigate the case of δ-limits of filters under the three types of continuity. We will do the same investigation for δ-cluster points of filters.

3.4.1 Almost-Continuous Functions

We introduce almost-continuous functions in order to study this class of functions, we state several characterizations of almost-continuous functions and the notion of a function that has an almost-strongly closed graph.

Definition 3.4.1. [48] A function $f : (X, \tau) \to (Y, \sigma)$ is *almost-continuous* at $x \in X$ if for every open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq \overline{V}^{\circ}$. Equivalently, for each $G \in \mathrm{RO}_{\sigma}(f(x))$, there exists $U \in \tau(x)$ such that $f(U) \subseteq G$. If this condition is satisfied at each $x \in X$, then f is said to be almost-continuous on X.

Theorem 3.4.1. [106] A function $f : (X, \tau) \to (Y, \sigma)$ is almost-continuous if and only if the inverse image of any regular open set in Y is open in X.

Proof. The proof is a direct consequence of Definition 3.4.1. □

Remark 3.3. *Every continuous function is almost-continuous.*

Theorem 3.4.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is almost-continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \longrightarrow x$, then $f(\mathcal{F}) \xrightarrow{\delta} f(x)$ in Y.

Proof. Assume that $\mathcal{F} \longrightarrow x$ in X and $V \in \sigma(f(x))$. Since f is almost-continuous at x, then there exists $U \in \tau(x)$ such that $f(U) \subseteq \overline{V}^{\circ}$. But since $\mathcal{F} \longrightarrow x$, then $U \in \mathcal{F}$. Thus, $\overline{V}^{\circ} \in f(\mathcal{F})$. So, $f(\mathcal{F}) \xrightarrow{\delta} f(x)$ in Y.

Conversely, let $V \in \sigma(f(x))$. Since $\mathcal{U}_{\tau}(x) \subseteq \mathcal{U}_{\tau}(x)$, then $\mathcal{U}_{\tau}(x) \longrightarrow x$. By hypothesis, we have $f(\mathcal{U}_{\tau}(x)) \xrightarrow{\delta} f(x)$. That is, $\mathcal{U}_{\sigma_s}(f(x)) \subseteq f(\mathcal{U}_{\tau}(x))$. But

$\overline{V}^{\circ} \in \mathfrak{U}_{\sigma_s}(f(x))$, then $\overline{V}^{\circ} \in f(\mathfrak{U}_{\tau}(x))$, so there exists $U \in \tau(x)$ such that $f(U) \subseteq \overline{V}^{\circ}$. Therefore, f is almost-continuous at $x \in X$. $\square$

Theorem 3.4.3. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is almost-continuous at $x \in X$ if and only if whenever $\mathfrak{F}$ is a filter on X with $\mathfrak{F} \propto x$, then $f(\mathfrak{F}) \overset{\delta}{\propto} f(x)$ in Y.

Proof. Suppose that $\mathfrak{F} \propto x \in X$ and $V \in \sigma(f(x))$. Since f is almost-continuous at x, then there exists $U \in \tau(x)$ such that $f(U) \subseteq \overline{V}^{\circ}$. But $U \cap F \neq \emptyset$ for all $F \in \mathfrak{F}$ since $\mathfrak{F} \propto x$. So, $\emptyset \neq f(U \cap F) \subseteq f(U) \cap f(F) \subseteq \overline{V}^{\circ} \cap f(F)$ for all $F \in \mathfrak{F}$. That is, $\overline{V}^{\circ} \cap f(F) \neq \emptyset$ for all $F \in \mathfrak{F}$. Hence, $f(\mathfrak{F}) \overset{\delta}{\propto} f(x)$ in Y.

Conversely, suppose, by the way of contradiction, that f is not almost-continuous at $x \in X$, then there exists $G \in \mathrm{RO}_{\sigma}(f(x))$ such that $f(U) \nsubseteq G$ for any $U \in \tau(x)$. So, $U \nsubseteq f^{-1}(G)$ for any $U \in \tau(x)$. This implies, $V \nsubseteq f^{-1}(G)$ for any $V \in \mathfrak{U}_{\tau}(x)$. Thus, $V \cap F \neq \emptyset$ for any $V \in \mathfrak{U}_{\tau}(x)$ where $F = X - f^{-1}(G)$. By Proposition 1.1.6, $\mathfrak{F} = \langle \mathfrak{U}_{\tau}(x)\big|_{F} \rangle$ is a filter on X such that $F \in \mathfrak{F}$ and $\mathfrak{U}_{\tau}(x) \subseteq \mathfrak{F}$. This implies $\mathfrak{F} \longrightarrow x$ and by Proposition 2.1.1, $\mathfrak{F} \propto x$. We claim that $f(\mathfrak{F}) \overset{\delta}{\not\propto} f(x)$. Since $F \in \mathfrak{F}$, then $f(F) \in f(\mathfrak{F})$. Now, $G \cap f(F) = G \cap f(X - f^{-1}(G)) = G \cap f(f^{-1}(Y - G)) \subseteq G \cap (Y - G) = \emptyset$. Hence, we have $G \in \mathrm{RO}_{\sigma}(f(x))$, $f(F) \in f(\mathfrak{F})$ and $G \cap f(F) = \emptyset$. Therefore, $f(\mathfrak{F}) \overset{\delta}{\not\propto} f(x)$ in Y. $\square$

3.4.2 Super-Continuous Functions

Definition 3.4.2. [96] A function $f : (X, \tau) \to (Y, \sigma)$ is *super-continuous* at $x \in X$ if for every open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\overline{U}^{\circ}) \subseteq V$. Equivalently, for each $V \in \sigma(f(x))$, there exists $H \in \mathrm{RO}_{\tau}(x)$ such that $f(H) \subseteq V$. If this condition is satisfied at each $x \in X$, then f is said to be super-continuous on X.

Theorem 3.4.4. [76] Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is super-continuous at $x \in X$ if and only if whenever $\mathfrak{F}$ is a filter on X with $\mathfrak{F} \overset{\delta}{\longrightarrow} x$, then $f(\mathfrak{F}) \longrightarrow f(x)$ in Y.

Proof. Assume that $\mathcal{F} \xrightarrow{\delta} x$ in X and $V \in \mathcal{U}(f(x))$. Since f is super-continuous at x, then there exists $U \in \tau(x)$ such that $f(\overline{U}^{\circ}) \subseteq V$. But $\mathcal{F} \xrightarrow{\delta} x$, so $\overline{U}^{\circ} \in \mathcal{F}$. Thus, $V \in f(\mathcal{F})$, Therefore, $f(\mathcal{F}) \longrightarrow f(x)$ in Y.

Conversely, let $V \in \sigma(f(x))$. Since $\mathcal{U}_{\tau_s}(x) \subseteq \mathcal{U}_{\tau_s}(x)$, then $\mathcal{U}_{\tau_s}(x) \xrightarrow{\delta} x$. By hypothesis, we have $f(\mathcal{U}_{\tau_s}(x)) \longrightarrow f(x)$. So, $\mathcal{U}_{\sigma}(f(x)) \subseteq f(\mathcal{U}_{\tau_s}(x))$. But $V \in \mathcal{U}_{\sigma}(f(x))$, then $V \in f(\mathcal{U}_{\tau_s}(x))$, so there exists $U \in \tau(x)$ such that $f(\overline{U}^{\circ}) \subseteq V$. Therefore, f is super-continuous at $x \in X$. $\square$

Theorem 3.4.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is super-continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \overset{\delta}{\propto} x$, then $f(\mathcal{F}) \propto f(x)$ in Y.

Proof. Suppose that $\mathcal{F} \overset{\delta}{\propto} x$ in X and $V \in \sigma(f(x))$. Since f is super-continuous at x, then there exists $U \in \tau(x)$ such that $f(\overline{U}^{\circ}) \subseteq V$. But $\overline{U}^{\circ} \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ since $\mathcal{F} \overset{\delta}{\propto} x$. So, $\emptyset \neq f(\overline{U}^{\circ} \cap F) \subseteq f(\overline{U}^{\circ}) \cap f(F) \subseteq V \cap f(F)$ for all $F \in \mathcal{F}$. That is, $V \cap f(F) \neq \emptyset$ for all $F \in \mathcal{F}$. Hence, $f(\mathcal{F}) \propto f(x)$ in Y.

Conversely, suppose, by the way of contradiction, that f is not super-continuous at $x \in X$, then there exists $V \in \sigma(f(x))$ such that $f(U) \nsubseteq V$ for any $U \in \mathrm{RO}_{\tau}(x)$. So, $U \nsubseteq f^{-1}(V)$ for any $U \in \mathrm{RO}_{\tau}(x)$. Let $\mathcal{B} = \{U - f^{-1}(V) : U \in \mathrm{RO}_{\tau}(x)\}$, then $\mathcal{B}$ is a filter base in X. Let $\mathcal{F} = \langle \mathcal{B} \rangle_X$, then $\mathcal{F}$ is a filter on X. We claim that $\mathcal{F} \overset{\delta}{\propto} x$ but $f(\mathcal{F}) \not\propto f(x)$. Let $U \in \mathrm{RO}_{\tau}(x)$ and $F \in \mathcal{F}$, then $F \supseteq B$ for some $B \in \mathcal{B}$. This implies that $F \supseteq W - f^{-1}(V)$ for some $W \in \mathrm{RO}_{\tau}(x)$. Since $U \cap W \in \mathrm{RO}_{\tau}(x)$, then $(U \cap W) - f^{-1}(V) \neq \emptyset$ but $U \cap F \supseteq U \cap (W - f^{-1}(V)) = (U \cap W) - f^{-1}(V) \neq \emptyset$. Hence, $\mathcal{F} \overset{\delta}{\propto} x$. Next, since $X \in \tau(x)$, then $B = X - f^{-1}(V) \in \mathcal{B} \subseteq \mathcal{F}$, so $f(B) \in f(\mathcal{F})$. We claim that $V \cap f(B) = \emptyset$. For if $f(b) \in V$ for some $b \in B$, then $b \in f^{-1}(V)$ and $b \in X - f^{-1}(V)$, so $b \in f^{-1}(V) \cap (X - f^{-1}(V)) = \emptyset$ which is a contradiction. Since $V \in \sigma(f(x))$, $f(B) \in f(\mathcal{F})$ and $V \cap f(B) = \emptyset$, then $f(\mathcal{F}) \not\propto f(x)$ in Y. $\square$

3.4.3 δ-Continuous Functions

Definition 3.4.3. [86] A function $f : (X, \tau) \to (Y, \sigma)$ is *δ-continuous* at $x \in X$ if for every open set V in Y containing $f(x)$, there exists an open set U in X

containing x such that $f(\overline{U}^\circ) \subseteq \overline{V}^\circ$. Equivalently, for each $G \in \mathrm{RO}_\sigma(f(x))$, there exists $H \in \mathrm{RO}_\tau(x)$ such that $f(H) \subseteq G$. If this condition is satisfied at each $x \in X$, then f is said to be δ-continuous on X.

A δ-continuous function preserves δ-convergence.

Theorem 3.4.6. [86] Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is δ-continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \xrightarrow{\delta} x$, then $f(\mathcal{F}) \xrightarrow{\delta} f(x)$ in Y.

Proof. Assume that $\mathcal{F} \xrightarrow{\delta} x$ in X and $V \in \sigma(f(x))$. Since f is δ-continuous at x, then there exists $U \in \tau(x)$ such that $f(\overline{U}^\circ) \subseteq \overline{V}^\circ$. But since $\mathcal{F} \xrightarrow{\delta} x$, then $\overline{U}^\circ \in \mathcal{F}$. So, $\overline{V}^\circ \in f(\mathcal{F})$. Thus, $f(\mathcal{F}) \xrightarrow{\delta} f(x)$ in Y.

Conversely, let $V \in \sigma(f(x))$. Since $\mathcal{U}_{\tau_s}(x) \subseteq \mathcal{U}_{\tau_s}(x)$, then $\mathcal{U}_{\tau_s}(x) \xrightarrow{\delta} x$. By hypothesis, we have $f\left(\mathcal{U}_{\tau_s}(x)\right) \xrightarrow{\delta} f(x)$. That is, $\mathcal{U}_{\sigma_s}\left(f(x)\right) \subseteq f\left(\mathcal{U}_{\tau_s}(x)\right)$. But $\overline{V}^\circ \in \mathcal{U}_{\sigma_s}\left(f(x)\right)$, then $\overline{V}^\circ \in f(\mathcal{U}_s\left(x\right))$, so there exists $U \in \tau(x)$ such that $f(\overline{U}^\circ) \subseteq \overline{V}^\circ$. Therefore, f is δ-continuous at $x \in X$. $\square$

Theorem 3.4.7. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is δ-continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \overset{\delta}{\propto} x$, then $f(\mathcal{F}) \overset{\delta}{\propto} f(x)$ in Y.

Proof. Suppose that $\mathcal{F} \overset{\delta}{\propto} x$ in X and $V \in \sigma(f(x))$. Since f is δ-continuous at x, there exists $U \in \tau(x)$ such that $f(\overline{U}^\circ) \subseteq \overline{V}^\circ$. But $\overline{U}^\circ \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ since $\mathcal{F} \overset{\delta}{\propto} x$. So, $\emptyset \neq f(\overline{U}^\circ \cap F) \subseteq f(\overline{U}^\circ) \cap f(F) \subseteq \overline{V}^\circ \cap f(F)$ for all $F \in \mathcal{F}$. Hence, $\overline{V}^\circ \cap f(F) \neq \emptyset$ for all $F \in \mathcal{F}$. Therefore, $f(\mathcal{F}) \overset{\delta}{\propto} f(x)$ in Y.

Conversely, suppose, by the way of contradiction, that f is not δ-continuous at $x \in X$, then there exists $V \in \mathrm{RO}_\sigma(f(x))$ such that $f(U) \nsubseteq V$ for any $U \in \mathrm{RO}_\tau(x)$. So, $U \nsubseteq f^{-1}(V)$ for any $U \in \mathrm{RO}_\tau(x)$. Let $\mathcal{B} = \{U - f^{-1}(V) : U \in \mathrm{RO}_\tau(x)\}$, then $\mathcal{B}$ is a filter base in X. Let $\mathcal{F} = \langle \mathcal{B} \rangle_X$, then $\mathcal{F}$ is a filter on X. We claim that $\mathcal{F} \overset{\delta}{\propto} x$ but $f(\mathcal{F}) \overset{\delta}{\not\propto} f(x)$. Let $U \in \mathrm{RO}_\tau(x)$ and $F \in \mathcal{F}$, then $F \supseteq B$ for some $B \in \mathcal{B}$. This implies that $F \supseteq W - f^{-1}(V)$ for some $W \in \mathrm{RO}_\tau(x)$. Since $U \cap W \in \mathrm{RO}_\tau(x)$, then $(U \cap W) - f^{-1}(V) \neq \emptyset$ but $U \cap F \supseteq U \cap (W - f^{-1}(V)) = (U \cap W) - f^{-1}(V) \neq \emptyset$. Hence, $\mathcal{F} \overset{\delta}{\propto} x$. Next, since $X \in \mathrm{RO}_\tau(x)$, then $B = X - f^{-1}(V) \in \mathcal{B} \subseteq \mathcal{F}$, so

$f(B) \in f(\mathcal{F})$. We claim that $V \cap f(B) = \emptyset$. For if $f(b) \in V$ for some $b \in B$, then $b \in f^{-1}(V)$ and $b \in X - f^{-1}(V)$, so $b \in f^{-1}(V) \cap (X - f^{-1}(V)) = \emptyset$, which is a contradiction. Since $V \in \mathrm{RO}_\sigma(f(x))$, $f(B) \in f(\mathcal{F})$ and $V \cap f(B) = \emptyset$, then $f(\mathcal{F}) \overset{\delta}{\not\to} f(x)$ in Y. □

The following two examples show that the concepts of δ-continuity and continuity are independent of each other.

Example 3.4.1. *[86] Consider the identity function* $\mathrm{id}_\mathbb{R} : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$, *where* τ *and* σ *are the usual and co-countable topologies on* $\mathbb{R}$, *respectively. Then* $\mathrm{id}_\mathbb{R}$ *is* δ*-continuous but not continuous.*

Example 3.4.2. *[86] Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ *and* $\sigma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, Y\}$. *Consider the identity function* $\mathrm{id}_X : (X, \tau) \to (Y, \sigma)$. *Then* id_X *is continuous but not* δ*-continuous since* $\{a\} \in \mathrm{RO}(Y, \sigma)$ *but* $f^{-1}(\{a\}) = \{a\} \notin \mathrm{RO}(X, \tau)$.

In the following proposition, we give the relation between the different types of continuity which have already been used.

Proposition 3.4.1. [8, 86, 106] Super-continuity $\Longrightarrow$ δ-continuity $\Longrightarrow$ almost-continuity.

3.4.4 More on Functions and δ-Convergence

Definition 3.4.4. [120] A function $f : (X, \tau) \to (Y, \sigma)$ is *open* if for each $U \in \tau$, $f(U) \in \sigma$.

Definition 3.4.5. [106] A function $f : (X, \tau) \to (Y, \sigma)$ is *almost-open* if for each $G \in \mathrm{RO}(X)$, $f(G) \in \sigma$.

Theorem 3.4.8. [64] A function $f : (X, \tau) \to (Y, \sigma)$ is open if and only if $f(A^\circ) \subseteq (f(A))^\circ$ for any $A \subseteq X$.

Proposition 3.4.2. Every open continuous function is δ-continuous.

Proof. Let V be an open set in Y containing $f(x)$. Then by continuity of f, there exists an open set U in X containing x such that $f(U) \subseteq V$. Also, by continuity of f, we have $f(\overline{U}) \subseteq \overline{f(U)} \subseteq \overline{V}$. So $(f(\overline{U}))^{\circ} \subseteq \overline{V}^{\circ}$. But f is open, so $f(\overline{U}^{\circ}) \subseteq (f(\overline{U}))^{\circ} \subseteq \overline{V}^{\circ}$. Therefore, f is δ-continuous. $\square$

Theorem 3.4.9. [86] Let $f : X \to Y$ be a function and X be semi-regular. Then f is almost-continuous if and only if f is δ-continuous.

Proof. Let $x \in X$ and $\mathfrak{F}$ be a filter on X such that $\mathfrak{F} \xrightarrow{\delta} x$, then $\mathfrak{F} \longrightarrow x$ by semi-regularity of X. Since f is almost-continuous, then $f(\mathfrak{F}) \xrightarrow{\delta} f(x)$ by Theorem 3.4.2. Therefore, f is δ-continuous at x by Theorem 3.4.6. Thus, f is δ-continuous on X since x was arbitrary. The converse from Proposition 3.4.1 part(i). $\square$

Theorem 3.4.10. [106] Let $f : X \to Y$ be a function and Y be semi-regular. Then f is almost-continuous if and only if f is continuous.

Proof. Let $x \in X$ and $\mathfrak{F}$ be a filter on X such that $\mathfrak{F} \longrightarrow x$. Since f is almost-continuous, then $f(\mathfrak{F}) \xrightarrow{\delta} f(x)$ by Theorem 3.4.2. But then $f(\mathfrak{F}) \longrightarrow f(x)$ by semi-regularity of Y. Therefore, f is continuous at x. Thus, f is continuous on X since x was arbitrary. The converse follows from Proposition 3.4.1 part(ii). $\square$

Corollary 3.4.1. *[86] Let $f : X \to Y$ be a function and Y be semi-regular. If f is δ-continuous, then f is continuous.*

Proof. This follows from Theorem 3.4.10 and the fact that every δ-continuous is almost-continuous. $\square$

Theorem 3.4.11. [96] Let $f : X \to Y$ be a function and X be semi-regular. If f is super-continuous, then f is continuous.

Proof. Let $x \in X$ and $\mathfrak{F}$ be a filter on X such that $\mathfrak{F} \longrightarrow x$, then $\mathfrak{F} \xrightarrow{\delta} x$ by semi-regularity of X. Since f is super-continuous, then $f(\mathfrak{F}) \longrightarrow f(x)$ by Theorem

3.4.4. Therefore, f is continuous at x. Thus, f is continuous on X since x was arbitrary. $\square$

Corollary 3.4.2. *[86] If (X, τ) and (Y, σ) are semi-regular spaces, then the following concepts on a function $f : (X, \tau) \to (Y, \sigma)$: δ-continuity, continuity and almost-continuity are equivalent.*

Lemma 3.4.1. *[64] Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of topological spaces and consider the product space $\prod\limits_{\alpha\in\Delta} X_\alpha$. Let $A_\alpha \subseteq X_\alpha$ for each $\alpha \in \Delta$. Then*

(i) $\overline{\prod\limits_{\alpha\in\Delta} A_\alpha} = \prod\limits_{\alpha\in\Delta} \overline{A}_\alpha$.

(ii) If Δ is finite, then $\left(\prod\limits_{\alpha\in\Delta} A_\alpha\right)^\circ = \prod\limits_{\alpha\in\Delta} A_\alpha^\circ$.

Theorem 3.4.12. [51] Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of topological spaces and let $\mathcal{F}$ be a filter on $X = \prod\limits_{\alpha\in\Delta} X_\alpha$. Then $\mathcal{F} \xrightarrow{\delta} x$ in X if and only if $\pi_\alpha(\mathcal{F}) \xrightarrow{\delta} \pi_\alpha(x)$ in X_α for all $\alpha \in \Delta$.

Proof. Assume that $\mathcal{F} \xrightarrow{\delta} x$ in X. Since π_α is open continuous for all $\alpha \in \Delta$, then by Proposition 3.4.1, π_α is δ-continuous for all $\alpha \in \Delta$. So, by Theorem 3.4.6, $\pi_\alpha(\mathcal{F}) \xrightarrow{\delta} \pi_\alpha(x)$ for all $\alpha \in \Delta$.

Conversely, suppose that $\pi_\alpha(\mathcal{F}) \xrightarrow{\delta} \pi_\alpha(x)$ for all $\alpha \in \Delta$. Let U be any neighborhood of x in X. Then $x \in \bigcap\limits_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i) \subseteq U$, where $U_i \in \mathcal{U}(\pi_{\alpha_i}(x))$ for all $i = 1, \ldots, n$. So, $\overline{U}_i^\circ \in \pi_{\alpha_i}(\mathcal{F})$ for all $i = 1, \ldots, n$, and hence for all $i = 1, \ldots, n$, there exists $F_i \in \mathcal{F}$ such that $\pi_{\alpha_i}(F_i) \subseteq \overline{U}_i^\circ$. Then, $F_i \subseteq \pi_{\alpha_i}^{-1}\left(\overline{U}_i^\circ\right)$ for all $i = 1, \ldots, n$. So, $\bigcap\limits_{i=1}^{n} F_i \subseteq \bigcap\limits_{i=1}^{n} \pi_{\alpha_i}^{-1}\left(\overline{U}_i^\circ\right)$. Since $\bigcap\limits_{i=1}^{n} F_i \in \mathcal{F}$, then $\bigcap\limits_{i=1}^{n} \pi_{\alpha_i}^{-1}\left(\overline{U}_i^\circ\right) \in \mathcal{F}$. But by Lemmas 2.3.1 and 3.4.1, $\bigcap\limits_{i=1}^{n} \pi_{\alpha_i}^{-1}\left(\overline{U}_i^\circ\right) = \prod\limits_{i=1}^{n} \overline{U}_i^\circ = \overline{\prod\limits_{i=1}^{n} U_i}^\circ = \overline{\bigcap\limits_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i)}^\circ \subseteq \overline{U}^\circ$. So, $\overline{U}^\circ \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{\delta} x$ in X. $\square$

Theorem 3.4.13. Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of topological spaces and let $\mathcal{F}$ be a filter on $X = \prod\limits_{\alpha\in\Delta} X_\alpha$. If $\mathcal{F} \overset{\delta}{\propto} x$ in X, then $\pi_\alpha(\mathcal{F}) \overset{\delta}{\propto} \pi_\alpha(x)$ in X_α for all $\alpha \in \Delta$.

Proof. Assume that $\mathcal{F} \overset{\delta}{\propto} x$. Since π_α is δ-continuous for all $\alpha \in \Delta$, then by Theorem 3.4.3, $\pi_\alpha(\mathcal{F}) \overset{\delta}{\propto} \pi_\alpha(x)$ for all $\alpha \in \Delta$. $\square$

3.5 Nearly Compact Spaces

In [103], M. K. Singal and Asha Mathur introduced a new class of topological spaces called the nearly compact spaces. This class of spaces is properly contained between the compact spaces and the quasi-H-closed spaces. Several characterizations and properties of these spaces were obtained.

3.5.1 Characterizations of Nearly Compact Spaces

Definition 3.5.1. [32] A topological space (X, τ) is called *nearly compact* iff each open cover of X has a finite subfamily whose interiors of the closures of its members cover the space X.

Definition 3.5.2. [32] A subset A of a topological space X is said to be

(i) a *nearly compact subspace* if the space (A, τ_A) is nearly compact.

(ii) an *N-closed relative* to X if for every cover $\{V_\alpha : \alpha \in \Delta\}$ of A by open sets in X, there exists a finite subset Ω of Δ such that $A \subseteq \bigcup\limits_{\alpha\in\Omega} \overline{V}^\circ_\alpha$.

Recall that, let X be a topological space and $A \subseteq Y \subseteq X$. If Y is open in X, then $\mathrm{Int}_Y(A) = A^\circ \cap Y$ [92].

Lemma 3.5.1. *Let (X, τ) be a topological space and $A \subseteq X$. If A is open in X, then $\mathrm{Int}_A\mathrm{Cl}_A(U \cap A) = \overline{U}^\circ \cap A$ for any $U \subseteq X$.*

Proof. Let A be open in X and $U \subseteq X$, then $A \subseteq \overline{A}^\circ$ and by Theorem 1.2.5, $\overline{U \cap A}^\circ = \overline{U}^\circ \cap \overline{A}^\circ$. Now,

$$\begin{aligned}
\mathrm{Int}_A\mathrm{Cl}_A(U \cap A) &= \mathrm{Int}_A(\overline{U \cap A} \cap A) \\
&= (\overline{U \cap A} \cap A)^\circ \cap A \\
&= \overline{U \cap A}^\circ \cap A^\circ \cap A \\
&= \overline{U}^\circ \cap \overline{A}^\circ \cap A \\
&= \overline{U}^\circ \cap A.
\end{aligned}$$

□

Theorem 3.5.1. [32] Let (X,τ) be a topological space and $A \subseteq X$ be open. Then A is nearly compact if and only if A is an N-closed relative to X.

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A by open sets in X, then $\{U_\alpha \cap A : \alpha \in \Delta\}$ is a cover of A by open sets in A. But A is nearly compact, then there exist $\alpha_1, \ldots, \alpha_n \in \Delta$ such that $A = \bigcup\limits_{i=1}^{n} \mathrm{Int}_A\mathrm{Cl}_A(U_{\alpha_i} \cap A)$. By Lemma 3.5.1 and since A is open in X, then $\mathrm{Int}_A\mathrm{Cl}_A(U_{\alpha_i} \cap A) = \overline{U}^{\circ}_{\alpha_i} \cap A$ for all $i = 1, \ldots, n$, so $A = \bigcup\limits_{i=1}^{n} (\overline{U}^{\circ}_{\alpha_i} \cap A)$. Hence, $A \subseteq \bigcup\limits_{i=1}^{n} \overline{U}^{\circ}_{\alpha_i}$. Therefore, A is N-closed relative to X.

Conversely, let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of A by open sets in A. Since V_α is open in A for every $\alpha \in \Delta$ and A is open in X, then V_α is open in X for every $\alpha \in \Delta$. So, $\{V_\alpha : \alpha \in \Delta\}$ is a cover of A by open sets in X. But A is N-closed relative to X, then there exist $\alpha_1, \ldots, \alpha_n \in \Delta$ such that $A \subseteq \bigcup\limits_{i=1}^{n} \overline{V}^{\circ}_{\alpha_i}$. So, $A = \bigcup\limits_{i=1}^{n} (\overline{V}^{\circ}_{\alpha_i} \cap A)$ but by Lemma 3.5.1 and since A is open in X, then $\overline{V}^{\circ}_{\alpha_i} \cap A = \mathrm{Int}_A\mathrm{Cl}_A(V_{\alpha_i} \cap A) = \mathrm{Int}_A\mathrm{Cl}_A V_{\alpha_i}$ for any $i = 1, \ldots, n$ since $V_{\alpha_i} \subseteq A$ for any $i = 1, \ldots, n$. Hence, $A = \bigcup\limits_{i=1}^{n} \mathrm{Int}_A\mathrm{Cl}_A V_{\alpha_i}$. Therefore, A is nearly compact. $\square$

Theorem 3.5.2. [104] For a topological space (X,τ), the following are equivalent:

(i) X is nearly compact.

(ii) Every regular open cover of X has a finite subcover.

(iii) Every δ-open cover of X has a finite subcover.

Proof.

(i) $\Longrightarrow$ (ii) Let $\mathcal{U}$ be a regular open cover of X. Then it is an open cover of X, but X is nearly compact, so there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $\bigcup\limits_{i=1}^{n} \overline{U_i}^{\circ} = X$ but for each $i = 1, \ldots, n$, U_i is regular open, hence for each $i = 1, \ldots, n$, $\overline{U_i}^{\circ} = U_i$. Thus, $\bigcup\limits_{i=1}^{n} U_i = X$. Therefore, $\mathcal{U}' = \{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$.

(ii) $\Longrightarrow$ (iii) Let $\mathcal{A}$ be a δ-open cover of X, then by Proposition 3.1.5, for each $A \in \mathcal{A}$, $A \in \tau_s$. Now, for each $x \in X = \bigcup\limits_{A \in \mathcal{A}} A$, there exists $A_x \in \mathcal{A}$ such that $x \in A_x$. But for each $x \in X$, $A_x \in \tau_s(x)$. Hence, for each $x \in X$, there exists $U_x \in \mathrm{RO}(X)$ such that $x \in U_x \subseteq A_x$. Thus, $\mathcal{U} = \{U_x : x \in X\}$ is a regular open cover of X. By hypothesis, $\mathcal{U}$ has a finite subcover, that is, there exist $x_1, \dots x_n \in X$ such that $\bigcup\limits_{i=1}^{n} U_{x_i} = X$. But $U_{x_i} \subseteq A_{x_i}$ for each $i = 1, \dots, n$. Thus, $X = \bigcup\limits_{i=1}^{n} U_{x_i} \subseteq \bigcup\limits_{i=1}^{n} A_{x_i} \subseteq X$. Hence, $X = \bigcup\limits_{i=1}^{n} A_{x_i}$. Therefore, $\mathcal{A}' = \{A_{x_1}, \dots, A_{x_n}\}$ is a finite subcover of $\mathcal{A}$.

(iii) $\implies$ (i) Let $\mathfrak{U}$ be an open cover of X. Then $\mathcal{A} = \{\overline{U}^{\circ} : U \in \mathfrak{U}\} \subseteq \mathrm{RO}(X) \subseteq \tau_\delta$. Since U is open for all $U \in \mathfrak{U}$, then $U \subseteq \overline{U}^{\circ}$ for all $U \in \mathfrak{U}$. Thus, $X = \bigcup\limits_{U \in \mathfrak{U}} U \subseteq \bigcup\limits_{U \in \mathfrak{U}} \overline{U}^{\circ} \subseteq X$. So, $X = \bigcup\limits_{U \in \mathfrak{U}} \overline{U}^{\circ}$. Hence, $\mathcal{A}$ is a δ-open cover of X. By hypothesis, there exist $U_1, \dots, U_n \in \mathfrak{U}$ such that $\bigcup\limits_{i=1}^{n} \overline{U_i}^{\circ} = X$. Therefore, X is nearly compact.

□

Proposition 3.5.1. Every compact space is nearly compact.

Proof. Assume that X is compact. Let $\mathfrak{U}$ be a regular open cover of X, then it is an open cover of X but X is compact, so $\mathfrak{U}$ has a finite subcover. Hence, by Theorem 3.5.2, X is nearly compact. □

The converse of the above proposition need not be true, as the following example shows.

Example 3.5.1. *Consider the topological space* $(\mathbb{R}, \tau)$ *where* τ *is the left ray topology. Then* $\mathrm{RO}(\mathbb{R}, \tau) = \{\emptyset, \mathbb{R}\}$. *So, every regular open cover of* $(\mathbb{R}, \tau)$ *has a finite subcover. Thus, by Theorem 3.5.2,* $(\mathbb{R}, \tau)$ *is nearly compact. Yet,* $(\mathbb{R}, \tau)$ *is not compact.*

We are now ready to make characterizations of nearly compact spaces using the δ-convergence of filters.

Theorem 3.5.3. [70, 112] For a topological space (X, τ), the following are equivalent:

(i) X is nearly compact.

(ii) Every filter on X has a δ-cluster point.

(iii) Every ultrafilter on X δ-converges.

(iv) For every family $\mathcal{C}$ of regular closed sets of X such that $\bigcap\limits_{C\in\mathcal{C}} C = \emptyset$, there exists a finite subfamily $\mathcal{C}'$ of $\mathcal{C}$ such that $\bigcap\limits_{C\in\mathcal{C}'} C = \emptyset$.

Proof.

(i) $\Longrightarrow$ (ii) Assume that there exists a filter $\mathcal{F}$ on X such that $\mathcal{F} \overset{\delta}{\not\propto} x$ for all $x \in X$. This means that for all $x \in X$, there exists $U_x \in \mathrm{RO}(x)$ and $F_x \in \mathcal{F}$ such that $F_x \cap U_x = \emptyset$. Consider $\mathcal{U} = \{U_x : x \in X\}$. Then $\mathcal{U}$ is a regular open cover of X. But X is nearly compact, then by Theorem 3.5.2, there exist $x_1, \dots, x_n \in X$ such that $X = \bigcup\limits_{i=1}^{n} U_{x_i}$. Now, for all $i = 1, \dots, n$, choose F_{x_i} such that $F_{x_i} \cap U_{x_i} = \emptyset$ and let $F_\circ = \bigcap\limits_{i=1}^{n} F_{x_i}$. Then, $F_\circ \in \mathcal{F}$ and

$$F_\circ = F_\circ \cap X = F_\circ \cap (\bigcup_{i=1}^{n} U_{x_i}) = \bigcup_{i=1}^{n} (F_\circ \cap U_{x_i}) \subseteq \bigcup_{i=1}^{n} (F_{x_i} \cap U_{x_i}) = \emptyset.$$

This implies $F_\circ = \emptyset$, which is a contradiction. Therefore, $\mathcal{F}$ must have a δ-cluster point in X.

(ii) $\Longrightarrow$ (iii) Let $\mathcal{F}$ be an ultrafilter on X. Then by hypothesis, $\mathcal{F}$ has a δ-cluster point $x \in X$. But then, $\mathcal{F}$ δ-converges to x by Theorem 3.1.11 and since $\mathcal{F}$ is an ultrafilter on X.

(iii) $\Longrightarrow$ (iv) Let $\mathcal{C} = \{F_\alpha : \alpha \in \Delta\}$ be a family of regular closed subsets of X such that $\bigcap\limits_{\alpha\in\Delta} F_\alpha = \emptyset$. Suppose, by the way of contradiction, that for each finite subset Ω of Δ, $\bigcap\limits_{\alpha\in\Omega} F_\alpha \neq \emptyset$. Let $\mathcal{B} = \{\bigcap\limits_{\alpha\in\Omega} F_\alpha : \Omega$ is a finite subset of $\Delta\}$. Then $\mathcal{B}$ is a filter base in X. Hence, by Theorem 1.1.4, the filter $\mathcal{F} = \langle\mathcal{B}\rangle_X$ is contained in some ultrafilter $\mathcal{F}'$ on X. So, by hypothesis, $\mathcal{F}' \xrightarrow{\delta} x \in X$. But then $\mathcal{F}' \overset{\delta}{\propto} x$. Hence, $\mathcal{F} \overset{\delta}{\propto} x$ by Theorem 3.1.10. Therefore, we have constructed a filter $\mathcal{F}$ on X which has $x \in X$ as a δ-cluster point. Now, since $x \notin \emptyset = \bigcap\limits_{\alpha\in\Delta} F_\alpha$. So, $x \notin F_{\alpha_\circ}$ for some $\alpha_\circ \in \Delta$. But $X - F_{\alpha_\circ} \in \mathrm{RO}(x)$ and $F_{\alpha_\circ} \in \mathcal{F}$. Since $\mathcal{F} \overset{\delta}{\propto} x$, then $(X - F_{\alpha_\circ}) \cap F_{\alpha_\circ} \neq \emptyset$, which is a contradiction. Therefore, there exists a finite subset Ω of Δ such that $\bigcap\limits_{\alpha\in\Omega} F_\alpha \neq \emptyset$.

(iv) $\Longrightarrow$ (i) Let $\mathcal{U}$ be a regular open cover of X. Then $\mathcal{C} = \{X - U : U \in \mathcal{U}\}$ is a family of regular closed subsets of X with $\bigcap\limits_{U\in\mathcal{U}} (X - U) = X - \bigcup\limits_{U\in\mathcal{U}} U = X - X = \emptyset$. So, by hypothesis, there exist $U_1, \dots, U_n \in \mathcal{U}$ such that $\bigcap\limits_{i=1}^{n} (X - U_i) = \emptyset$. So,

$\emptyset = \bigcap_{i=1}^{n} (X - U_i) \; = X - \bigcup_{i=1}^{n} U_i$. Hence, $X = \bigcup_{i=1}^{n} U_i$. Thus, every regular open cover of X has a finite subcover. Therefore, by Theorem 3.5.2, X is nearly compact.

□

Theorem 3.5.4. Let X be a topological space and $A \subseteq X$. Then the following are equivalent:

(i) A is N-closed relative to X.

(ii) Every filter on X which meets A δ-accumulates at some point of A.

(iii) Every ultrafilter on X which meets A δ-converges to some point of A.

(iv) For every family $\mathcal{C}$ of regular closed sets of X such that $\left(\bigcap_{C \in \mathcal{C}} C\right) \cap A = \emptyset$, there exists a finite subfamily $\mathcal{C}'$ of $\mathcal{C}$ such that $\left(\bigcap_{C \in \mathcal{C}'} C\right) \cap A = \emptyset$.

Proof. Similar to the proof of Theorem 3.5.3. □

Theorem 3.5.5. [73] Let (X, τ) be a topological space. Then (X, τ) is nearly compact if and only if (X, τ_s) is compact.

Proof. (X, τ) is nearly compact if and only if every ultrafilter δ-converges if and only if every ultrafilter τ_s-converges if and only if (X, τ_s) is compact. □

Corollary 3.5.1. *[103] Let (X, τ) be a semi-regular space. Then (X, τ) is nearly compact if and only if (X, τ) is compact.*

Proof. If (X, τ) is semi-regular, then $\tau_s = \tau$. Thus, by Theorem 3.5.5, (X, τ) is nearly compact if and only if (X, τ_s) is compact if and only if (X, τ) is compact. □

Theorem 3.5.6. The property of a topological space being nearly compact is a semi-regular property. That is, (X, τ) is nearly compact if and only if (X, τ_s) is nearly compact.

Proof.

$$\begin{aligned}(X,\tau_s)\ \text{ is nearly compact iff } & (X,(\tau_s)_s) \text{ is compact by Theorem 3.5.5} \\ \text{iff } & (X,\tau_s) \text{ is compact by Proposition 1.3.2} \\ \text{iff } & (X,\tau) \text{ is nearly compact by Theorem 3.5.5.}\end{aligned}$$

□

Theorem 3.5.7. A δ-closed subset of a nearly compact space X is N-closed relative to X.

Proof. Let X be nearly compact and $A \subseteq X$ be δ-closed. Let $\mathfrak{F}$ be an ultrafilter on X which meets A. Then by Theorem 3.5.3, $\mathfrak{F} \xrightarrow{\delta} x$ for some $x \in X$. Now, since $\mathfrak{F}$ is an ultrafilter on X and $\mathfrak{F}$ meets A, then $A \in \mathfrak{F}$. So, we have $\mathfrak{F} \xrightarrow{\delta} x$ and $A \in \mathfrak{F}$ but A is δ-closed, so by Theorem 3.1.5, $x \in A$. Hence, every ultarfilter $\mathfrak{F}$ on X which meets A δ-converges to some point of A. Therefore, A is N-closed relative to X by Theorem 3.5.4. □

Corollary 3.5.2. *[70] Every regular closed subset of a nearly compact space is N-closed relative to X.*

Proof. This follows from Theorem 3.5.7 and the fact that every regular closed set is δ-closed. □

Theorem 3.5.8. Let X be a Hausdorff space and $A \subseteq X$. If A is an N-closed relative to X, then A is δ-closed.

Proof. Let A be an N-closed relative to X and X be a Hausdorff space. Let $x \in \delta\text{-Cl}(A)$. Then by Theorem 3.1.4, there exists a filter on X which meets A such that $\mathfrak{F} \xrightarrow{\delta} x$. But since A is an N-closed relative to X, then by Theorem 3.5.4, $\mathfrak{F} \overset{\delta}{\propto} a$ for some $a \in A$. But also by Theorem 3.1.9, $\mathfrak{F}$ has a subfilter $\mathfrak{F}'$ such that $\mathfrak{F}' \xrightarrow{\delta} a$. Also, by Theorem 3.1.7, $\mathfrak{F}' \xrightarrow{\delta} x$ since $\mathfrak{F}'$ is a subfilter of $\mathfrak{F}$ and $\mathfrak{F} \xrightarrow{\delta} x$. Now, X is Hausdorff implies $x = a$ by Theorem 3.2.1. Therefore, $x \in A$. So, $\delta\text{-Cl}(A) \subseteq A$. Hence, $\delta\text{-Cl}(A) = A$. Therefore, A is δ-closed. □

3.5.2 Near-Compactness and Functions

Theorem 3.5.9. Let $f : (X, \tau) \to (Y, \sigma)$ be a δ-continuous function. If $A \subseteq X$ is an N-closed relative to X, then $f(A) \subseteq Y$ is an N-closed relative to Y.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be δ-continuous. Let $A \subseteq X$ be an N-closed relative to X. Let $\mathcal{G}$ be a filter on Y which meets $f(A)$. Then $f^{-1}(\mathcal{G})$ is a filter on X which meets A. But A is an N-closed relative to X. Then by Theorem 3.5.4, $f^{-1}(\mathcal{G}) \overset{\delta}{\propto} a$ for some $a \in A$. But f is δ-continuous, then Theorem 3.4.3, $ff^{-1}(\mathcal{G}) \overset{\delta}{\propto} f(a)$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$. So, by Theorem 3.1.10, $\mathcal{G} \overset{\delta}{\propto} f(a)$. Therefore, $f(A)$ is an N-closed relative to Y by Theorem 3.5.4. □

Theorem 3.5.10. [103] Let $f : (X, \tau) \to (Y, \sigma)$ be an almost-continuous function. If $A \subseteq X$ is compact, then $f(A) \subseteq Y$ is an N-closed relative to Y.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be almost-continuous. Let $A \subseteq X$ be compact. Let $\mathcal{G}$ be a filter on Y which meets $f(A)$. Then $f^{-1}(\mathcal{G})$ is a filter on X which meets A. But A is compact in X. Then by Theorem 2.4.3, $f^{-1}(\mathcal{G}) \propto a$ for some $a \in A$. But f is almost-continuous, then Theorem 3.4.7, $ff^{-1}(\mathcal{G}) \overset{\delta}{\propto} f(a)$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$. So, by Theorem 3.1.10, $\mathcal{G} \overset{\delta}{\propto} f(a)$. Therefore, $f(A)$ is an N-closed relative to Y by Theorem 3.5.4. □

Theorem 3.5.11. Let $f : (X, \tau) \to (Y, \sigma)$ be a super-continuous function. If $A \subseteq X$ is an N-closed relative to X, then $f(A) \subseteq Y$ is compact.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be strongly super-continuous. Let $A \subseteq X$ be an N-closed relative to X. Let $\mathcal{G}$ be a filter on Y which meets $f(A)$. Then $f^{-1}(\mathcal{G})$ is a filter on X which meets A. But A is an N-closed relative to X. Then by Theorem 3.5.4, $f^{-1}(\mathcal{G}) \overset{\delta}{\propto} a$ for some $a \in A$. But f is super-continuous, then Theorem 3.4.5, $ff^{-1}(\mathcal{G}) \propto f(a)$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$. So, by Theorem 3.1.10, $\mathcal{G} \propto f(a)$. Therefore, $f(A)$ is compact by Theorem 2.4.3. □

Theorem 3.5.12. [104] A topological space (X, τ) is nearly compact if and only if it is an almost-continuous image of a compact space.

Proof. If (X, τ) is an almost-continuous image of a compact space, then by Theorem 3.5.10, (X, τ) is nearly compact. Conversely, let (X, τ) be nearly compact, then (X, τ_s) is compact by Theorem 3.5.5. Consider the identity function $\mathrm{id}_X : (X, \tau_s) \to (X, \tau)$, then id_X is an onto almost-continuous function. Thus, there exist a compact space and an almost-continuous function such that (X, τ) is the image of that compact space under this almost-continuous function. $\square$

Theorem 3.5.13. [61] The product $X = \prod\limits_{\alpha \in \Delta} X_\alpha$ is nearly compact if and only if each space X_α, $\alpha \in \Delta$ is nearly compact.

Proof. Assume that X is nearly compact. For each $\alpha \in \Delta$, the projection function π_α is δ-continuous and onto. From Theorem 3.5.9, it follows that $X_\alpha = \pi_\alpha(X)$ is nearly compact for all $\alpha \in \Delta$.

Conversely, let $\mathcal{F}$ be an ultrafilter on X. Since π_α is onto for each $\alpha \in \Delta$, then by Theorem 1.1.6, $\pi_\alpha(\mathcal{F})$ is an ultrafilter on X_α for all $\alpha \in \Delta$. But X_α is nearly compact for all $\alpha \in \Delta$. Hence $\pi_\alpha(\mathcal{F}) \xrightarrow{\delta} x_\alpha \in X_i$ for all $\alpha \in \Delta$, by Theorem 3.5.3. Let $x = (x_\alpha)_{\alpha \in \Delta}$, then $x \in X$ and $\pi_\alpha(x) = x_\alpha$ for all $\alpha \in \Delta$. So, $\pi_\alpha(\mathcal{F}) \xrightarrow{\delta} \pi_\alpha(x)$ for all $\alpha \in \Delta$. Hence, by Theorem 3.4.12, $\mathcal{F} \xrightarrow{\delta} x$. Therefore, $X = \prod\limits_{\alpha \in \Delta} X_\alpha$ is nearly compact by Theorem 3.5.3. $\square$

3.5.3 Almost-Strongly Closed Graphs

Definition 3.5.3. [51] A function $f : (X, \tau) \to (Y, \sigma)$ is said to have an *almost-strongly closed graph* if for each $(x, y) \notin \Gamma_f$, there exist $U \in \tau(x)$ and $G \in \mathrm{RO}_\sigma(y)$ such that $(U \times G) \cap \Gamma_f = \emptyset$.

Theorem 3.5.14. A function $f : (X, \tau) \to (Y, \sigma)$ has an almost-strongly closed graph if and only if for every $x \in X$ and $y \in Y$ such that $y \neq f(x)$, there exist $U \in \tau(x)$ and $G \in \mathrm{RO}_\sigma(y)$ such that $f(U) \cap G = \emptyset$.

Proof. This follows from Definition 3.5.3 and Proposition 2.4.1. $\square$

Remark 3.4. *A function with an almost-strongly closed graph has a closed graph, but the converse is not true as shown by the following example.*

Example 3.5.2. *Consider the function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$, where τ and σ are the usual and co-finite topologies on $\mathbb{R}$, respectively, given by $f(x) = 2$ for each $x \in \mathbb{R}$. Then f has a closed graph but it doesn't have an almost-strongly closed graph.*

We first characterize the concept of almost-strongly closed graph in terms of δ-convergence of filters.

Theorem 3.5.15. [51] A function $f : (X, \tau) \to (Y, \sigma)$ has an almost-strongly closed graph if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \longrightarrow x$ in X and $f(\mathcal{F}) \xrightarrow{\delta} y$ in Y, then $(x, y) \in \Gamma_f$.

Proof. Assume that f has an almost-strongly closed graph. Let $\mathcal{F} \longrightarrow x$ and $f(\mathcal{F}) \xrightarrow{\delta} y$. Suppose on the contrary that $(x, y) \notin \Gamma_f$. Then by hypothesis, there exist $U \in \tau(x)$ and $G \in \mathrm{RO}_\sigma(y)$ such that $f(U) \cap G = \emptyset$. But since $\mathcal{F} \longrightarrow x$ and $U \in \tau(x)$, then $U \in \mathcal{F}$, and hence $f(U) \in f(\mathcal{F})$. On the other hand, $f(\mathcal{F}) \xrightarrow{\delta} y$ and $G \in \mathrm{RO}_\sigma(y)$, so $G \in f(\mathcal{F})$. Thus, $f(U) \cap G \neq \emptyset$, which is a contradiction. Therefore, $(x, y) \in \Gamma_f$.

Conversely, suppose on the contrary that f does not have an almost-strongly closed graph. Then there exist $x \in X$ and $y \in Y$ with $(x, y) \notin \Gamma_f$ such that $f(U) \cap G \neq \emptyset$ for all $U \in \tau(x)$ and all $G \in \mathrm{RO}_\sigma(y)$. This implies that $U \cap f^{-1}(G) \neq \emptyset$ for all $U \in \tau(x)$ and all $G \in \mathrm{RO}_\sigma(y)$. Let $\mathcal{F} = \{F \subseteq X : F \supseteq U \cap f^{-1}(G), U \in \tau(x), G \in \mathrm{RO}_\sigma(y)\}$, then $\mathcal{F}$ is a filter on X. We claim that $\mathcal{F} \longrightarrow x$ and $f(\mathcal{F}) \xrightarrow{\delta} y$. First, let $U_\circ \in \tau(x)$. Then $U_\circ \supseteq U_\circ \cap f^{-1}(G)$ for each $G \in \mathrm{RO}_\sigma(y)$. Hence, $U_\circ \in \mathcal{F}$. Next, let $G_\circ \in \mathrm{RO}_\sigma(y)$. Then $G_\circ \supseteq f(f^{-1}(G_\circ)) \supseteq f(U \cap f^{-1}(G_\circ))$ for each $U \in \tau(x)$, but $U \cap f^{-1}(G_\circ) \in \mathcal{F}$ for each $U \in \tau(x)$. So, $G_\circ \in f(\mathcal{F})$. Therefore, we have constructed a filter $\mathcal{F} \longrightarrow x$ in X for which $f(\mathcal{F}) \xrightarrow{\delta} y$ in Y. By hypothesis, $(x, y) \in \Gamma_f$, which is a contradiction. Thus, f must be of almost-strongly closed graph. □

Corollary 3.5.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be any function, where Y is a semi-regular space. Then the following are equivalent:*

(i) f has an almost-strongly closed graph.

(ii) If a filter $\mathcal{F} \longrightarrow x$ in X and $f(\mathcal{F}) \xrightarrow{\delta} y$ in Y, then $(x, y) \in \Gamma_f$.

(iii) If a filter $\mathcal{F} \longrightarrow x$ in X and $f(\mathcal{F}) \longrightarrow y$ in Y, then $(x, y) \in \Gamma_f$.

(iv) f has a closed graph.

Proof.

(i) $\Longrightarrow$ (ii) Follows from Theorem 3.5.15.

(ii) $\Longrightarrow$ (iii) Follows from Theorem 3.3.1 and the fact that Y is a semi-regular space.

(iii) $\Longrightarrow$ (iv) Follows from Theorem 2.4.9.

(iv) $\Longrightarrow$ (i) Suppose that f has a closed graph. Let $\mathcal{F}$ be a filter on X with $\mathcal{F} \longrightarrow x$ in X and $f(\mathcal{F}) \xrightarrow{\delta} y$ in Y. Since Y is semi-regular, then by Theorem 3.3.1, $f(\mathcal{F}) \longrightarrow y$. But f has a closed graph, so by Theorem 2.4.9, $(x, y) \in \Gamma_f$. Therefore, f has an almost-strongly closed graph by Theorem 3.5.15.

□

The graph of an almost-continuous function need not be almost-strongly closed as it is shown in the next example.

Example 3.5.3. *In Example 2.4.2, we have the function f is continuous, and hence f is almost-continuous. But the graph Γ_f is not almost-strongly closed graph since $(1, 1) \notin \Gamma_f$ but for any $U \in \tau(1)$ and $G \in \mathrm{RO}_\sigma(1)$, we have $(1, -1) \in (U \times G) \cap \Gamma_f$.*

We now turn to gathering some more facts about the functions with almost-strongly closed graph and their relations to other functions. In theorem 2.4.10, we have proved that a continuous function has closed graph if the codomain is Hausdorff. We are now ready to give a sufficient condition on the codomain of an almost-continuous function f to insure that it has an almost-strongly closed graph.

Theorem 3.5.16. [41] Let $f : (X, \tau) \to (Y, \sigma)$ be almost-continuous, where (Y, σ) is Hausdorff. Then f has an almost-strongly closed graph.

Proof. Suppose that $\mathcal{F}$ is a filter on X with $\mathcal{F} \longrightarrow x$ in X and $f(\mathcal{F}) \xrightarrow{\delta} y$ in Y. Since f is almost-continuous, then by Theorem 3.4.2, $f(\mathcal{F}) \xrightarrow{\delta} f(x)$ in Y. But Y is Hausdorff implies $f(x) = y$ by Theorem 3.2.1. So, $(x, y) \in \Gamma_f$. Hence, by Theorem 3.5.15, f has an almost-strongly closed graph. □

Example 3.5.4. *Consider the identity function $f:(\mathbb{R},\tau)\to(\mathbb{R},\sigma)$, where τ and σ are the usual and discrete topologies on $\mathbb{R}$, respectively. Then f has an almost-strongly closed graph but f is not almost-continuous.*

We are now ready to give a sufficient condition on the codomain of a function f has an almost-strongly closed graph to insure that it is almost-continuous.

Theorem 3.5.17. [51] Let (Y,σ) be a nearly compact space. For every topological space (X,τ), each function $f:(X,\tau)\to(Y,\sigma)$ with an almost-strongly closed graph is almost-continuous.

Proof. Let $x\in X$ and $V\in\sigma(f(x))$. For each $y\in Y-V$, we have $y\neq f(x)$, this means for each $y\in Y-V$, $(x,y)\notin\Gamma_f$. But f has an almost-strongly closed graph, then by Theorem 3.5.14, there exist $U_y\in\tau(x)$ and $V_y\in\sigma(y)$ such that $f(U_y)\cap\overline{V}^{\circ}_y=\emptyset$. Let $\mathcal{V}=\{V\}\cup\{V_y : y\in Y-V\}$. Then $\mathcal{V}$ is an open cover for Y. But Y is nearly compact, then there exist $y_1,\ldots,y_n\in Y-V$ such that $Y=\overline{V}^{\circ}\cup\bigcup\limits_{i=1}^{n}\overline{V}^{\circ}_{y_i}$. Let $U=\bigcap\limits_{i=1}^{n}U_{y_i}$. Then $U\in\tau(x)$ and $U\subseteq U_{y_i}$ for all $i=1,\ldots,n$. Now,

$$f(U)\cap\Big(\bigcup_{i=1}^{n}\overline{V}^{\circ}_{y_i}\Big)=\bigcup_{i=1}^{n}\Big(f(U)\cap\overline{V}^{\circ}_{y_i}\Big)\subset\bigcup_{i=1}^{n}\Big(f(U_{y_i})\cap\overline{V}^{\circ}_{y_i}\Big)=\emptyset.$$

This implies that, $f(U)\subseteq Y-\bigcup\limits_{i=1}^{n}\overline{V}^{\circ}_{y_i}\subseteq\overline{V}^{\circ}$. Thus, f is almost-continuous at the arbitrary point $x\in X$, and so f is almost-continuous. □

Lemma 3.5.2. *Let $X=Z\cup\{p\}$ where Z is a set with $p\notin Z$, (Y,σ) be a topological space and $y\in Y$. Let $g:Z\to(Y,\sigma)$ be a function and $\mathcal{F}$ be a filter on Z. Define a function $\widetilde{g}:(X,\tau_p)\to(Y,\sigma)$ by $\widetilde{g}(z)=g(z)$ for any $z\in Z$ and $\widetilde{g}(p)=y$. Then $g(\mathcal{F})\xrightarrow{\delta}y$ in (Y,σ) if and only if $\widetilde{g}$ is almost-continuous on X.*

Proof. Similar to the proof of Lemma 2.4.1. □

Theorem 3.5.18. [51] Let (Y,σ) be a Hausdorff space. Then (Y,σ) is nearly compact if and only if for any space $(X,\tau)\in\mathcal{S}$, each function $f:(X,\tau)\to(Y,\sigma)$, that has an almost-strongly closed graph, is almost-continuous.

Proof. The first direction follows from Theorem 3.5.17. Conversely, Suppose, by the way of contradiction, that Y is not nearly compact, then there is a filter $\mathcal{F}$ on Y such that δ-Adh$(\mathcal{F}) = \emptyset$. Let $X = Y \cup \{p\}$ where $p \notin Y$. Consider the topological space (X, τ_p). Then by Theorem 2.4.12, (X, τ_p) is Hausdorff. Also, by Theorems 1.4.2 and 1.4.4, (X, τ_p) is completely normal and fully normal. This implies that $(X, \tau_p) \in \mathcal{S}$. Fix a point $b \in Y$ and define $\widetilde{\mathrm{id}}_Y : (X, \tau_p) \to (Y, \sigma)$ by $\widetilde{\mathrm{id}}_Y(x) = \mathrm{id}_Y(x) = x$ for any $x \in Y$ and $\widetilde{\mathrm{id}}_Y(p) = b$. Let $(x, y) \in X \times Y$ and $(x, y) \notin \Gamma_{\widetilde{\mathrm{id}}_Y}$. Consider the case when $x \neq p$. Since $\widetilde{\mathrm{id}}_Y(x) \neq y$ and (Y, σ) is Hausdorff, then there exists $V_y \in \sigma(y)$ such that $\widetilde{\mathrm{id}}_Y(x) \notin \overline{V}_y$. So, $\widetilde{\mathrm{id}}_Y(x) \notin \overline{V}^{\circ}_y$. Hence, $\{x\} \in \tau_p(x)$, $V_y \in \sigma(y)$ and $\widetilde{\mathrm{id}}_Y(\{x\}) \cap \overline{V}^{\circ}_y = \{\widetilde{\mathrm{id}}_Y(x)\} \cap \overline{V}^{\circ}_y = \emptyset$. Consider the case when $x = p$. Then $b = \widetilde{\mathrm{id}}_Y(p) \neq y$. Again, since (Y, σ) is Hausdorff, then there exists $V_y \in \sigma(y)$ such that $b \notin \overline{V}_y$. So, $b \notin \overline{V}^{\circ}_y$. Moreover, since δ-Adh$(\mathcal{F}) = \emptyset$, then by Theorem 3.1.2, we have $\mathcal{F} \overset{\delta}{\not\propto} y$, so there exist $W_y \in \sigma(y)$ and $F \in \mathcal{F}$ such that $F \cap \overline{W}^{\circ}_y = \emptyset$. Let $Z_y = V_y \cap W_y$. Then $Z_y \in \sigma(y)$, $b \notin \overline{Z}^{\circ}_y$ and $F \cap \overline{Z}^{\circ}_y = \emptyset$. Thus, $F \cup \{p\} \in \tau_p(x)$, $Z_y \in \sigma(y)$ and $\widetilde{\mathrm{id}}_Y(F \cup \{p\}) \cap \overline{Z}^{\circ}_y = (\mathrm{id}_Y(F) \cup \{b\}) \cap \overline{Z}^{\circ}_y = \emptyset$. We have shown, in both cases, that for each $(x, y) \in (X \times Y) - \Gamma_{\widetilde{\mathrm{id}}_Y}$, there exist $U_x \in \tau_p(x)$ and $G_y \in \sigma(y)$ such that $\widetilde{\mathrm{id}}_Y(U_x) \cap \overline{G}^{\circ}_y = \emptyset$. Thus, by Theorem 3.5.14, $\widetilde{\mathrm{id}}_Y$ has an almost-strongly closed graph. By hypothesis, $\widetilde{\mathrm{id}}_Y$ is almost-continuous, and so by Lemma 3.5.2, $\mathrm{id}_Y(\mathcal{F}) \xrightarrow{\delta} b$ in (Y, σ) implies $\mathcal{F} \xrightarrow{\delta} b$ in (Y, σ), and hence by Proposition 3.1.2 part (i), $\mathcal{F} \overset{\delta}{\propto} b$, so by Theorem 3.1.2, δ-Adh$(\mathcal{F}) \neq \emptyset$, which is a contradiction. Therefore, (Y, σ) is nearly compact. □

Chapter 4

θ-Convergence of Filters

We study θ-convergence of filters. We will start by introducing the definition of a θ-limit of a filter and define a θ-cluster point of a filter. A number of results in regular spaces have been achieved. weakly-θ-continuous, strongly-θ-continuous, and θ-continuous functions are all characterized. As well, the connections between these functions and θ-limits (θ-cluster points) of filters are investigated. Several important notions, such as Urysohn and quasi-H-closed spaces, can be characterized with the help of filters. The concept of a strongly closed graph is defined and characterized by filters.

4.1 θ-Limit and θ-Cluster Points of Filters

Definition 4.1.1. [48] Let (X, τ) be a topological space. For each $x \in X$, let $\mathcal{C}^\tau(x) = \{\overline{U} : x \in U \in \tau\}$. Then $\mathcal{C}^\tau(x)$ is a filter base in X. Let $\langle \mathcal{C}^\tau(x) \rangle$ be the filter generated by $\mathcal{C}^\tau(x)$. We call $\langle \mathcal{C}^\tau(x) \rangle$ the *θ-neighborhood filter* of x.

Notation 8. *For a topological space (X, τ) and $x \in X$, when there is no confusion, we will just write the filter base "$\mathcal{C}(x)$" instead of "$\mathcal{C}^\tau(x)$".*

Definition 4.1.2. [115] Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. We say that $\mathcal{F}$ *θ-converges* to x, written $\mathcal{F} \xrightarrow{\theta} x$ iff $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}$. In such a case, x is called the *θ-limit* of $\mathcal{F}$.

Definition 4.1.3. [115] Let (X,τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F}$ *θ-accumulates* at x, written $\mathcal{F} \overset{\theta}{\propto} x$, iff $\mathcal{F}(\cap)[\mathcal{C}(x)]$. Equivalently, for each $F \in \mathcal{F}$ and for each $G \in [\mathcal{C}(x)]$, $F \cap G \neq \emptyset$. In such a case, x is called the *θ-cluster* point of $\mathcal{F}$.

Proposition 4.1.1. Let X be a topological space and $x \in X$. Since $\mathcal{C}(x)$ is a filter base, then $\mathcal{F} \overset{\theta}{\propto} x$ if and only if for each $F \in \mathcal{F}$ and for each $G \in \mathcal{C}(x)$, $F \cap G \neq \emptyset$.

Proposition 4.1.2. Let (X,τ) be a topological space, $\mathcal{F}$ be a filter on (X,τ) and $x \in X$.

(i) If $\mathcal{F} \xrightarrow{\theta} x$ then $\mathcal{F} \overset{\theta}{\propto} x$.

(ii) If $\mathcal{F} \longrightarrow x$, then $\mathcal{F} \xrightarrow{\theta} x$.

(iii) If $\mathcal{F} \propto x$, then $\mathcal{F} \overset{\theta}{\propto} x$.

Proof. (i) Let $G \in \mathcal{C}(x)$. Since $\mathcal{F} \xrightarrow{\theta} x$, then $G \in \mathcal{F}$. But then, $G \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. So, $\mathcal{F} \overset{\theta}{\propto} x$.

(ii) Let $G \in \mathcal{C}(x)$, then $G = \overline{U}$ for some open U in X containing x. Since $\mathcal{F} \longrightarrow x$, then $U \in \mathcal{F}$. But since $U \subseteq \overline{U} = G$, then $G \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{\theta} x$.

(iii) Let $G \in \mathcal{C}(x)$ and $F \in \mathcal{F}$, then $G = \overline{U}$ for some open U in X containing x. Since $\mathcal{F} \propto x$, then $U \cap F \neq \emptyset$. But then, $G \cap F = \overline{U} \cap F \supseteq U \cap F \neq \emptyset$. So, $\mathcal{F} \overset{\theta}{\propto} x$.

$\square$

The converse of each statement in proposition 4.1.2 need not be true as the following example shows.

Example 4.1.1. *Let* $X = \{a,b,c\}$, $\tau_X = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ *and* $\mathcal{F} = \{X, \{a,c\}\}$. *Then*

$$\begin{aligned}
\mathcal{U}(a) &= \{\{a\},\{a,b\},\{a,c\},X\}, & \langle \mathcal{C}(a) \rangle &= \{\{a,c\},X\} \\
\mathcal{U}(b) &= \{\{b\},\{a,b\},\{b,c\},X\}, & \langle \mathcal{C}(b) \rangle &= \{\{b,c\},X\} \\
\mathcal{U}(c) &= \{X\}, & \langle \mathcal{C}(c) \rangle &= \{X\}.
\end{aligned}$$

Now, we can sum-up in the following table:

Limits of $\mathcal{F}$	θ-Limits of $\mathcal{F}$	Cluster points of $\mathcal{F}$	θ-Cluster points of $\mathcal{F}$
	a	a	a
			b
c	c	c	c

Table 4.1 – Limits, θ-limits, cluster points and θ-cluster points

From Table 4.1, we have

(i) $\mathcal{F} \overset{\theta}{\propto} b$ *but* $\mathcal{F} \overset{\theta}{\not\to} b$.

(ii) $\mathcal{F} \xrightarrow{\theta} a$ *but* $\mathcal{F} \not\to a$.

(iii) $\mathcal{F} \overset{\theta}{\propto} b$ *but* $\mathcal{F} \not\propto b$.

Definition 4.1.4. [115] Let (X,τ) be a topological space, $E \subseteq X$ and $x \in X$. Then x is a *θ-adherent* point of E iff for all $G \in \mathcal{C}(x)$, $G \cap E \neq \emptyset$. Equivalently, for every open set U in X containing x, $\overline{U} \cap E \neq \emptyset$. The set of all θ-adherent points of a set E is called the *θ-closure* of the set E and denoted by θ-Cl(E).

Definition 4.1.5. [115] A subset E of a topological space (X,τ) is called *θ-closed* if θ-Cl$(E) = E$. The complement of a θ-closed set is called a *θ-open* set.

Proposition 4.1.3. [58] For any subset E of a topological space X, $\overline{E} \subseteq \theta$-Cl$(E)$.

Proof. Let $x \in \overline{E}$ and U be open in X containing x. Then $U \cap E \neq \emptyset$. But $U \subseteq \overline{U}$. so $U \cap E \subseteq \overline{U} \cap E$. Thus, $\overline{U} \cap E \neq \emptyset$. Therefore, $x \in \theta$-Cl(E). □

Theorem 4.1.1. [115] Let (X,τ) be a topological space and $E \subseteq X$. If E is open in X, then $\overline{E} = \theta$-Cl(E).

Proof. $\overline{E} \subseteq \theta$-Cl$(E)$ by Proposition 4.1.3. Let $x \in \theta$-Cl(E), then $\overline{U} \cap E \neq \emptyset$ for all $U \in \tau(x)$. By Corollary 1.2.1, $U \cap E \neq \emptyset$ for all $U \in \tau(x)$. So, $x \in \overline{E}$. Hence, θ-Cl$(E) \subseteq \overline{E}$. □

Proposition 4.1.4. [67] Let A be a subset of a topological space X. Then A is θ-open in X if and only if for each $x \in A$, there exists an open set V in X such that $x \in V \subseteq \overline{V} \subseteq A$.

Proof. suppose that A is θ-open in X. Let $x \in A$. Then $x \notin X - A = \theta\text{-Cl}(X - A)$. So, there exists an open set V in X containing x such that $\overline{V} \cap (X - A) = \emptyset$. So, $\overline{V} \subseteq A$. Hence, there exists an open set V in X such that $x \in V \subseteq \overline{V} \subseteq A$.

Conversely, suppose, by the way of contradiction, that A is not θ-open. Then there exists $x \in \theta\text{-Cl}(X - A)$ such that $x \notin X - A$. So, $x \in A$. By hypothesis, there exists an open set V in X such that $x \in V \subseteq \overline{V} \subseteq A$. But $x \in \theta\text{-Cl}(X - A)$ implies $\overline{V} \cap (X - A) \neq \emptyset$, which is a contradiction since $\overline{V} \cap (X - A) \subseteq A \cap (X - A) = \emptyset$. Therefore, A is θ-open in X. □

Proposition 4.1.5. The family of all θ-open sets in (X, τ) is a new topology on X denoted by τ_θ.

Proposition 4.1.6. [115] For any subset E of a topological space (X, τ), $\overline{E} \subseteq \delta\text{-Cl}(E) \subseteq \theta\text{-Cl}(E)$.

Proof. Let $x \in \overline{E}$ and $U \in \tau(x)$. Then $U \cap E \neq \emptyset$. But $U \subseteq \overline{U}^\circ$ since U is open. So $U \cap E \subseteq \overline{U}^\circ \cap E$. Hence, $\overline{U}^\circ \cap E \neq \emptyset$. Thus, $x \in \delta\text{-Cl}(E)$. Next, let $x \in \delta\text{-Cl}(E)$ and let $U \in \tau(x)$. Then $\overline{U}^\circ \cap E \neq \emptyset$. But $\overline{U}^\circ \subseteq \overline{U}$, then $\overline{U} \cap E \neq \emptyset$. Thus, $x \in \theta\text{-Cl}(E)$. □

Remark 4.1. *Let (X, τ) be a topological space. Then $\tau_\theta \subseteq \tau_\delta \subseteq \tau$.*

Proof. Let $U \in \tau_\theta$. Then $X - U = \theta\text{-Cl}(X - U)$. By Propositions 3.1.3 and 4.1.6, we have $X - U \subseteq \delta\text{-Cl}(X - U) \subseteq \theta\text{-Cl}(X - U) = X - U$. So, $X - U = \delta\text{-Cl}(X - U)$. Thus, $U \in \tau_\delta$. Next, as $\tau_\delta = \tau_s$ and $\tau_s \subseteq \tau$, then $\tau_\delta \subseteq \tau$. □

Theorem 4.1.2. [67] Let (X, τ) be a topological space. Then (X, τ) is almost-regular if and only if $\tau_s = \tau_\theta$.

Proof. Let U be a regular open set in X and let $x \in U$. Then by almost-regularity of X, there is an open set V in X such that $x \in V \subseteq \overline{V} \subseteq U$. Hence, by Proposition 4.1.4, U is θ-open in X. So, $\mathrm{RO}(X, \tau) \subseteq \tau_\theta$, and hence $\tau_s \subseteq \tau_\theta$ but $\tau_\theta \subseteq \tau_s$. Therefore, $\tau_s = \tau_\theta$.

Conversely, suppose that $\tau_s = \tau_\theta$. Let U be a regular open set in X containing x. Then $U \in \tau_s = \tau_\theta$. So, by Proposition 4.1.4, there is an open set V in X such that $x \in V \subseteq \overline{V} \subseteq U$. Let $W = \overline{V}^\circ$. Then W is regular open in X, $x \in V \subseteq \overline{V}^\circ = W$, and so $x \in W \subseteq \overline{W} = \overline{\overline{V}^\circ} \subseteq \overline{V} \subseteq U$. Hence, there is a regular open set W in X such that $x \in W \subseteq \overline{W} \subseteq U$. Therefore, X is almost-regular. $\square$

Definition 4.1.6. [115] Let $\mathfrak{F}$ be a filter on a topological space (X, τ). A point $x \in X$ is said to be a *θ-adherent* point of $\mathfrak{F}$ if x is a θ-adherent point of every set in $\mathfrak{F}$. The *θ-adherence* of $\mathfrak{F}$, θ-$\mathrm{Adh}(\mathfrak{F})$, is the set of all θ-adherent points of $\mathfrak{F}$.

Remark 4.2. *[115] Let X be a topological space. If $\mathfrak{F}$ is a filter on X, then* $\theta\text{-}\mathrm{Adh}(\mathfrak{F}) = \bigcap\limits_{F \in \mathfrak{F}} \theta\text{-}\mathrm{Cl}(F)$.

Theorem 4.1.3. [115] Let $\mathfrak{F}$ be a filter on a topological space X and $x \in X$. Then $x \in \theta$-$\mathrm{Adh}(\mathfrak{F})$ if and only if $\mathfrak{F} \overset{\theta}{\propto} x$.

Proof.

$$
\begin{aligned}
x \in \theta\text{-}\mathrm{Adh}(\mathfrak{F}) \text{ iff } & x \in \bigcap_{F \in \mathfrak{F}} \theta\text{-}\mathrm{Cl}(F) \\
\text{iff } & x \in \theta\text{-}\mathrm{Cl}(F) \text{ for all } F \in \mathfrak{F} \\
\text{iff } & G \cap F \neq \emptyset \text{ for all } G \in \mathfrak{C}(x) \text{ and all } F \in \mathfrak{F} \\
\text{iff } & \mathfrak{F}(\cap)[\mathfrak{C}(x)] \quad \text{by Proposition 4.1.1.} \\
\text{iff } & \mathfrak{F} \overset{\theta}{\propto} x.
\end{aligned}
$$

$\square$

Theorem 4.1.4. Let X be a topological space, $E \subseteq X$ and $x \in X$. Then $x \in \theta$-$\mathrm{Cl}(E)$ if and only if there exists a filter $\mathfrak{F}$ on X such that $E \in \mathfrak{F}$ and $\mathfrak{F} \xrightarrow{\theta} x$.

Proof. Suppose that there exists a filter $\mathcal{F}$ on X such that $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\theta} x$. We will show that $x \in \theta\text{-Cl}(E)$. Let $G \in \mathcal{C}(x)$. But $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}$, so $G \in \mathcal{F}$. Hence, $G \cap E \neq \emptyset$. Therefore, $x \in \theta\text{-Cl}(E)$.

Conversely, suppose that $x \in \theta\text{-Cl}(E)$, then $G \cap E \neq \emptyset$ for all $G \in \mathcal{C}(x)$. So, $G \cap E \neq \emptyset$ for all $G \in \mathcal{G}$ where $\mathcal{G} = \langle \mathcal{C}(x) \rangle$. Consider the filter $\mathcal{F} = \langle \mathcal{G}\big|_E \rangle$. Then by Proposition 1.1.6, $E \in \mathcal{F}$ and $\langle \mathcal{C}(x) \rangle = \mathcal{G} \subseteq \mathcal{F}$. Therefore, $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\theta} x$. $\square$

Theorem 4.1.5. Let X be a topological space, $E \subseteq X$ and $x \in X$. Then $x \in \theta\text{-Cl}(E)$ if and only if there exists a filter $\mathcal{F}$ on X, which meets E such that $\mathcal{F} \xrightarrow{\theta} x$.

Proof. Suppose that $x \in \theta\text{-Cl}(E)$. Then by Theorem 4.1.4, there exists a filter $\mathcal{F}$ on X such that $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\theta} x$. Since $E \in \mathcal{F}$, then for all $F \in \mathcal{F}$, $F \cap E \neq \emptyset$. That is, $\mathcal{F}$ meets E. Conversely, suppose that $\mathcal{F}$ is a filter on X such that $\mathcal{F} \xrightarrow{\theta} x$ and $F \cap E \neq \emptyset$ for all $F \in \mathcal{F}$. Since $\mathcal{F} \xrightarrow{\theta} x$, then $G \in \mathcal{F}$ for all $G \in \mathcal{C}(x)$. So, by hypothesis, $G \cap E \neq \emptyset$ for all $G \in \mathcal{C}(x)$. Therefore, $x \in \theta\text{-Cl}(E)$. $\square$

Theorem 4.1.6. Let (X, τ) be a topological space and $A \subseteq X$. Then A is θ-closed if and only if whenever a filter $\mathcal{F} \xrightarrow{\theta} x$ with $A \in \mathcal{F}$, then $x \in A$.

Proof. Assume that a filter $\mathcal{F} \xrightarrow{\theta} x$ and $A \in \mathcal{F}$. Then by Theorem 4.1.4, $x \in \theta\text{-Cl}(A)$. But $\theta\text{-Cl}(A) = A$ since A is θ-closed. So, $x \in A$. Conversely, let $x \in \theta\text{-Cl}(A)$. Then by Theorem 4.1.4, there is a filter $\mathcal{F}$ on X such that $\mathcal{F} \xrightarrow{\theta} x$ and $A \in \mathcal{F}$. So by hypothesis, $x \in A$. Thus, $\theta\text{-Cl}(A) \subseteq A$. But $A \subseteq \theta\text{-Cl}(A)$. Therefore, $\theta\text{-Cl}(A) = A$, and hence A is θ-closed. $\square$

Theorem 4.1.7. Let (X, τ) be a topological space and $A \subseteq X$. Then A is θ-open in X if and only if whenever a filter $\mathcal{F} \xrightarrow{\theta} x \in A$, then $A \in \mathcal{F}$.

Proof. Suppose that A is θ-open in X. let $\mathcal{F}$ be a filter on X such that $\mathcal{F} \xrightarrow{\theta} x \in A$. Since $x \in A$, then by Proposition 4.1.4, $A \in \langle \mathcal{C}(x) \rangle$. Hence, $A \in \mathcal{F}$. Conversely, suppose, by the way of contradiction, that A is not θ-open, then $X - A$ is not θ-closed, so there exists $x \in \theta\text{-Cl}(X - A)$ such that $x \notin X - A$. So, $x \in A$. Now, by Theorem 4.1.4, there exists a filter $\mathcal{F}$ on X such that $X - A \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\theta} x$.

Since $\mathcal{F} \xrightarrow{\theta} x \in A$, then by hypothesis, $A \in \mathcal{F}$. But then $\emptyset = A \cap (X - A) \in \mathcal{F}$, which is a contradiction. Therefore, A is θ-open. $\square$

Remark 4.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be filters on a topological space (X, τ) and $x \in X$.*

(i) The principal filter $\langle x \rangle \xrightarrow{\theta} x$.

(ii) If $\mathcal{F} \xrightarrow{\theta} x$ and $\mathcal{G} \xrightarrow{\theta} x$, then $\mathcal{F} \cap \mathcal{G} \xrightarrow{\theta} x$.

Theorem 4.1.8. Let X be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \xrightarrow{\theta} x$ if and only if for every subfilter $\mathcal{F}'$ of $\mathcal{F}$, $\mathcal{F}' \xrightarrow{\theta} x$.

Proof. If every subfilter of $\mathcal{F}$ θ-converges to $x \in X$, then so does $\mathcal{F}$ because it is a subfilter of itself. Conversely, suppose that $\mathcal{F} \xrightarrow{\theta} x$ and $\mathcal{F}'$ is a subfilter of $\mathcal{F}$, then $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{F}'$. So, $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}'$. Therefore, $\mathcal{F}' \xrightarrow{\theta} x$. $\square$

Theorem 4.1.9. Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \xrightarrow{\theta} x$ if and only if every subfilter $\mathcal{G}$ of $\mathcal{F}$ has a subfilter $\mathcal{H}$ such that $\mathcal{H} \xrightarrow{\theta} x$.

Proof. Suppose, by the way of contradiction, that $\mathcal{F} \overset{\theta}{\not\longrightarrow} x$, then there is an open set U in X containing x such that $\overline{U} \notin \mathcal{F}$. Then $(X - \overline{U}) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. So, $\mathcal{G} = \langle \mathcal{F}\big|_{X-\overline{U}} \rangle$ is a subfilter of $\mathcal{F}$ containing $X - \overline{U}$. By hypothesis, $\mathcal{G}$ has a subfilter $\mathcal{H}$ which θ-converges to x. Since U is open in X containing x, then $\overline{U} \in \mathcal{H}$ but $X - \overline{U} \in \mathcal{G} \subseteq \mathcal{H}$. So, $\emptyset \in \mathcal{H}$ which is a contradiction. Therefore, $\mathcal{F} \xrightarrow{\theta} x$. The converse follows from Theorem 4.1.8. $\square$

Theorem 4.1.10. [115] Let X be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \overset{\theta}{\propto} x$ if and only if there exists a subfilter $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \xrightarrow{\theta} x$.

Proof. Suppose there exists a subfilter $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \xrightarrow{\theta} x$. Then $\mathcal{F} \subseteq \mathcal{F}'$ and $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}'$. Let $F \in \mathcal{F}$ and $G \in \mathcal{C}(x)$, then $F \in \mathcal{F}'$ and $G \in \mathcal{F}'$. So, $F \cap G \neq \emptyset$. Hence, $\mathcal{F} \overset{\theta}{\propto} x$.

Conversely, assume that $\mathcal{F} \overset{\theta}{\propto} x$. We will construct a subfilter $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \xrightarrow{\theta} x$. Since $\mathcal{F} \overset{\theta}{\propto} x$, then $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and all $G \in \mathcal{C}(x)$. This implies $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and all $G \in \langle \mathcal{C}(x) \rangle$. Let $\mathcal{F}' = \mathcal{F} \vee \langle \mathcal{C}(x) \rangle$. Then $\mathcal{F}'$ is a filter on X such that $\mathcal{F} \subseteq \mathcal{F}'$ and $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}'$. Thus, $\mathcal{F}'$ is a subfilter of $\mathcal{F}$ and $\mathcal{F}' \xrightarrow{\theta} x$. $\square$

Theorem 4.1.11. Let X be a topological space, $\mathcal{F}'$ be a subfilter of $\mathcal{F}$ and $x \in X$. If $\mathcal{F}' \overset{\theta}{\propto} x$, then $\mathcal{F} \overset{\theta}{\propto} x$.

Proof. Suppose that $\mathcal{F}' \overset{\theta}{\propto} x$. Let $G \in \mathcal{C}(x)$ and $F \in \mathcal{F}$, But $\mathcal{F} \subseteq \mathcal{F}'$, so $F \in \mathcal{F}'$. Thus, $G \cap F \neq \emptyset$. Hence, $\mathcal{F} \overset{\theta}{\propto} x$. $\square$

Theorem 4.1.12. [115] Let $\mathcal{F}$ be an ultrafilter on a topological space X and $x \in X$. Then $\mathcal{F} \xrightarrow{\theta} x$ if and only if $\mathcal{F} \overset{\theta}{\propto} x$.

Proof. If $\mathcal{F} \xrightarrow{\theta} x$, then $\mathcal{F} \overset{\theta}{\propto} x$ by Proposition 4.1.2 part (i). Conversely, suppose that $\mathcal{F} \overset{\theta}{\propto} x$. Let $G \in \mathcal{C}(x)$. Then $G \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. So, $\mathcal{F}$ meets G. But $\mathcal{F}$ is an ultrafilter on X, then by the proof of Theorem 1.1.5, $G \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{\theta} x$. $\square$

4.2 θ-Convergence in Urysohn Spaces

Definition 4.2.1. [120] A topological space X is called *Urysohn* if for each $x_1 \neq x_2$ in X, there exist open sets U and V in X containing x_1 and x_2, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. Equivalently, for each $x_1 \neq x_2$ in X, there exist $G_1 \in \mathcal{C}(x_1)$ and $G_2 \in \mathcal{C}(x_2)$ such that $G_1 \cap G_2 = \emptyset$.

The following implications hold.

Proposition 4.2.1. [31] Urysohn $\Longrightarrow$ Hausdorff $\Longrightarrow$ weakly-T_2.

Now, we offer the following characterization.

Theorem 4.2.1. [61] A topological space X is Urysohn if and only if each filter $\mathcal{F}$ on X θ-converges to at most one point in X.

Proof. Suppose that X is a Urysohn space and $\mathcal{F}$ is a filter on X such that $\mathcal{F} \xrightarrow{\theta} x$ and $\mathcal{F} \xrightarrow{\theta} y$. Assume that $x \neq y$. But X is Urysohn, so there exists $G \in \mathcal{C}(x)$ and $H \in \mathcal{C}(y)$ such that $G \cap H = \emptyset$. Now, since $\mathcal{F} \xrightarrow{\theta} x$, then $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}$, and so $G \in \mathcal{F}$. Also, since $\mathcal{F} \xrightarrow{\theta} y$, then $\langle \mathcal{C}(y) \rangle \subseteq \mathcal{F}$, and so $H \in \mathcal{F}$. Thus, $G \cap H \neq \emptyset$, which is a contradiction. So, we must have $x = y$.

Conversely, suppose, by the way of contradiction, that X is not an Urysohn space. So, there exist $x \neq y$ in X such that $G \cap H \neq \emptyset$ for all $G \in \mathcal{C}(x)$ and for all $H \in \mathcal{C}(y)$. Since $\mathcal{C}(x)$ and $\mathcal{C}(y)$ are filter bases for $\langle \mathcal{C}(x) \rangle$ and $\langle \mathcal{C}(y) \rangle$, respectively, then $G \cap H \neq \emptyset$ for all $G \in \langle \mathcal{C}(x) \rangle$ and for all $H \in \langle \mathcal{C}(y) \rangle$. Then $\mathcal{F} = \langle \mathcal{C}(x) \rangle \vee \langle \mathcal{C}(y) \rangle$ is a filter on X such that $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}$ and $\langle \mathcal{C}(y) \rangle \subseteq \mathcal{F}$. Thus, the filter $\mathcal{F}$ θ-converges to both x and y. But then, by hypothesis, $x = y$, which is a contradiction. So, X must be Urysohn. □

Theorem 4.2.2. [61] Let X be an Urysohn space, $\mathcal{F}$ be a filter on X and $x \in X$. If $\mathcal{F} \xrightarrow{\theta} x$, then x is the unique θ-cluster point of $\mathcal{F}$.

Proof. If $\mathcal{F} \xrightarrow{\theta} x$, then x is a θ-cluster point of $\mathcal{F}$ by Proposition 4.1.2 part (i). Now, suppose that $y \in X$ is a θ-cluster point of $\mathcal{F}$ with $x \neq y$. But X is Urysohn. So, there exist $G \in \mathcal{C}(x)$ and $H \in \mathcal{C}(y)$ such that $G \cap H = \emptyset$. Now, since $\mathcal{F} \xrightarrow{\theta} x$, then $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}$, and so $G \in \mathcal{F}$. But then, $G \cap H \neq \emptyset$ since $\mathcal{F} \overset{\theta}{\propto} y$ and $H \in \mathcal{C}(y)$, which is a contradiction. Therefore, $x = y$. □

4.3 θ-Convergence in Regular Spaces

We will see immediately that, in regular spaces, θ-convergence of filters is equivalent to convergence of filters and in this case equivalence is also valid for cluster and θ-cluster points.

Theorem 4.3.1. Let (X, τ) be a regular space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \xrightarrow{\theta} x$ if and only if $\mathcal{F} \longrightarrow x$.

Proof. If $\mathcal{F} \longrightarrow x$, then by Propostion 4.1.2 part (ii), $\mathcal{F} \xrightarrow{\theta} x$. Conversely, suppose that $\mathcal{F} \xrightarrow{\theta} x$. Let $U \in \tau(x)$. Since X is regular, then there exists $V \in \tau(x)$ such that $\overline{V} \subseteq U$. Since $V \in \tau(x)$, then $\overline{V} \in \mathcal{C}(x)$ but $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}$. So, $\overline{V} \in \mathcal{F}$ and thus, $U \in \mathcal{F}$. Therefore, $\mathcal{F} \longrightarrow x$. $\square$

Theorem 4.3.2. [115] Let X be a regular space and $E \subseteq X$. Then $\overline{E} = \theta\text{-Cl}(E)$.

Proof. By Proposition 4.1.3, $\overline{E} \subseteq \theta\text{-Cl}(E)$. Next, let $x \in \theta\text{-Cl}(E)$, then by Theorem 4.1.4, there exists a filter on X such that $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\theta} x$. But X is regular, so $\mathcal{F} \longrightarrow x$ by Theorem 4.3.1. Thus, by Theorem 2.1.2, $x \in \overline{E}$. $\square$

Theorem 4.3.3. Let (X, τ) be a regular space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \overset{\theta}{\propto} x$ if and only if $\mathcal{F} \propto x$.

Proof. If $\mathcal{F} \propto x$, then by Proposition 4.1.2 part (iii), $\mathcal{F} \overset{\theta}{\propto} x$. Conversely, suppose that $\mathcal{F} \overset{\theta}{\propto} x$. Let $U \in \tau(x)$ and $F \in \mathcal{F}$. Since X is regular, then there exists $V \in \tau(x)$ such that $\overline{V} \subseteq U$. But $\overline{V} \in \mathcal{C}(x)$ and $\mathcal{F} \overset{\theta}{\propto} x$, so $\overline{V} \cap F \neq \emptyset$, thus, $U \cap F \neq \emptyset$. Therefore, $\mathcal{F} \propto x$. $\square$

Theorem 4.3.4. Let (X, τ) be a topological space. Then X is a regular space if and only if for each $x \in X$ and for each filter $\mathcal{F}$ on X, $\mathcal{F} \longrightarrow x$ whenever $\mathcal{F} \xrightarrow{\theta} x$.

Proof. Suppose that for each filter $\mathcal{F}$ on X, $\mathcal{F} \longrightarrow x$ whenever $\mathcal{F} \xrightarrow{\theta} x$. To show that X is regular, let $x \in X$ be an arbitrary. Since $\langle \mathcal{C}(x) \rangle \xrightarrow{\theta} x$, then by hypothesis, $\langle \mathcal{C}(x) \rangle \longrightarrow x$. That is, $\mathcal{U}(x) \subseteq \langle \mathcal{C}(x) \rangle$, so for all $U \in \mathcal{U}(x)$, there exists $H \in \mathcal{C}(x)$ such that $H \subseteq U$. This implies, for all $U \in \tau(x)$, there exists $V \in \tau(x)$ such that $\overline{V} \subseteq U$. Thus, X is regular. The converse follows from Theorem 4.3.1.

$\square$

4.4 θ-Convergent Filters and Functions

We will investigate the case of θ-limits of filters under the three types of continuity. We will do the same investigation for θ-cluster points of filters.

4.4.1 Weakly-θ-Continuous Functions

We introduce weakly-θ-continuous functions in order to study this class of functions, we state several characterizations of weakly-θ-continuous functions and the notion of a function that has a strongly closed graph.

Definition 4.4.1. [28, 81] A function $f:(X,\tau)\to(Y,\sigma)$ is *weakly-θ-continuous* at $x\in X$ if for every open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U)\subseteq\overline{V}$. Equivalently, for all $G\in\mathcal{C}^{\sigma}(f(x))$, there exists $U\in\tau(x)$ such that $f(U)\subseteq G$. If this condition is satisfied at each $x\in X$, then f is said to be weakly-θ-continuous on X.

We now present the following theorem (to be proved later in section 6.3, theorem 6.3.5).

Theorem 4.4.1. [106] Every almost-continuous function is weakly-θ-continuous.

Theorem 4.4.2. [28] Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then f is weakly-θ-continuous at $x\in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F}\longrightarrow x$, then $f(\mathcal{F})\xrightarrow{\theta}f(x)$ in Y.

Proof. Assume that $\mathcal{F}\longrightarrow x$ and $G\in\mathcal{C}^{\sigma}(f(x))$. Since f is weakly-θ-continuous at x, then there exists $U\in\tau(x)$ such that $f(U)\subseteq G$. But $\mathcal{F}\longrightarrow x$, then $U\in\mathcal{F}$, so $G\in f(\mathcal{F})$. Hence, $\langle\mathcal{C}^{\sigma}(f(x))\rangle\subseteq f(\mathcal{F})$. Therefore, $f(\mathcal{F})\xrightarrow{\theta}f(x)$ in Y.

Conversely, let $G\in\mathcal{C}^{\sigma}(f(x))$. Since $\mathcal{U}_{\tau}(x)\subseteq\mathcal{U}_{\tau}(x)$, then $\mathcal{U}_{\tau}(x)\longrightarrow x$. By hypothesis, we have $f(\mathcal{U}_{\tau}(x))\xrightarrow{\theta}f(x)$. That is, $\langle\mathcal{C}^{\sigma}(f(x))\rangle\subseteq f(\mathcal{U}_{\tau}(x))$. Hence, there exists $U\in\tau(x)$ such that $f(U)\subseteq G$. Therefore, f is weakly-θ-continuous at $x\in X$. □

Theorem 4.4.3. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then f is weakly-θ-continuous at $x\in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F}\propto x$, then $f(\mathcal{F})\overset{\theta}{\propto}f(x)$ in Y.

Proof. Suppose that $\mathcal{F}\propto x$ in X and $G\in\mathcal{C}^{\sigma}(f(x))$. Since f is weakly-θ-continuous, then there exists $U\in\tau(x)$ such that $f(U)\subseteq G$. But $U\cap F\neq\emptyset$ for all $F\in\mathcal{F}$

since $\mathcal{F} \propto x$. So, $\emptyset \neq f(U \cap F) \subseteq f(U) \cap f(F) \subseteq G \cap f(F)$ for all $F \in \mathcal{F}$. That is, $G \cap f(F) \neq \emptyset$ for all $F \in \mathcal{F}$. Hence, $f(\mathcal{F}) \overset{\theta}{\propto} f(x)$.

Conversely, suppose, by the way of contradiction, that f is not weakly-θ-continuous at $x \in X$, then there exists $K \in \mathcal{C}^{\sigma}(f(x))$ such that $f(U) \nsubseteq K$ for any $U \in \tau(x)$. So, $U \nsubseteq f^{-1}(K)$ for any $U \in \tau(x)$. This implies, $V \nsubseteq f^{-1}(K)$ for any $V \in \mathcal{U}_{\tau}(x)$. Thus, $V \cap F \neq \emptyset$ for any $V \in \mathcal{U}_{\tau}(x)$ where $F = X - f^{-1}(K)$. By Proposition 1.1.6, $\mathcal{F} = \langle \mathcal{U}_{\tau}(x)\big|_{F} \rangle$ is a filter on X such that $F \in \mathcal{F}$ and $\mathcal{U}_{\tau}(x) \subseteq \mathcal{F}$. This implies $\mathcal{F} \longrightarrow x$ and by Proposition 2.1.1, $\mathcal{F} \propto x$. We claim that $f(\mathcal{F}) \overset{\theta}{\not\propto} f(x)$. Since $F \in \mathcal{F}$, then $f(F) \in f(\mathcal{F})$. Now, $K \cap f(F) = K \cap f(X - f^{-1}(K)) = K \cap f(f^{-1}(Y - K)) \subseteq K \cap (Y - K) = \emptyset$. Hence, we have $K \in \mathcal{C}^{\sigma}(f(x))$, $f(F) \in f(\mathcal{F})$ and $K \cap f(F) = \emptyset$. Therefore, $f(\mathcal{F}) \overset{\theta}{\not\propto} f(x)$ in Y. □

4.4.2 Strongly-θ-Continuous Functions

Definition 4.4.2. [66] A function $f : (X, \tau) \to (Y, \sigma)$ is *strongly-θ-continuous* at $x \in X$ if for every open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\overline{U}) \subseteq V$. Equivalently, for every $V \in \sigma(f(x))$, there exists $H \in \mathcal{C}^{\tau}(x)$ such that $f(H) \subseteq V$. If this condition is satisfied at each $x \in X$, then f is said to be strongly-θ-continuous on X.

Theorem 4.4.4. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is strongly-θ-continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \xrightarrow{\theta} x$, then $f(\mathcal{F}) \longrightarrow f(x)$ in Y.

Proof. Assume that $\mathcal{F} \xrightarrow{\theta} x$ in X and $V \in \mathcal{U}_{\sigma}(f(x))$. Since f is strongly-θ-continuous at x, then there exists $H \in \mathcal{C}^{\tau}(x)$ such that $f(H) \subseteq V$. But since $\mathcal{F} \xrightarrow{\theta} x$, then $H \in \mathcal{F}$, and hence $V \in f(\mathcal{F})$. Thus, $\mathcal{U}_{\sigma}(f(x)) \subseteq f(\mathcal{F})$, Therefore, $f(\mathcal{F}) \longrightarrow f(x)$.

Conversely, let $V \in \mathcal{U}_{\sigma}(f(x))$. Since $\langle \mathcal{C}^{\tau}(x) \rangle \subseteq \langle \mathcal{C}^{\tau}(x) \rangle$, then $\langle \mathcal{C}^{\tau}(x) \rangle \xrightarrow{\theta} x$. By hypothesis, we have $f(\langle \mathcal{C}^{\tau}(x) \rangle) \longrightarrow f(x)$. So, $\mathcal{U}_{\sigma}(f(x)) \subseteq f(\langle \mathcal{C}^{\tau}(x) \rangle)$. Therefore, there exists $H \in \mathcal{C}^{\tau}(x)$ such that $f(H) \subseteq V$. That is, f is strongly-θ-continuous at $x \in X$. □

Theorem 4.4.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is strongly-θ-continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \overset{\theta}{\propto} x$, then $f(\mathcal{F}) \propto f(x)$ in Y.

Proof. Suppose that $\mathcal{F} \overset{\theta}{\propto} x$ in X and $V \in \mathcal{U}_\sigma(f(x))$. Since f is strongly-θ-continuous at x, then there exists $H \in \mathcal{C}^\tau(x)$ such that $f(H) \subseteq V$. But $H \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ since $\mathcal{F} \overset{\theta}{\propto} x$. So, $\emptyset \neq f(H \cap F) \subseteq f(H) \cap f(F) \subseteq V \cap f(F)$ for all $F \in \mathcal{F}$. That is, $V \cap f(F) \neq \emptyset$ for all $F \in \mathcal{F}$. Hence, $f(\mathcal{F}) \propto f(x)$ in Y.

Conversely, suppose, by the way of contradiction, that f is not strongly-θ-continuous at $x \in X$, then there exists $V \in \sigma(f(x))$ such that $f(\overline{U}) \nsubseteq V$ for any $U \in \tau(x)$. So, $\overline{U} \nsubseteq f^{-1}(V)$ for any $U \in \tau(x)$. Let $\mathcal{B} = \{\overline{U} - f^{-1}(V) : U \in \tau(x)\}$, then $\mathcal{B}$ is a filter base in X. Let $\mathcal{F} = \langle \mathcal{B} \rangle_X$, then $\mathcal{F}$ is a filter on X. We claim that $\mathcal{F} \overset{\theta}{\propto} x$ but $f(\mathcal{F}) \not\propto f(x)$. Let $U \in \tau(x)$ and $F \in \mathcal{F}$, then $F \supseteq B$ for some $B \in \mathcal{B}$. This implies that $F \supseteq \overline{W} - f^{-1}(V)$ for some $W \in \tau(x)$. Since $U \cap W \in \tau(x)$, then $\overline{U \cap W} - f^{-1}(V) \neq \emptyset$ but $\overline{U} \cap F \supseteq \overline{U} \cap (\overline{W} - f^{-1}(V)) = (\overline{U} \cap \overline{W}) - f^{-1}(V)$ and $\overline{U} \cap \overline{W} \supseteq \overline{U \cap W}$. So, $\overline{U} \cap F \supseteq \overline{U \cap W} - f^{-1}(\overline{V}) \neq \emptyset$. Hence, $\mathcal{F} \overset{\theta}{\propto} x$. Next, since $X \in \tau(x)$, then $B = \overline{X} - f^{-1}(V) = X - f^{-1}(V) \in \mathcal{B} \subseteq \mathcal{F}$, so $f(B) \in f(\mathcal{F})$. We claim that $V \cap f(B) = \emptyset$. For if $f(b) \in V$ for some $b \in B$, then $b \in f^{-1}(V)$ and $b \in X - f^{-1}(V)$, so $b \in f^{-1}(V) \cap (X - f^{-1}(V)) = \emptyset$, which is a contradiction. Since $V \in \sigma(f(x))$, $f(B) \in f(\mathcal{F})$ and $V \cap f(B) = \emptyset$, then $f(\mathcal{F}) \not\propto f(x)$ in Y. $\square$

4.4.3 θ-Continuous Functions

Definition 4.4.3. [37] A function $f : (X, \tau) \to (Y, \sigma)$ is *θ-continuous* at $x \in X$ if for every open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\overline{U}) \subseteq \overline{V}$. Equivalently, for all $G \in \mathcal{C}^\sigma(f(x))$, there exists $H \in \mathcal{C}^\tau(x)$ such that $f(H) \subseteq G$. If this condition is satisfied at each $x \in X$, then f is said to be θ-continuous on X.

A θ-continuous function preserves θ-convergence.

Theorem 4.4.6. [28, 82] Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is θ-continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \xrightarrow{\theta} x$, then $f(\mathcal{F}) \xrightarrow{\theta} f(x)$ in Y.

Proof. Assume that $\mathcal{F} \xrightarrow{\theta} x$ and $G \in \mathcal{C}^{\sigma}(f(x))$. Since f is θ-continuous at x, then there exists $H \in \mathcal{C}^{\tau}(x)$ such that $f(H) \subseteq G$. Also, since $\mathcal{F} \xrightarrow{\theta} x$ and $H \in \mathcal{C}^{\tau}(x)$, then $H \in \mathcal{F}$. So, $G \in f(\mathcal{F})$. Thus, $\langle \mathcal{C}^{\sigma}(f(x)) \rangle \subseteq f(\mathcal{F})$. That is, $f(\mathcal{F}) \xrightarrow{\theta} f(x)$.

Conversely, let $G \in \mathcal{C}^{\sigma}(f(x))$. Since $\langle \mathcal{C}^{\tau}(x) \rangle \subseteq \langle \mathcal{C}^{\tau}(x) \rangle$, then $\langle \mathcal{C}^{\tau}(x) \rangle \xrightarrow{\theta} x$. By hypothesis, we have $f(\langle \mathcal{C}^{\tau}(x) \rangle) \xrightarrow{\theta} f(x)$. That is, $\langle \mathcal{C}^{\sigma}(f(x)) \rangle \subseteq f(\langle \mathcal{C}^{\tau}(x) \rangle)$. Hence, there exists $H \in \mathcal{C}^{\tau}(x)$ such that $f(H) \subseteq G$. Therefore, f is θ-continuous at $x \in X$. □

Theorem 4.4.7. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is θ-continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \overset{\theta}{\propto} x$, then $f(\mathcal{F}) \overset{\theta}{\propto} f(x)$ in Y.

Proof. Suppose that $\mathcal{F} \overset{\theta}{\propto} x$ and $G \in \mathcal{C}^{\sigma}(f(x))$. Since f is θ-continuous, then there exists $H \in \mathcal{C}^{\tau}(x)$ such that $f(H) \subseteq G$. But $H \cap F \neq \emptyset$, for all $F \in \mathcal{F}$ since $\mathcal{F} \overset{\theta}{\propto} x$. So, $\emptyset \neq f(H \cap F) \subseteq f(H) \cap f(F) \subseteq G \cap f(F)$ for all $F \in \mathcal{F}$. That is, $G \cap f(F) \neq \emptyset$ for all $F \in \mathcal{F}$. Hence, $f(\mathcal{F}) \overset{\theta}{\propto} f(x)$.

Conversely, suppose, by the way of contradiction, that f is not θ-continuous at $x \in X$, then there exists $V \in \sigma(f(x))$ such that $f(\overline{U}) \nsubseteq \overline{V}$ for any $U \in \tau(x)$. So, $\overline{U} \nsubseteq f^{-1}(\overline{V})$ for any $U \in \tau(x)$. Let $\mathcal{B} = \{\overline{U} - f^{-1}(\overline{V}) : U \in \tau(x)\}$, then $\mathcal{B}$ is a filter base in X. Let $\mathcal{F} = \langle \mathcal{B} \rangle_X$, then $\mathcal{F}$ is a filter on X. We claim that $\mathcal{F} \overset{\theta}{\propto} x$ but $f(\mathcal{F}) \overset{\theta}{\not\propto} f(x)$. Let $U \in \tau(x)$ and $F \in \mathcal{F}$, then $F \supseteq B$ for some $B \in \mathcal{B}$. This implies that $F \supseteq \overline{W} - f^{-1}(\overline{V})$ for some $W \in \tau(x)$. Since $U \cap W \in \tau(x)$, then $\overline{U \cap W} - f^{-1}(\overline{V}) \neq \emptyset$ but $\overline{U} \cap F \supseteq \overline{U} \cap (\overline{W} - f^{-1}(\overline{V})) = (\overline{U} \cap \overline{W}) - f^{-1}(\overline{V})$ and $\overline{U} \cap \overline{W} \supseteq \overline{U \cap W}$. So, $\overline{U} \cap F \supseteq \overline{U \cap W} - f^{-1}(\overline{V}) \neq \emptyset$. Hence, $\mathcal{F} \overset{\theta}{\propto} x$. Next, since $X \in \tau(x)$, then $B = \overline{X} - f^{-1}(\overline{V}) = X - f^{-1}(\overline{V}) \in \mathcal{B} \subseteq \mathcal{F}$, so $f(B) \in f(\mathcal{F})$. We claim that $\overline{V} \cap f(B) = \emptyset$. For if $f(b) \in \overline{V}$ for some $b \in B$, then $b \in f^{-1}(\overline{V})$ and $b \in X - f^{-1}(\overline{V})$, so $b \in f^{-1}(\overline{V}) \cap (X - f^{-1}(\overline{V})) = \emptyset$, which is a contradiction. Since $V \in \sigma(f(x))$, $f(B) \in f(\mathcal{F})$ and $\overline{V} \cap f(B) = \emptyset$, then $f(\mathcal{F}) \overset{\theta}{\not\propto} f(x)$ in Y. □

In the following proposition, we give the relation between the different types of continuity which have already been used.

Proposition 4.4.1. Strongly-θ-continuity $\implies$ continuity $\implies$ θ-continuity $\implies$ weakly-θ-continuity.

4.4.4 More on Functions and θ-Convergence

Theorem 4.4.8. Let $f : X \to Y$ be a function and X be regular. Then f is continuous if and only if f is strongly-θ-continuous.

Proof. Suppose that f is continuous and X is regular. Let $x \in X$, and $\mathcal{F}$ be a filter on X such that $\mathcal{F} \xrightarrow{\theta} x$. Since X is regular, then by Theorem 4.3.1, $\mathcal{F} \longrightarrow x$. But f is continuous at x, so by Theorem 2.3.1, $f(\mathcal{F}) \longrightarrow f(x)$. Hence, by Theorem 4.4.4, f is strongly-θ-continuous at x. Therefore, f is strongly-θ-continuous since x was arbitrary. The converse follows from Proposition 4.4.1. $\square$

Theorem 4.4.9. Let $f : X \to Y$ be a function and Y be regular. Then f is weakly-θ-continuous if and only if f is continuous.

Proof. Suppose that f is weakly-θ-continuous and Y is regular. Let $x \in X$ and $\mathcal{F}$ be a filter on X such that $\mathcal{F} \longrightarrow x$. Since f is weakly-θ-continuous, then $f(\mathcal{F}) \xrightarrow{\theta} f(x)$ by Theorem 4.4.2. But Y is regular, then by Theorem 4.3.1, $f(\mathcal{F}) \longrightarrow f(x)$. Therefore, by Theorem 2.3.1, f is continuous at x. Thus, f is continuous since x was arbitrary. The converse follows from Proposition 4.4.1. $\square$

Corollary 4.4.1. *Let $f : X \to Y$ be a function and Y be regular. Then f is θ-continuous if and only if f is continuous.*

Proof. This follows from Theorem 4.4.9 and the fact that every θ-continuous function is weakly-θ-continuous. $\square$

Theorem 4.4.10. [51] Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of topological spaces and let $\mathcal{F}$ be a filter on $X = \prod\limits_{\alpha \in \Delta} X_\alpha$. Then $\mathcal{F} \xrightarrow{\theta} x$ in X if and only if $\pi_\alpha(\mathcal{F}) \xrightarrow{\theta} \pi_\alpha(x)$ in X_α for all $\alpha \in \Delta$.

Proof. Assume that $\mathcal{F} \xrightarrow{\theta} x$ in X. Since π_α is continuous for all $\alpha \in \Delta$, then π_α is θ-continuous for all $\alpha \in \Delta$. So, by Theorem 4.4.6, $\pi_\alpha(\mathcal{F}) \xrightarrow{\theta} \pi_\alpha(x)$ for all $\alpha \in \Delta$.

Conversely, suppose that $\pi_\alpha(\mathcal{F}) \xrightarrow{\theta} \pi_\alpha(x)$ for all $\alpha \in \Delta$. Let U be any neighborhood of x in X. Then $x \in \bigcap\limits_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i) \subseteq U$, where $U_i \in \mathcal{U}(\pi_{\alpha_i}(x))$ for all

$i = 1, \ldots, n$. So, $\overline{U}_i \in \pi_{\alpha_i}(\mathcal{F})$ for all $i = 1, \ldots, n$, and hence for all $i = 1, \ldots, n$, there exists $F_i \in \mathcal{F}$ such that $\pi_{\alpha_i}(F_i) \subseteq \overline{U}_i$. Then, $F_i \subseteq \pi_{\alpha_i}^{-1}\left(\overline{U}_i\right)$ for all $i = 1, \ldots, n$. So, $\bigcap_{i=1}^{n} F_i \subseteq \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}\left(\overline{U}_i\right)$. Since $\bigcap_{i=1}^{n} F_i \in \mathcal{F}$, then $\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}\left(\overline{U}_i\right) \in \mathcal{F}$. But by Lemmas 2.3.1 and 3.4.1, $\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}\left(\overline{U}_i\right) = \prod_{i=1}^{n} \overline{U}_i = \overline{\prod_{i=1}^{n} U_i} = \overline{\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i)} \subseteq \overline{U}$. So, $\overline{U} \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{\theta} x$ in X. □

Theorem 4.4.11. Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of topological spaces and let $\mathcal{F}$ be a filter on $X = \prod_{\alpha \in \Delta} X_\alpha$. If $\mathcal{F} \overset{\theta}{\propto} x$ in X, then $\pi_\alpha(\mathcal{F}) \overset{\theta}{\propto} \pi_\alpha(x)$ in X_α for all $\alpha \in \Delta$.

Proof. Assume that $\mathcal{F} \overset{\theta}{\propto} x$. Since π_α is continuous for all $\alpha \in \Delta$, then π_α is θ-continuous for all $\alpha \in \Delta$. By Theorem 4.4.5, $\pi_\alpha(\mathcal{F}) \overset{\theta}{\propto} \pi_\alpha(x)$ for all $\alpha \in \Delta$. □

4.5 Quasi-H-Closed Spaces

4.5.1 Characterizations of Quasi-H-Closed Spaces

Definition 4.5.1. [43, 74] A topological space X is called *quasi-H-closed* iff every open cover of X has a finite subfamily whose closures cover X. A quasi-H-closed Hausdorff space is called an *H-closed* space.

Recall that a set U is regular open if $\overline{U}^{\circ} = U$. In view of the fact that for any open set U, $\overline{U}^{\circ}$ is regular open by Proposition 1.2.7 part (i), it follows immediately that the open sets in Definition 4.5.1 may be replaced with regular open sets and an equivalent definition obtained.

Theorem 4.5.1. [18] The property of a topological space being quasi-H-closed is a semi-regular property. That is, (X, τ) is quasi-H-closed if and only if (X, τ_s) is quasi-H-closed.

Proof. Let $\mathcal{U} \subseteq \tau_s$ be a cover of X, then $\mathcal{U} \subseteq \tau$ is a cover of X but (X, τ) is quasi-H-closed, so there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $X = \bigcup_{i=1}^{n} \mathrm{Cl}_\tau(U_i)$. But since

$U_i \in \tau$ for any $i = 1, \ldots, n$, then by Lemma 1.3.1 part (i), $\mathrm{Cl}_\tau(U_i) = \mathrm{Cl}_{\tau_s}(U_i)$ for any $i = 1, \ldots, n$, thus $X = \bigcup_{i=1}^{n} \mathrm{Cl}_{\tau_s}(U_i)$. Hence, (X, τ_s) is quasi-H-closed.

Conversely, let $\mathcal{U} \subseteq \mathrm{RO}(X, \tau)$ be a cover of X. Then $\mathcal{U} \subseteq \tau_s$ is a cover of X but (X, τ_s) is quasi-H-closed, so there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $X = \bigcup_{i=1}^{n} \mathrm{Cl}_{\tau_s}(U_i)$. But since $U_i \in \tau$ for any $i = 1, \ldots, n$, then by Lemma 1.3.1 part (i), $\mathrm{Cl}_{\tau_s}(U_i) = \mathrm{Cl}_\tau(U_i)$ for any $i = 1, \ldots, n$, thus $X = \bigcup_{i=1}^{n} \mathrm{Cl}_\tau(U_i)$. Hence, (X, τ) is quasi-H-closed. □

Definition 4.5.2. [34] A subset A of a topological space X is said to be

(i) a *quasi-H-closed subspace* if the space (A, τ_A) is quasi-H-closed.

(ii) a *quasi-H-closed relative* to X if for every cover $\{V_\alpha : \alpha \in \Delta\}$ of A by open sets in X, there exists a finite subset Ω of Δ such that $A \subseteq \bigcup_{\alpha \in \Omega} \overline{V}_\alpha$.

If the space X is Hausdorff and $A \subseteq X$ is a quasi-H-closed subspace (quasi-H-closed relative to X), then A is called an H-closed subspace (H-closed relative to X).

Remark 4.4. *Every quasi-H-closed subspace of a topological space X is a quasi-H-closed relative to X, but the converse doesn't hold in general [93, p. 161].*

Proposition 4.5.1. Every compact space is quasi-H-closed.

Proof. If X is compact and $\mathcal{U}$ is any open cover for X, then there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $X = \bigcup_{i=1}^{n} U_i$. So, $X = \bigcup_{i=1}^{n} U_i \subseteq \bigcup_{i=1}^{n} \overline{U}_i \subseteq X$. Thus, $X = \bigcup_{i=1}^{n} \overline{U}_i$. Hence, X is quasi-H-closed. □

It should be noted that the converse of the above proposition need not be true, as the following example shows.

Example 4.5.1. *Let $X = \mathbb{R}$ with the left ray topology. Let $\mathcal{U}$ be any open cover of $\mathbb{R}$. Then for any $U \in \mathcal{U}$, we have $\overline{U} = \mathbb{R}$. Select $U_\circ \in \mathcal{U}$ so that $U_\circ \neq \emptyset$. Then $\overline{U}_\circ = \mathbb{R}$. So, $\{U_\circ\}$ is a finite subfamily of $\mathcal{U}$ such that $\overline{U}_\circ = \mathbb{R}$. Thus, X with this topology is quasi-H-closed. Yet, X is not compact.*

We are now ready to make characterizations of quasi-H-closed spaces using our main concept, namely, the θ-convergence of filters.

Theorem 4.5.2. [47, 74, 115] For a topological space (X, τ), the following are equivalent:

(i) X is quasi-H-closed.

(ii) Every filter on X has a θ-cluster point.

(iii) Every ultrafilter on X θ-converges.

(iv) For every family $\mathcal{C}$ of closed sets of X such that $\bigcap_{C\in\mathcal{C}} C = \emptyset$, there exists a finite subfamily $\mathcal{C}'$ of $\mathcal{C}$ such that $\bigcap_{C\in\mathcal{C}'} C^\circ = \emptyset$.

Proof.

(i) $\Longrightarrow$ (ii) Assume that there exists a filter $\mathcal{F}$ on X such that $\mathcal{F} \overset{\theta}{\not\propto} x$ for all $x \in X$. This means that for all $x \in X$, there exist $G_x \in \tau(x)$ and $F_x \in \mathcal{F}$ such that $F_x \cap \overline{G}_x = \emptyset$. Let $\mathcal{U} = \{G_x : x \in X\}$, then $\mathcal{U}$ is an open cover of X. But X is quasi-H-closed, so there exist $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} \overline{G}_{x_i}$. Now, for all $i = 1, \ldots, n$, choose $F_{x_i} \in \mathcal{F}$ such that $F_{x_i} \cap \overline{G}_{x_i} = \emptyset$ and let $F_\circ = \bigcap_{i=1}^{n} F_{x_i}$. Then, $F_\circ \in \mathcal{F}$ and $F_\circ \subseteq F_{x_i}$ for each $i = 1, \ldots, n$. So,

$$F_\circ = F_\circ \cap X = F_\circ \cap (\bigcup_{i=1}^{n} \overline{G}_{x_i}) = \bigcup_{i=1}^{n} (F_\circ \cap \overline{G}_{x_i}) \subseteq \bigcup_{i=1}^{n} (F_{x_i} \cap \overline{G}_{x_i}) = \emptyset.$$

This implies $F_\circ = \emptyset$, which is a contradiction. Therefore, $\mathcal{F} \overset{\theta}{\propto} x$ for some $x \in X$.

(ii) $\Longrightarrow$ (iii) Let $\mathcal{F}$ be an ultrafilter on X. Then, by hypothesis, $\mathcal{F} \overset{\theta}{\propto} x \in X$. But then, $\mathcal{F} \xrightarrow{\theta} x$ by Theorem 4.1.12 and since $\mathcal{F}$ is an ultrafilter on X.

(iii) $\Longrightarrow$ (iv) Let $\mathcal{C} = \{F_\alpha : \alpha \in \Delta\}$ be a family of closed subsets of X with $\bigcap_{\alpha\in\Delta} F_\alpha = \emptyset$. Suppose, by the way of contradiction, that for each finite subset Ω of Δ, $\bigcap_{\alpha\in\Omega} F_\alpha^\circ \neq \emptyset$. Let $\mathcal{B} = \{\bigcap_{\alpha\in\Omega} F_\alpha^\circ : \Omega \text{ is a finite subset of } \Delta\}$. Then, $\mathcal{B}$ is a filter base in X. Hence, the filter $\mathcal{F} = \langle\mathcal{B}\rangle_X$ is contained in some ultrafilter $\mathcal{F}'$ in X by Theorem 1.1.4. So, by hypothesis, $\mathcal{F}' \xrightarrow{\theta} x \in X$. But then $\mathcal{F}' \overset{\theta}{\propto} x$. Hence $\mathcal{F} \overset{\theta}{\propto} x$ by Theorem 4.1.11. Therefore, we have constructed a filter $\mathcal{F}$ on X which has $x \in X$ as a θ-cluster point. Now, $x \notin \emptyset = \bigcap_{\alpha\in\Delta} F_\alpha$, then $x \notin F_{\alpha_\circ}$ for some $\alpha_\circ \in \Delta$. So, $U_\circ = X - F_{\alpha_\circ} \in \tau(x)$. But $F^\circ_{\alpha_\circ} \in \mathfrak{F}$ by the construction of $\mathfrak{F}$. So, $\overline{U}_\circ \cap F^\circ_{\alpha_\circ} \neq \emptyset$ since $\mathfrak{F} \overset{\theta}{\propto} x$. On the other hand,

$$\overline{U}_\circ \cap F^\circ_{\alpha_\circ} = \overline{X - F_{\alpha_\circ}} \cap F^\circ_{\alpha_\circ} = (X - F^\circ_{\alpha_\circ}) \cap F^\circ_{\alpha_\circ} = \emptyset,$$

which is a contradiction. Therefore, there exists a finite subset Ω of Δ such that $\bigcap\limits_{\alpha \in \Omega} F_\alpha \neq \emptyset$.

(iv) $\Longrightarrow$ (i) Let $\mathfrak{U}$ be an open cover of X. Then $\mathfrak{C} = \{X - U : U \in \mathfrak{U}\}$ is a family of closed subsets of X with $\bigcap\limits_{U \in \mathfrak{U}} (X - U) = X - \bigcup\limits_{U \in \mathfrak{U}} U = X - X = \emptyset$. So, by hypothesis, there exist $U_1, \dots, U_n \in \mathfrak{U}$ such that $\bigcap\limits_{i=1}^{n} (X - U_i)^\circ = \emptyset$. So, $\emptyset = \bigcap\limits_{i=1}^{n} (X - U_i)^\circ = \bigcap\limits_{i=1}^{n} (X - \overline{U}_i) = X - \bigcup\limits_{i=1}^{n} \overline{U}_i$. Hence, $X = \bigcup\limits_{i=1}^{n} \overline{U}_i$. Therefore, X is quasi-H-closed.

□

Theorem 4.5.3. [28, 97] Let X be a topological space and $A \subseteq X$. Then the following are equivalent:

(i) A is a quasi-H-closed relative to X.

(ii) Every filter on X which meets A θ-accumulates at some point of A.

(iii) Every ultrafilter on X which meets A θ-converges to some point of A.

(iv) For every family $\mathfrak{C}$ of closed sets of X such that $\left(\bigcap\limits_{C \in \mathfrak{C}} C\right) \cap A = \emptyset$, there exists a finite subfamily $\mathfrak{C}'$ of $\mathfrak{C}$ such that $\left(\bigcap\limits_{C \in \mathfrak{C}'} C^\circ\right) \cap A = \emptyset$.

Proof. Similar to the proof of Theorem 4.5.2. □

Theorem 4.5.4. Let X be a regular space. Then X is quasi-H-closed if and only if X is compact.

Proof. Assume that X is quasi-H-closed. Let $\mathfrak{F}$ be a filter on X. Then $\mathfrak{F}$ has a θ-cluster point $x \in X$ by Theorem 4.5.2. But X is regular. So, x is a cluster point of $\mathfrak{F}$ by Theorem 4.3.3. But then, X is compact by Theorem 2.4.7. The converse follows from Proposition 4.5.1. □

Theorem 4.5.5. A θ-closed subset of a quasi-H-closed space X is a quasi-H-closed relative to X.

Proof. Let X be quasi-H-closed and $A \subseteq X$ be θ-closed. Let $\mathcal{F}$ be an ultrafilter on X which meets A. Then by Theorem 4.5.2, $\mathcal{F} \xrightarrow{\theta} x$ for some $x \in X$ since X is quasi-H-closed. Now, since $\mathcal{F}$ is an ultrafilter on X and $\mathcal{F}$ meets A, then $A \in \mathcal{F}$. So, we have $\mathcal{F} \xrightarrow{\theta} x$ and $A \in \mathcal{F}$ but A is θ-closed, so by Theorem 4.1.6, $x \in A$. Hence, every ultarfilter $\mathcal{F}$ on X which meets A θ-converges to some point of A Therefore, A is is a quasi-H-closed relative to X by Theorem 4.5.3. $\square$

We point out that Uryshon spaces need not be regular. This fact makes the following theorem far from being trivial.

Theorem 4.5.6. Let X be an Urysohn space and $A \subseteq X$. If A is a quasi-H-closed relative to X, then A is θ-closed.

Proof. Let A be a quasi-H-closed relative to X and X be an Urysohn space. Let $x \in \theta\text{-Cl}(A)$. Then, by Theorem 4.1.5, there exists a filter on X which meets A such that $\mathcal{F} \xrightarrow{\theta} x$. But since A is a quasi-H-closed relative to X, then by Theorem 4.5.3, $\mathcal{F} \overset{\theta}{\propto} a$ for some $a \in A$. But also by Theorem 4.1.10, $\mathcal{F}$ has a subfilter $\mathcal{F}'$ such that $\mathcal{F}' \xrightarrow{\theta} a$. Also, by Theorem 4.1.8, $\mathcal{F}' \xrightarrow{\theta} x$ since $\mathcal{F}'$ is a subfilter of $\mathcal{F}$ and $\mathcal{F} \xrightarrow{\theta} x$. Now, X is Urysohn implies $x = a$ by Theorem 4.2.1. Therefore, $x \in A$. So, $\theta\text{-Cl}(A) \subseteq A$. Hence, $A \subseteq \theta\text{-Cl}(A) \subseteq A$. Thus, $\theta\text{-Cl}(A) = A$. Therefore, A is θ-closed. $\square$

4.5.2 Quasi-H-Closedness and Functions

Theorem 4.5.7. Let $f : X \to Y$ be a θ-continuous function. If $A \subseteq X$ is a quasi-H-closed relative to X, then $f(A) \subseteq Y$ is a quasi-H-closed relative to Y.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be θ-continuous. Let $A \subseteq X$ be a quasi-H-closed relative to X. Let $\mathcal{G}$ be a filter on Y which meets $f(A)$. Then $f^{-1}(\mathcal{G})$ is a filter on X which meets A. But A is a quasi-H-closed relative to X. Then by Theorem 4.5.3, $f^{-1}(\mathcal{G}) \overset{\theta}{\propto} a$ for some $a \in A$. But f is θ-continuous, then Theorem 4.4.7,

$ff^{-1}(\mathcal{G}) \overset{\theta}{\propto} f(a)$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$. So, by Theorem 4.1.11, $\mathcal{G} \overset{\theta}{\propto} f(a)$. Therefore, $f(A)$ is a quasi-H-closed relative to Y by Theorem 4.5.3. $\square$

Theorem 4.5.8. Let $f : X \to Y$ be a weakly-θ-continuous function. If $A \subseteq X$ is compact, then $f(A) \subseteq Y$ is a quasi-H-closed relative to Y.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be weakly-θ-continuous. Let $A \subseteq X$ be compact. Let $\mathcal{G}$ be a filter on Y which meets $f(A)$. Then $f^{-1}(\mathcal{G})$ is a filter on X which meets A. But A is compact in X. Then by Theorem 2.4.3, $f^{-1}(\mathcal{G}) \propto a$ for some $a \in A$. But f is weakly-θ-continuous, then Theorem 4.4.3, $ff^{-1}(\mathcal{G}) \overset{\theta}{\propto} f(a)$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$. So, by Theorem 4.1.11, $\mathcal{G} \overset{\theta}{\propto} f(a)$. Therefore, $f(A)$ is a quasi-H-closed relative to Y by Theorem 4.5.3. $\square$

Theorem 4.5.9. Let $f : X \to Y$ be a strongly-θ-continuous function. If $A \subseteq X$ is a quasi-H-closed relative to X, then $f(A) \subseteq Y$ is compact.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be strongly-θ-continuous. Let $A \subseteq X$ be a quasi-H-closed relative to X. Let $\mathcal{G}$ be a filter on Y which meets $f(A)$. Then $f^{-1}(\mathcal{G})$ is a filter on X which meets A. But A is a quasi-H-closed relative to X. Then by Theorem 4.5.3, $f^{-1}(\mathcal{G}) \overset{\theta}{\propto} a$ for some $a \in A$. But f is strongly-θ-continuous, then Theorem 4.4.5, $ff^{-1}(\mathcal{G}) \propto f(a)$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$. So, by Theorem 2.1.9, $\mathcal{G} \propto f(a)$. Therefore, $f(A)$ is compact by Theorem 2.4.3. $\square$

Theorem 4.5.10. [43] The product $X = \prod\limits_{\alpha \in \Delta} X_\alpha$ is quasi-H-closed if and only if each space X_α, $\alpha \in \Delta$ is quasi-H-closed.

Proof. Assume that X is quasi-H-closed. The projection π_α is continuous and onto for all $\alpha \in \Delta$. But any continuous function is θ-continuous. So, π_α is θ-continuous for all $\alpha \in \Delta$. From Theorem 4.5.7, it follows that $\pi_\alpha(X) = X_\alpha$ is quasi-H-closed for all $\alpha \in \Delta$.

Conversely, let $\mathcal{F}$ be an ultrafilter on X. Since π_α is onto for all $\alpha \in \Delta$, then by Theorem 1.1.6, $\pi_\alpha(\mathcal{F})$ is an ultrafilter on X_α for all $\alpha \in \Delta$. But X_α is quasi-H-closed for all $\alpha \in \Delta$, so $\pi_\alpha(\mathcal{F}) \xrightarrow{\theta} x_\alpha \in X_\alpha$ for all $\alpha \in \Delta$, by Theorem 4.5.2. Let $x = (x_\alpha)_{\alpha \in \Delta}$, then $x \in X$ and $\pi_\alpha(x) = x_\alpha$ for all $\alpha \in \Delta$. So, $\pi_\alpha(\mathcal{F}) \xrightarrow{\theta} \pi_\alpha(x)$

for all $\alpha \in \Delta$. Hence, $\mathcal{F} \xrightarrow{\theta} x$ by Theorem 4.4.10. Therefore, $X = \prod_{\alpha \in \Delta} X_\alpha$ is quasi-H-closed by Theorem 4.5.2. □

4.5.3 Strongly Closed Graphs

Definition 4.5.3. [65] A function $f : (X, \tau) \to (Y, \sigma)$ is said to have a *strongly closed graph* if for each $(x, y) \notin \Gamma_f$, there exist $U \in \tau(x)$ and $G \in \mathcal{C}^\sigma(y)$ such that $(U \times G) \cap \Gamma_f = \emptyset$.

Theorem 4.5.11. [65] Let $f : X \to Y$ be a function, then f has a strongly closed graph if and only if for each $x \in X$ and each $y \in Y$, with $(x, y) \notin \Gamma_f$, there exist $U \in \tau(x)$ and $G \in \mathcal{C}^\sigma(y)$ such that $f(U) \cap G = \emptyset$.

Proof. The straightforward proof follows from Definition 4.5.3 and is omitted. □

Of course, a function with a strongly closed graph has a closed graph, but the converse is not true as shown by the following example.

Example 4.5.2. *[65] Let X be the closed unit interval $[0, 1]$ and let Y be the upper half-plane $\{(x, y) : y \geq 0\}$ with the half-disc topology [109, p. 96]. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(x) = (x, 0)$ if $x \neq 0$ and $f(0) = (1, 1)$. Then f has a closed graph but it doesn't have a strongly closed graph.*

The following theorem and corollary give a characterization of functions with strongly closed graph in terms of θ-convergence of filters.

Theorem 4.5.12. [52] A function $f : (X, \tau) \to (Y, \sigma)$ has a strongly closed graph if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \longrightarrow x$ and $f(\mathcal{F}) \xrightarrow{\theta} y$ in Y, then $(x, y) \in \Gamma_f$.

Proof. Assume that f has a strongly closed graph. Let $\mathcal{F} \longrightarrow x$ and $f(\mathcal{F}) \xrightarrow{\theta} y$. Suppose on the contrary that $(x, y) \notin \Gamma_f$. Since f has a strongly closed graph, then by Theorem 4.5.11, there exist $U \in \tau(x)$ and $G \in \mathcal{C}^\sigma(y)$ such that $f(U) \cap G = \emptyset$.

But since $\mathcal{F} \longrightarrow x$ and $U \in \tau(x)$, then $U \in \mathcal{F}$, and hence $f(U) \in f(\mathcal{F})$. On the other hand, $f(\mathcal{F}) \xrightarrow{\theta} y$ and $G \in \mathcal{C}^\sigma(y)$, so $G \in f(\mathcal{F})$. Thus, $f(U) \cap G \neq \emptyset$, which is a contradiction. Therefore, $(x, y) \in \Gamma_f$.

Conversely, suppose on the contrary that f does not have a strongly closed graph, then there exists $x \in X$, $y \in Y$ with $(x, y) \notin \Gamma_f$ such that $f(U) \cap G \neq \emptyset$ for all $U \in \tau(x)$ and all $G \in \mathcal{C}^\sigma(y)$. This implies that $U \cap f^{-1}(G) \neq \emptyset$ for all $U \in \tau(x)$ and all $G \in \mathcal{C}^\sigma(y)$. Let $\mathcal{F} = \{F \subseteq X : F \supseteq U \cap f^{-1}(G), U \in \tau(x), G \in \mathcal{C}^\sigma(y)\}$, then $\mathcal{F}$ is a filter on X. We claim that $\mathcal{F} \longrightarrow x$ and $f(\mathcal{F}) \xrightarrow{\theta} y$. First, let $U_\circ \in \tau(x)$. Then $U_\circ \supseteq U_\circ \cap f^{-1}(G)$ for each $G \in \mathcal{C}^\sigma(y)$. Hence, $U_\circ \in \mathcal{F}$. Next, let $G_\circ \in \mathcal{C}^\sigma(y)$. Then $G_\circ \supseteq f(f^{-1}(G_\circ)) \supseteq f(U \cap f^{-1}(G_\circ))$ for each $U \in \tau(x)$, but $U \cap f^{-1}(G_\circ) \in \mathcal{F}$ for each $U \in \tau(x)$. So, $G_\circ \in f(\mathcal{F})$. Therefore, we have constructed a filter $\mathcal{F} \longrightarrow x$ in X for which $f(\mathcal{F}) \xrightarrow{\theta} y$ in Y. By hypothesis, $(x, y) \in \Gamma_f$ which is a contradiction. Therefore, Γ_f is a strongly closed graph. □

Corollary 4.5.1. *[41] Let $f : (X, \tau) \to (Y, \sigma)$ be any function, where Y is a regular space. Then the following are equivalent:*

(i) f has a strongly closed graph.

(ii) If a filter $\mathcal{F} \longrightarrow x$ in X and $f(\mathcal{F}) \xrightarrow{\theta} y$ in Y, then $(x, y) \in \Gamma_f$.

(iii) If a filter $\mathcal{F} \longrightarrow x$ in X and $f(\mathcal{F}) \longrightarrow y$ in Y, then $(x, y) \in \Gamma_f$.

(iv) f has a closed graph.

Proof.

(i) $\Longrightarrow$ (ii) Follows from Theorem 4.5.12.

(ii) $\Longrightarrow$ (iii) Follows from Theorem 4.3.1 and the fact that Y is a regular space.

(iii) $\Longrightarrow$ (iv) Follows from Theorem 2.4.9.

(iv) $\Longrightarrow$ (i) Suppose that f has a closed graph. Let $\mathcal{F}$ be a filter on X with $\mathcal{F} \longrightarrow x$ and $f(\mathcal{F}) \xrightarrow{\theta} y$. Since Y is regular, then by Theorem 4.3.1, $f(\mathcal{F}) \longrightarrow y$. But f has a closed graph, so by Theorem 2.4.9, $(x, y) \in \Gamma_f$. Therefore, f has a strongly closed graph by Theorem 4.5.12.

□

The graph of a weakly-θ-continuous function need not be strongly closed as it is shown in the next example.

Example 4.5.3. *In Example 2.4.2, we have the function f is continuous, and hence f is weakly-θ-continuous. But the graph Γ_f is not strongly closed graph since $(1,1) \notin \Gamma_f$ but for any $U \in \tau(1)$ and $W \in \mathcal{C}^{\sigma}(1)$, we have $(1,-1) \in (U \times W) \cap \Gamma_f$.*

We are now ready to give a sufficient condition on the codomain of a weakly-θ-continuous function f to insure that it has a strongly closed graph.

Theorem 4.5.13. [44, 65] If $f : (X, \tau) \to (Y, \sigma)$ is weakly-θ-continuous and (Y, σ) is Urysohn, then f has a strongly closed graph.

Proof. Suppose that $\mathcal{F}$ is a filter on X with $\mathcal{F} \longrightarrow x \in X$ and $f(\mathcal{F}) \xrightarrow{\theta} y \in Y$. Since f is weakly-θ-continuous, then by Theorem 4.4.2, $f(\mathcal{F}) \xrightarrow{\theta} f(x)$. But Y is Urysohn implies $f(x) = y$ by Theorem 4.2.1. So, $(x, y) \in \Gamma_f$. Hence, by Theorem 4.5.12, f has a strongly closed graph. $\square$

Example 4.5.4. *Consider the identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$, where τ and σ are the usual and discrete topologies on $\mathbb{R}$, respectively. Then f has a strongly closed graph but f is not weakly-θ-continuous.*

We are now ready to give a sufficient condition on the codomain of a function f has a strongly closed graph to insure that it is weakly-θ-continuous.

Theorem 4.5.14. [47] Let (Y, σ) be an H-closed space. For every topological space (X, τ), each function $f : (X, \tau) \to (Y, \sigma)$ with a strongly closed graph is weakly-θ-continuous.

Proof. Let $x \in X$ and $V \in \sigma(f(x))$. For each $y \in Y - V$, we have $y \neq f(x)$, this means for each $y \in Y - V$, $(x, y) \notin \Gamma_f$. But Γ_f is strongly closed, then by Theorem 4.5.11, there exist $U_y \in \tau(x)$ and $V_y \in \sigma(y)$ such that $f(U_y) \cap \overline{V}_y = \emptyset$. Let $\mathcal{V} = \{V\} \cup \{V_y : y \in Y - V\}$, then $\mathcal{V}$ is an open cover for Y. But Y is H-closed, then there is a finite subfamily, say $\mathcal{V}' = \left\{V, V_{y_1}, \ldots, V_{y_n}\right\}$, of $\mathcal{V}$ such that $Y = \overline{V} \cup \bigcup\limits_{i=1}^{n} \overline{V}_{y_i}$. Let $U = \bigcap\limits_{i=1}^{n} U_{y_i}$. Then $U \in \tau(x)$ and $U \subseteq U_{y_i}$ for all $i = 1, \ldots, n$. Now, $f(U) \cap \Big(\bigcup\limits_{i=1}^{n} \overline{V}_{y_i}\Big) = \bigcup\limits_{i=1}^{n} \Big(f(U) \cap \overline{V}_{y_i}\Big) \subseteq \bigcup\limits_{i=1}^{n} \Big(f(U_{y_i}) \cap \overline{V}_{y_i}\Big) = \emptyset$. This implies that, $f\big(U\big) \subseteq Y - \bigcup\limits_{i=1}^{n} \overline{V}_{y_i} \subseteq \overline{V}$. Thus, f is weakly-θ-continuous at the arbitrary point $x \in X$, and so f is weakly-θ-continuous on X. $\square$

Lemma 4.5.1. *Let $X = Z \cup \{p\}$ where Z is a set with $p \notin Z$, (Y, σ) be a topological space and $y \in Y$. Let $g : Z \to (Y, \sigma)$ be a function and $\mathcal{F}$ be a filter on Z. Define a function $\widetilde{g} : (X, \tau_p) \to (Y, \sigma)$ by $\widetilde{g}(z) = g(z)$ for any $z \in Z$ and $\widetilde{g}(p) = y$. Then $g(\mathcal{F}) \xrightarrow{\theta} y$ in (Y, σ) if and only if $\widetilde{g}$ is weakly-θ-continuous on X.*

Proof. Similar to the proof of Lemma 2.4.1. □

Theorem 4.5.15. [47, 52] A Hausdorff space (Y, σ) is H-closed if and only if for every topological space $(X, \tau) \in \mathcal{S}$, each function $f : (X, \tau) \to (Y, \sigma)$ with a strongly closed graph is weakly-θ-continuous.

Proof. The first direction follows by Theorem 4.5.14. Suppose, by the way of contradiction, that Y is not H-closed, then by Proposition 4.1.2 part (iii), there is a filter $\mathcal{F}$ on Y such that $\theta\text{-Adh}_\sigma(\mathcal{F}) = \emptyset$. Let $X = Y \cup \{p\}$ where $p \notin Y$. Consider the topological space (X, τ_p). Then by Theorem 2.4.12, (X, τ_p) is Hausdorff. Also, by Theorems 1.4.2 and 1.4.4, (X, τ_p) is completely normal and fully normal. This implies that $(X, \tau_p) \in \mathcal{S}$. Fix a point $b \in Y$ and define $\widetilde{\text{id}}_Y : (X, \tau_p) \to (Y, \sigma)$ by $\widetilde{\text{id}}_Y(x) = \text{id}_Y(x) = x$ for any $x \in Y$ and $\widetilde{\text{id}}_Y(p) = b$. Let $(x, y) \in X \times Y$ and $(x, y) \notin \Gamma_{\widetilde{\text{id}}_Y}$. Consider the case when $x \neq p$. Since $\widetilde{\text{id}}_Y(x) \neq y$ and (Y, σ) is Hausdorff, then there exists $V_y \in \sigma(y)$ such that $\widetilde{\text{id}}_Y(x) \notin \overline{V}_y$. Hence, $\{x\} \in \tau_p(x)$, $V_y \in \sigma(y)$ and $\widetilde{\text{id}}_Y(\{x\}) \cap \overline{V}_y = \{\widetilde{\text{id}}_Y(x)\} \cap \overline{V}_y = \emptyset$. Consider the case when $x = p$. Then $b = \widetilde{\text{id}}_Y(p) \neq y$. Again, since (Y, σ) is Hausdorff, then there exists $V_y \in \sigma(y)$ such that $b \notin \overline{V}_y$. Moreover, since $\theta\text{-Adh}_\sigma(\mathcal{F}) = \emptyset$, then by Theorem 4.1.3, we have $\mathcal{F} \overset{\theta}{\not\propto} y$, so there exist $W_y \in \sigma(y)$ and $F \in \mathcal{F}$ such that $F \cap \overline{W}_y = \emptyset$. Let $Z_y = V_y \cap W_y$. Then $Z_y \in \sigma(y)$, $b \notin \overline{Z}_y$ and $F \cap \overline{Z}_y = \emptyset$. So, $F \cup \{p\} \in \tau_p(x)$, $Z_y \in \sigma(y)$ and $\widetilde{\text{id}}_Y(F \cup \{p\}) \cap \overline{Z}_y = (\text{id}(F) \cup \{b\}) \cap \overline{Z}_y = F \cap \overline{Z}_y = \emptyset$. We have shown, in both cases, that for each $(x, y) \in (X \times Y) - \Gamma_{\widetilde{\text{id}}_Y}$, there exist $U_x \in \tau_p(x)$ and $G_y \in \sigma(y)$ such that $\widetilde{\text{id}}_Y(U_x) \cap \overline{G}_y = \emptyset$. Thus, by Theorem 4.5.11, $\widetilde{\text{id}}_Y$ has a strongly closed graph. By hypothesis, $\widetilde{\text{id}}_Y$ is weakly-θ-continuous, so by Lemma 4.5.1, $\text{id}_Y(\mathcal{F}) \xrightarrow{\theta} b$ in (Y, σ) implies $\mathcal{F} \xrightarrow{\theta} b$ in (Y, σ), and hence by Proposition 4.1.2 part (i), $\mathcal{F} \overset{\theta}{\propto} b$, and so Theorem 4.1.3, $\theta\text{-Adh}_\sigma(\mathcal{F}) \neq \emptyset$, which is a contradiction. Therefore, (Y, σ) is H-closed. □

Chapter 5

rc-Convergence of Filters

We study rc-convergence of filters. We will start by introducing the definition of an rc-limit of a filter and define an rc-cluster point of a filter. Characterizations of various topological properties in terms of filters have been discovered. Also, new results in regular, extremally disconnected, and semi-Urysohn spaces have been obtained. Various functions: rc-continuous, θs-continuous, and S-continuous are all characterized by filters. As well, the connections between these functions and rc-limits (rc-cluster points) of filters are investigated. Several important notions, such as S-closed spaces and rc-strongly closed graphs, can be characterized with the help of filters.

5.1 rc-Limit and rc-Cluster Points of Filters

Definition 5.1.1. [48] Let (X, τ) be a topological space. For each $x \in X$, let $\mathcal{C}_r^\tau(x) = \{\overline{U} : x \in \overline{U}, U \in \tau\}$. Then $\mathcal{C}_r^\tau(x) = \mathrm{RC}_\tau(x)$ and $\mathcal{C}_r^\tau(x)$ has the finite intersection property. Thus, $\mathcal{C}_r^\tau(x)$ is a filter subbase on X. Let $\langle \mathcal{C}_r^\tau(x) \rangle$ be the filter generated by $\mathcal{C}_r^\tau(x)$. We call $\langle \mathcal{C}_r^\tau(x) \rangle$ the *rc-neighborhood filter* of x.

Notation 9. *For a topological space (X, τ) and $x \in X$, the subbase $\mathcal{C}_r^\tau(x)$ will be simply be denoted by $\mathcal{C}_r(x)$ when there is no confusion.*

Proposition 5.1.1. Let (X, τ) be a topological space. Then

(i) $\mathcal{C}(x) \subseteq \mathcal{C}_r(x)$ for any $x \in X$.

(ii) (X, τ) is extremally disconnected if and only if $\mathcal{C}(x) = \mathcal{C}_r(x)$ for any $x \in X$.

Proof. (i) Let $x \in X$ and $H \in \mathcal{C}(x)$, then $H = \overline{U}$ where $x \in U$ and $U \in \tau$ but $U \subseteq \overline{U}$, so $H = \overline{U}$ where $x \in \overline{U}$ and $U \in \tau$, thus $H \in \mathcal{C}_r(x)$. Therefore, $\mathcal{C}(x) \subseteq \mathcal{C}_r(x)$ for any $x \in X$.

(ii) Suppose that X is extremally disconnected and $x \in X$. By part (i), $\mathcal{C}(x) \subseteq \mathcal{C}_r(x)$. Let $R \in \mathcal{C}_r(x)$, then $R = \overline{U}$, $U \in \tau$ and $x \in \overline{U}$ but since X is extremally disconnected and $U \in \tau$, then $R = \overline{U} \in \tau$. As $x \in R \in \tau$, then $\overline{R} \in \mathcal{C}(x)$ but R is regular closed, so R is closed. Thus, $R = \overline{R}$, and hence $R \in \mathcal{C}(x)$. Therefore, $\mathcal{C}_r(x) \subseteq \mathcal{C}(x)$. Conversely, Let $U \in \tau$, then we show that $\overline{U} \in \tau$. Let $x \in \overline{U}$, then $\overline{U} \in \mathcal{C}_r(x) = \mathcal{C}(x)$, then there exists $V \in \tau(x)$ such that $\overline{U} = \overline{V}$ but $x \in V \subseteq \overline{V} = \overline{U}$. So, $\overline{U} \in \mathcal{U}(x)$ for any $x \in X$. Hence, $\overline{U} \in \tau$. Therefore, X is extremally disconnected.

□

Definition 5.1.2. [48] Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. We say that $\mathcal{F}$ *rc-converges* to x, written $\mathcal{F} \xrightarrow{rc} x$ if $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}$. In such a case, x is called the *rc-limit* of $\mathcal{F}$.

Definition 5.1.3. [111] Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. We say that $\mathcal{F}$ *s-converges* to x, written $\mathcal{F} \xrightarrow{s} x$ if for all $S \in \mathrm{SO}(x)$, $\overline{S} \in \mathcal{F}$. In such a case, x is called the *s-limit* of $\mathcal{F}$.

Proposition 5.1.2. [10] Let $\mathcal{F}$ be a filter on a topological space X. Then $\mathcal{F} \xrightarrow{s} x$ if and only if $\mathcal{F} \xrightarrow{rc} x$.

Proof. This follows from the fact that $\mathrm{RC}(x) = \{\overline{V} : V \in \mathrm{SO}(x)\}$. □

Definition 5.1.4. [10] Let (X, τ) be a topological space. We define the *τ_{rc}-topology* on X as the topology on X which has $\mathrm{RC}(X)$ as a subbase. It is to be noted that intersection of two regular closed sets may fail to be regular closed. Therefore these collections do not form a base for topology.

Proposition 5.1.3. Let (X, τ) be a topological space and $\mathcal{F}$ be a filter on X. Then $\mathcal{F} \xrightarrow{rc} x$ in (X, τ) if and only if $\mathcal{F} \longrightarrow x$ in (X, τ_{rc}).

Proof. Suppose that $\mathcal{F} \xrightarrow{rc} x$. Then $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}$. That is, $\langle \mathrm{RC}(x) \rangle \subseteq \mathcal{F}$. Let $U \in \tau_{rc}(x)$. Then there exist $R_1, \ldots, R_n \in \mathrm{RC}(x)$ such that $x \in \bigcap_{i=1}^{n} R_i \subseteq U$. Since $\bigcap_{i=1}^{n} R_i \in \langle \mathrm{RC}(x) \rangle$, then $\bigcap_{i=1}^{n} R_i \in \mathcal{F}$. So, $U \in \mathcal{F}$. Thus, $\mathcal{F} \longrightarrow x$ in (X, τ_{rc}).

Conversely, suppose that $\mathcal{F} \xrightarrow{\tau_{rc}} x$. Let $R \in \mathcal{C}_r(x) = \mathrm{RC}(x)$. But since $\mathrm{RC}(X) \subseteq \tau_{rc}$, then $R \in \tau_{rc}(x)$. As $\mathcal{F} \xrightarrow{\tau_{rc}} x$, then $R \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{rc} x$. $\square$

Definition 5.1.5. [48] Let (X, τ) be a topological space and $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F}$ *rc-accumulates* at x, written $\mathcal{F} \overset{rc}{\propto} x$, iff $\mathcal{F}(\cap)[\mathcal{C}_r(x)]$ iff for each $F \in \mathcal{F}$ and for each $R \in [\mathcal{C}_r(x)]$, $F \cap R \neq \emptyset$. The point x is then called the *rc-cluster* of $\mathcal{F}$.

Definition 5.1.6. [111] Let (X, τ) be a topological space and $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F}$ *s-accumulates* at x, written $\mathcal{F} \overset{s}{\propto} x$, iff for each $V \in \mathrm{SO}(x)$ and each $F \in \mathcal{F}$, $F \cap \overline{V} \neq \emptyset$. The point x is called the *s-cluster* of $\mathcal{F}$.

Proposition 5.1.4. Let (X, τ) be a topological space and $\mathcal{F}$ be a filter on X. Then $\mathcal{F} \overset{rc}{\propto} x$ in (X, τ) if and only if $\mathcal{F} \propto x$ in (X, τ_{rc}).

Proof. Suppose that $\mathcal{F} \overset{rc}{\propto} x$. Let $U \in \tau_{rc}(x)$. Then there exist $R_1, \ldots, R_n \in \mathrm{RC}(x)$ such that $x \in \bigcap_{i=1}^{n} R_i \subseteq U$. Since $R_i \in \mathrm{RC}(x)$ for any $i = 1, \ldots, n$, then $\bigcap_{i=1}^{n} R_i \in [\mathrm{RC}(x)] = [\mathcal{C}_r(x)]$. So, $(\bigcap_{i=1}^{n} R_i) \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. Hence, $U \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. Thus, $\mathcal{F} \propto x$ in (X, τ_{rc}).

Conversely, suppose that $\mathcal{F} \overset{\tau_{rc}}{\propto} x$. Let $R \in [\mathcal{C}_r(x)] = [\mathrm{RC}(x)]$. But since $\mathrm{RC}(X) \subseteq \tau_{rc}$, then $R \in \tau_{rc}(x)$. As $\mathcal{F} \overset{\tau_{rc}}{\propto} x$, then $R \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. Therefore, $\mathcal{F} \overset{rc}{\propto} x$. $\square$

Remark 5.1. *[10] Since $\mathcal{C}_r(x) = \mathrm{RC}(x) = \{\overline{V} : V \in \mathrm{SO}(x)\}$ by Proposition 1.2.7 part (iii). Then an equivalent formulation of Definition 5.1.6, is that a filter $\mathcal{F}$ s-accumulates at $x \in X$ if and only if for each $F \in \mathcal{F}$ and for each $R \in \mathrm{RC}(x)$, $F \cap R \neq \emptyset$.*

Proposition 5.1.5. [10] Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. If $\mathcal{F} \overset{rc}{\propto} x$, then $\mathcal{F} \overset{s}{\propto} x$.

The converse is not necessarily true follows from the following example.

Example 5.1.1. *Suppose that* $X = \{a, b, c, d\}$ *with* $\tau = \{\emptyset, X, \{a, b\}, \{c\}, \{a, b, c\}\}$ *and* $\mathcal{F} = \{\{a, b, c\}, X\}$. *Now,* $\mathcal{C}_r(d) = \{X, \{a, b, d\}, \{c, d\}\}$ *and* $[\mathcal{C}_r(d)] = \{X, \{a, b, d\}, \{c, d\}, \{d\}\}$. *Then* $\mathcal{F} \overset{s}{\propto} d$ *since* $\mathcal{C}_r(d)(\cap)\mathcal{F}$ *but* $\mathcal{F} \overset{rc}{\not\propto} d$ *since* $\{a, b, c\} \cap \{d\} = \emptyset$.

Proposition 5.1.6. Let X be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. If $\mathcal{F} \xrightarrow{rc} x$, then $\mathcal{F} \overset{rc}{\propto} x$.

Proof. Suppose that $\mathcal{F} \xrightarrow{rc} x$. Let $F \in \mathcal{F}$ and $R \in [\mathcal{C}_r(x)]$, then $R \in \mathcal{F}$ since $\mathcal{F} \xrightarrow{rc} x$. So, $F \cap R \neq \emptyset$. $\square$

Convergence of filters in the usual sense and rc-convergence are independent of each other.

Example 5.1.2. *Consider the topological space* (X, τ) *given in Example 5.1.1. Let* $\mathcal{F} = \{\{a, b, c\}, X\}$. *Then* $\mathcal{U}(d) = \{X\} \subseteq \mathcal{F}$, *so* $\mathcal{F} \longrightarrow d$. *But* $\mathcal{F} \overset{rc}{\not\longrightarrow} d$ *since* $\mathcal{C}_r(d) = \{\{c, d\}, \{a, b, d\}, X\} \not\subseteq \mathcal{F}$.

Example 5.1.3. *Consider the topological space* (X, τ) *given in Example 5.1.1. Let* $\mathcal{F} = \{\{a, b, d\}, X\}$. *Then* $\mathcal{U}(a) = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, X\} \not\subseteq \mathcal{F}$, *so* $\mathcal{F} \not\longrightarrow a$. *But* $\mathcal{F} \xrightarrow{rc} a$ *since* $\langle \mathcal{C}_r(a) \rangle = \{\{a, b, d\}, X\} \subseteq \mathcal{F}$.

Cluster and rc-cluster point of filters are independent of each other.

Example 5.1.4. *Consider the topological space* (X, τ) *given in Example 5.1.1. Let* $\mathcal{F} = \langle a \rangle$ *be the principal filter generated by* a. *Then* $\mathcal{U}(d) = \{X\} \subseteq \mathcal{F}$, *so* $\mathcal{F} \longrightarrow d$, *and hence* $\mathcal{F} \propto d$. *But* $\mathcal{F} \overset{rc}{\not\propto} d$ *since* $\{a\} \in \mathcal{F}$, $\{c, d\} \in \mathcal{C}_r(d)$, *but* $\{a\} \cap \{c, d\} = \emptyset$.

Example 5.1.5. *Consider the topological space (X, τ) given in Example 5.1.1. Let $\mathcal{F} = \langle d \rangle$ be the principal filter generated by d. Then $\mathcal{F} \not\propto a$ since $\{d\} \in \mathcal{F}$, $\{a, b\} \in \mathcal{U}(a)$, but $\{d\} \cap \{a, b\} = \emptyset$. Now, by Example 5.1.3, $[\mathcal{C}_r(a)] = \{\{a, b, d\}, X\}$. Clearly, $\mathcal{F}(\cap)[\mathcal{C}_r(a)]$. Hence, $\mathcal{F} \overset{rc}{\propto} a$.*

Definition 5.1.7. [59] Let (X, τ) be a topological space, $E \subseteq X$ and $x \in X$. Then x is a *θ-semi-adherent* point of E iff for all $R \in \mathcal{C}_r(x)$, $R \cap E \neq \emptyset$. The set of all θ-semi-adherent points of a set E is called the *θ-semiclosure* of the set E and denoted by θ-sCl(E).

Proposition 5.1.7. Let X be a topological space and $E \subseteq X$, then $E \subseteq \theta$-sCl$(E) \subseteq \theta$-Cl(E).

Proof. Let $x \in E$ and $R \in \mathcal{C}_r(x)$, then $x \in R$. So, $x \in R \cap E$, and hence $R \cap E \neq \emptyset$ and thus, $x \in \theta$-sCl(E). Hence, $E \subset \theta$-sCl(E). Next, let $x \in \theta$-sCl(E) and $G \in \mathcal{C}(x)$. By Proposition 5.1.1, $\mathcal{C}(x) \subseteq \mathcal{C}_r(x)$, so $G \in \mathcal{C}_r(x)$, and hence $G \cap E \neq \emptyset$ since $x \in \theta$-sCl(E). Thus, $x \in \theta$-Cl(E). □

Proposition 5.1.8. Let (X, τ) be a topological space and $E \subseteq X$. Then θ-sCl$(E) = \bigcap\{U \subseteq X : U \in \mathrm{RO}(X), E \subseteq U\}$.

Proof. Let $x \in \theta$-sCl(E) and suppose, by the way of contradiction, that $x \notin \bigcap\{U \subseteq X : U \in \mathrm{RO}(X), E \subseteq U\}$, then $x \notin U$ for some $U \in \mathrm{RO}(X)$ with $E \subseteq U$. Then $X - U \in \mathrm{RC}(X)$ and $x \in X - U$. So, $X - U \in \mathrm{RC}(x)$. As $x \in \theta$-sCl(E), then $(X - U) \cap E \neq \emptyset$ but $E \subseteq U$, then $E \cap (X - U) \subseteq U \cap (X - U) = \emptyset$, which is a contradiction. Hence, θ-sCl$(E) \subseteq \bigcap\{U \subseteq X : U \in \mathrm{RO}(X), E \subseteq U\}$.

Next, let $x \in \bigcap\{U \subseteq X : U \in \mathrm{RO}(X), E \subseteq U\}$ and suppose, by the way of contradiction, that $x \notin \theta$-sCl(E). Then there exists $R \in \mathrm{RC}(x)$ such that $E \cap R = \emptyset$. Let $U = X - R$. Then $U \in \mathrm{RO}(X)$ and $E \subseteq U$, so by hypothesis, $x \in U = X - R$. That is, $x \notin R$, which is a contradiction with $R \in \mathrm{RC}(x)$. Thus, $x \in \theta$-sCl(E). Hence, $\bigcap\{U \subseteq X : U \in \mathrm{RO}(X), E \subseteq U\} \subseteq \theta$-sCl$(E)$. Therefore, the equality holds. □

Definition 5.1.8. [59] A subset E of a topological space (X, τ) is called *θ-semiclosed* if θ-sCl$(E) = E$. The complement of a θ-semiclosed set is called a *θ-semiopen* set. The family of all θ-semiopen sets in (X, τ) is denoted by τ^+.

Proposition 5.1.9. [87, 89] Let A be a subset of a topological space X. Then A is θ-semiopen if and only if for each $x \in A$, there exists a semi-open set G in X such that $x \in G \subseteq \overline{G} \subseteq A$.

Proof.

$$\begin{aligned}
A \text{ is } \theta\text{-semiopen iff } X - A \text{ is } \theta\text{-semiclosed} & \\
&\text{iff } \theta\text{-sCl}(X - A) \subseteq X - A \\
&\text{iff } A \subseteq X - \theta\text{-sCl}(X - A) \\
&\text{iff } \forall\, x \in A,\ x \notin \theta\text{-sCl}(X - A) \\
&\text{iff } \forall\, x \in A,\ \exists\, R \in \text{RC}(x) \text{ such that } R \cap (X - A) = \emptyset \\
&\text{iff } \forall\, x \in A,\ \exists\, R \in \text{RC}(X) \text{ such that } x \in R \subseteq A \\
&\text{iff } \forall\, x \in A,\ \exists\, G \in \text{SO}(X) \text{ such that } x \in G \subseteq \overline{G} \subseteq A.
\end{aligned}$$

□

Proposition 5.1.10. [87] Let A be a subset of a topological space (X, τ). Then $A \in \tau^+$ if and only if A is the union of regular closed sets of (X, τ).

Proposition 5.1.11. [87] Let (X, τ) be a topological space and $A \subseteq X$.

(i) If A is regular closed in X, then A is θ-semiopen.

(ii) If A is regular open in X, then A is θ-semiclosed.

(iii) The family of all θ-semiopen subsets of X need not be a topology on X.

Proof.

(i) Let A be regular closed in X and $x \in A$, then take $R = A \in \text{RC}(X)$ such that $x \in R \subseteq A$. So, by Proposition 5.1.9, A is θ-semiopen.

(ii) Suppose that A is regular open in X, then $X - A$ is regular closed in X. So, by part (i), $X - A$ is θ-semiopen. Hence, A is θ-semiclosed.

(iii) τ^+ is not a topology in general. As the following example shows.

□

Example 5.1.6. *Let* $X = \{a, b, c\}$ *with* $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. *Then* $\mathrm{RC}(X) = \{\emptyset, X, \{a, c\}, \{b, c\}\}$. *Let* $A = \{a, c\}$ *and* $B = \{b, c\}$, *then clearly,* A *and* B *are* θ*-semiopen since* $A, B \in \mathrm{RC}(X)$. *But* $\{c\} = A \cap B$. *Thus,* $c \in A \cap B$ *but there is no* $R \in \mathrm{RC}(X)$ *such that* $c \in R \subseteq A \cap B$, *and hence by Proposition 5.1.9,* $A \cap B \notin \tau^+$, *that is* $A \cap B$ *is not* θ*-semiopen.*

Proposition 5.1.12. [87] The following are equivalent for a topological space (X, τ):

(i) (X, τ) is extremally disconnected.

(ii) $\mathrm{SO}(X, \tau)$ is a topology on X.

(iii) τ^+ is a topology on X.

Remark 5.2. *If* $\mathrm{Cl}_{\tau_{rc}}$ *denotes the closure operator in the* τ_{rc}*-topology, then* $\mathrm{Cl}_{\tau_{rc}}(A) \neq \theta\text{-sCl}(A)$ *as the following example shows.*

Example 5.1.7. *Consider the topological space* (X, τ) *given in Example 5.1.1. Then* $\mathrm{RC}(X) = \{\emptyset, X, \{a, b, d\}, \{c, d\}\}$ *and* $\tau_{rc} = \{\emptyset, X, \{a, b, d\}, \{c, d\}, \{d\}\}$. *Let* $A = \{a, c\}$, *then* $\mathrm{Cl}_{\tau_{rc}}(A) = \{a, b, c\}$ *and* $\theta\text{-sCl}(A) = X$. *Thus,* $\mathrm{Cl}_{\tau_{rc}}(A) \neq \theta\text{-sCl}(A)$.

Theorem 5.1.1. Let (X, τ) be a topological space and $E \subseteq X$. If E is open in X, then $\theta\text{-sCl}(E) \subseteq \overline{E}$.

Proof. Since E is open in X, then by Theorem 4.1.1, $\theta\text{-Cl}(E) = \overline{E}$. But $\theta\text{-sCl}(E) \subseteq \theta\text{-Cl}(E)$ by Proposition 5.1.7. Thus, $\theta\text{-sCl}(E) \subseteq \overline{E}$. $\square$

Theorem 5.1.2. Let (X, τ) be an extremally disconnected and $E \subseteq X$. Then $\overline{E} \subseteq \theta\text{-sCl}(E)$.

Proof. Let $x \in \overline{E}$ and let $\overline{U} \in \mathcal{C}_r(x)$. Since X is extremally disconnected, then $\overline{U} \in \tau(x)$. So, $\overline{U} \cap E \neq \emptyset$ since $x \in \overline{E}$. Hence, $x \in \theta\text{-sCl}(E)$. Therefore, $\overline{E} \subseteq \theta\text{-sCl}(E)$. $\square$

Corollary 5.1.1. *Let* (X, τ) *be an extremally disconnected space and* $E \subseteq X$. *If* E *is open in* X, *then* $\theta\text{-sCl}(E) = \overline{E}$.

Proof. This follows from Theorems 5.1.1 and 5.1.2. $\square$

Theorem 5.1.3. Let (X, τ) be a regular extremally disconnected space and $E \subseteq X$. Then $\overline{E} = \theta\text{-sCl}(E)$.

Proof. Since X is extremally disconnected, then by Theorem 5.1.2, $\overline{E} \subseteq \theta\text{-sCl}(E)$. Also, since X is regular, then by Theorem 4.3.2, $\overline{E} = \theta\text{-Cl}(E)$. Thus, $\overline{E} \subseteq \theta\text{-sCl}(E) \subseteq \theta\text{-Cl}(E) = \overline{E}$. Therefore, $\overline{E} = \theta\text{-sCl}(E)$. $\square$

Definition 5.1.9. [75] Let $\mathcal{F}$ be a filter on a topological space (X, τ). A point $x \in X$ is said to be a *θ-semi-adherent* point of $\mathcal{F}$ if x is a θ-semi-adherent point of every set in $\mathcal{F}$. The *θ-semiadherence* of $\mathcal{F}$, $\theta\text{-sAdh}(\mathcal{F})$, is the set of all θ-semi-adherent points of $\mathcal{F}$.

Remark 5.3. *[75] Let X be a topological space. If $\mathcal{F}$ is a filter on X, then $\theta\text{-sAdh}(\mathcal{F}) = \bigcap\limits_{F \in \mathcal{F}} \theta\text{-sCl}(F)$.*

Theorem 5.1.4. Let $\mathcal{F}$ be a filter on a topological space X and $x \in X$. Then $x \in \theta\text{-sAdh}(\mathcal{F})$ if and only if $\mathcal{F} \overset{s}{\propto} x$.

Proof.

$$
\begin{aligned}
x \in \theta\text{-sAdh}(\mathcal{F}) &\text{ iff } x \in \bigcap_{F \in \mathcal{F}} \theta\text{-sCl}(F) \\
&\text{ iff } x \in \theta\text{-sCl}(F) \text{ for all } F \in \mathcal{F} \\
&\text{ iff } R \cap F \neq \emptyset \text{ for all } R \in \mathrm{RC}(x) \text{ and all } F \in \mathcal{F} \\
&\text{ iff } \mathcal{F} \overset{s}{\propto} x.
\end{aligned}
$$

$\square$

Theorem 5.1.5. Let $\mathcal{F}$ be a filter on a topological space X and $x \in X$. If $\mathcal{F} \overset{rc}{\propto} x$, then $x \in \theta\text{-sAdh}(\mathcal{F})$.

Proof. If $\mathcal{F} \overset{rc}{\propto} x$, then by Proposition 5.1.5, $\mathcal{F} \overset{s}{\propto} x$. So, by Theorem 5.1.4, $x \in \theta\text{-sAdh}(\mathcal{F})$. $\square$

Example 5.1.8. *Consider the topological space* (X, τ) *and the filter* $\mathcal{F}$ *on* X *given in Example 5.1.1, we get* $\mathcal{F} \overset{s}{\propto} d$ *but* $\mathcal{F} \overset{rc}{\not\propto} d$. *So, by Theorem 5.1.4,* $d \in \theta\text{-sAdh}(\mathcal{F})$ *and by Proposition 5.1 and Theorem 2.1.1,* $d \notin \mathrm{Adh}_{\tau_{rc}}(\mathcal{F})$. *Hence,* $\theta\text{-sAdh}(\mathcal{F}) \neq \mathrm{Adh}_{\tau_{rc}}(\mathcal{F})$.

Theorem 5.1.6. Let X be a topological space, $E \subseteq X$ and $x \in X$. If $\mathcal{F}$ is a filter on X such that $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{rc} x$, then $x \in \theta\text{-sCl}(E)$.

Proof. Suppose that there is a filter $\mathcal{F}$ on X such that $E \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{rc} x$. We will show that $x \in \theta\text{-sCl}(E)$. Let $R \in \mathcal{C}_r(x)$. But $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}$, then $R \in \mathcal{F}$ and thus, $R \cap E \neq \emptyset$. So, $x \in \theta\text{-sCl}(E)$. □

Theorem 5.1.7. Let X be a topological space, $E \subseteq X$ and $x \in X$. If $\mathcal{F}$ is a filter on X such that $\mathcal{F} \xrightarrow{rc} x$ and $F \cap E \neq \emptyset$ for all $F \in \mathcal{F}$, then $x \in \theta\text{-sCl}(E)$.

Proof. Suppose that $\mathcal{F}$ is a filter on X such that $\mathcal{F} \xrightarrow{rc} x$ and $F \cap E \neq \emptyset$ for all $F \in \mathcal{F}$. Let $R \in \mathcal{C}_r(x)$, then $R \in \mathcal{F}$ since $\mathcal{F} \xrightarrow{rc} x$. So, by hypothesis, $R \cap E \neq \emptyset$. Therefore, $x \in \theta\text{-sCl}(E)$. □

Remark 5.4. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be filters on a topological space* (X, τ) *and* $x \in X$.

(i) *The principal filter* $\langle x \rangle \xrightarrow{rc} x$.

(ii) *If* $\mathcal{F} \xrightarrow{rc} x$ *and* $\mathcal{G} \xrightarrow{rc} x$, *then* $\mathcal{F} \cap \mathcal{G} \xrightarrow{rc} x$.

Theorem 5.1.8. Let X be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \xrightarrow{rc} x$ if and only if for every subfilter $\mathcal{F}'$ of $\mathcal{F}$, $\mathcal{F}' \xrightarrow{rc} x$.

Proof. If every subfilter $\mathcal{F}'$ of $\mathcal{F}$ rc-converges to x, then so does $\mathcal{F}$ because $\mathcal{F} \subseteq \mathcal{F}$. Conversely, if $\mathcal{F} \xrightarrow{rc} x$ and $\mathcal{F}'$ is a subfilter of $\mathcal{F}$, then $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{F}'$. Hence, $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}'$. Therefore, $\mathcal{F}' \xrightarrow{rc} x$. □

Theorem 5.1.9. Let (X, τ) be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \xrightarrow{rc} x$ if and only if every subfilter $\mathcal{G}$ of $\mathcal{F}$ has a subfilter $\mathcal{H}$ such that $\mathcal{H} \xrightarrow{rc} x$.

Proof. Suppose, by the way of contradiction, that $\mathcal{F} \overset{rc}{\not\longrightarrow} x$, then there is $R \in \mathcal{C}_r(x)$ such that $R \notin \mathcal{F}$. Then $(X - R) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Consider $\mathcal{G} = \langle \mathcal{F}\big|_{X-R} \rangle$, then $\mathcal{G}$ is a filter of $\mathcal{F}$ such that $\mathcal{F} \subseteq \mathcal{G}$ and $X - R \in \mathcal{G}$. So, $\mathcal{G}$ has a subfilter $\mathcal{F}$. By hypothesis, $\mathcal{G}$ has a subfilter $\mathcal{H}$ such that $\mathcal{H} \xrightarrow{rc} x$. Since $R \in \mathcal{C}_r(x)$, then $R \in \mathcal{H}$ but $X - R \in \mathcal{G} \subseteq \mathcal{H}$. So, $\emptyset \in \mathcal{H}$, which is a contradiction. Therefore, $\mathcal{F} \xrightarrow{rc} x$. The converse follows from Theorem 5.1.8. $\square$

Theorem 5.1.10. Let X be a topological space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \overset{rc}{\propto} x$ if and only if there exists a subfilter $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \xrightarrow{rc} x$.

Proof. Let $\mathcal{F}$ be a filter on X. Suppose that there exists a subfilter $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \xrightarrow{rc} x$. Then $\mathcal{F} \subseteq \mathcal{F}'$ and $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}'$. We show that $\mathcal{F} \overset{rc}{\propto} x$. Let $F \in \mathcal{F}$ and $R \in [\mathcal{C}_r(x)]$, then $F \in \mathcal{F}'$ and $R \in \mathcal{F}'$. So, $F \cap R \neq \emptyset$. Hence, $\mathcal{F} \overset{rc}{\propto} x$.

Conversely, assume that $\mathcal{F} \overset{rc}{\propto} x$. We will construct a subfilter $\mathcal{F}'$ of $\mathcal{F}$ that rc-converges to x. Since $\mathcal{F} \overset{rc}{\propto} x$, then $\mathcal{F}(\cap)[\mathcal{C}_r(x)]$. Since $[\mathcal{C}_r(x)] \subseteq \langle \mathcal{C}_r(x) \rangle$ and by Proposition 1.1.1 part (i), $\mathcal{F}(\cap)\langle \mathcal{C}_r(x) \rangle$. Let $\mathcal{F}' = \mathcal{F} \vee \langle \mathcal{C}_r(x) \rangle$. Then $\mathcal{F}'$ is a filter on X such that $\mathcal{F} \subseteq \mathcal{F}'$ and $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}'$. Thus, $\mathcal{F}'$ is a subfilter of $\mathcal{F}$ such that $\mathcal{F}' \xrightarrow{rc} x$. $\square$

Theorem 5.1.11. Let X be a topological space, $\mathcal{F}'$ be a subfilter of $\mathcal{F}$ on X and $x \in X$. If $\mathcal{F}' \overset{rc}{\propto} x$, then $\mathcal{F} \overset{rc}{\propto} x$.

Proof. Suppose that $\mathcal{F}' \overset{rc}{\propto} x$, then $\mathcal{F}'(\cap)[\mathcal{C}_r(x)]$ but since $\mathcal{F} \subseteq \mathcal{F}'$ and by Proposition 1.1.1 part (i), so $\mathcal{F}(\cap)[\mathcal{C}_r(x)]$. Hence, $\mathcal{F} \overset{rc}{\propto} x$. $\square$

Theorem 5.1.12. Let $\mathcal{F}$ be an ultrafilter on a topological space X and $x \in X$. Then $\mathcal{F} \xrightarrow{rc} x$ if and only if $\mathcal{F} \overset{rc}{\propto} x$.

Proof. If $\mathcal{F} \xrightarrow{rc} x$, then $\mathcal{F} \overset{rc}{\propto} x$ by Proposition 5.1.6. Conversely, suppose that $\mathcal{F} \overset{rc}{\propto} x$. Let $R \in \mathcal{C}_r(x)$. Then $R \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. So, $\mathcal{F}$ meets R. But $\mathcal{F}$ is an ultrafilter on X, then by the proof of Theorem 1.1.5, $R \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{rc} x$. $\square$

5.2 rc-Convergence in Semi-Urysohn Spaces

Definition 5.2.1. [31] A topological space (X, τ) is said to be *semi-Urysohn* if for each $x_1 \neq x_2$ in X, there exist semi-open sets U and V in X containing x_1 and x_2, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$.

Theorem 5.2.1. [31] A topological space X is semi-Urysohn if and only if for each $x_1 \neq x_2$ in X, there exist regular closed sets F_1 and F_2 in X containing x_1 and x_2, respectively, such that $F_1 \cap F_2 = \emptyset$.

The following implications hold.

Proposition 5.2.1. [31] Urysohn $\implies$ semi-Urysohn $\implies$ weakly-T_2.

Theorem 5.2.2. Let X be a topological space. If X is semi-Urysohn, then each filter $\mathcal{F}$ on X rc-converges to at most one point in X.

Proof. Suppose that X is semi-Urysohn and $\mathcal{F}$ is a filter on X. Assume that $\mathcal{F} \xrightarrow{rc} x$ and $\mathcal{F} \xrightarrow{rc} y$ with $x \neq y$ in X. But X is semi-Urysohn, so there exist $G \in \mathcal{C}_r(x)$ and $H \in \mathcal{C}_r(y)$ such that $G \cap H = \emptyset$. On the other hand, $G \in \mathcal{F}$ and $H \in \mathcal{F}$, so $G \cap H \neq \emptyset$, which is a contradiction. Thus, $\mathcal{F}$ rc-converges to at most one point in X. $\square$

Theorem 5.2.3. Let X be a semi-Urysohn space, $\mathcal{F}$ be a filter on X and $x \in X$. If $\mathcal{F} \xrightarrow{rc} x$ in X, then x is the unique rc-cluster point of $\mathcal{F}$.

Proof. If $\mathcal{F} \xrightarrow{rc} x$, then $\mathcal{F} \overset{rc}{\propto} x$ by Proposition 5.1.6. Now, suppose that $y \in X$ is an rc-cluster point of $\mathcal{F}$ with $y \neq x$. But X is semi-Urysohn, so there exist $G \in \mathcal{C}_r(x)$ and $H \in \mathcal{C}_r(y)$ such that $G \cap H = \emptyset$. But $\mathcal{F} \xrightarrow{rc} x$, so $G \in \mathcal{F}$. Also, since $\mathcal{F} \overset{rc}{\propto} y$, then $F \cap H \neq \emptyset$ for all $F \in \mathcal{F}$, but then $G \cap H \neq \emptyset$, which is a contradiction. Therefore, x is the unique rc-cluster point of $\mathcal{F}$. $\square$

5.3 rc-Convergence in Regular Extremally Disconnected Spaces

We will see immediately that, in regular extremally disconnected spaces, rc-convergence of filters is equivalent to convergence of filters and in this case equivalence is also valid for cluster and rc-cluster points.

Theorem 5.3.1. Let (X, τ) be a regular space and $\mathcal{F}$ be a filter on X and $x \in X$.

(i) If $\mathcal{F} \xrightarrow{rc} x$, then $\mathcal{F} \longrightarrow x$.

(ii) If $\mathcal{F} \overset{rc}{\propto} x$, then $\mathcal{F} \propto x$.

Proof. (i) Suppose that $\mathcal{F} \xrightarrow{rc} x$. Let $U \in \tau(x)$, then there exists $V \in \tau(x)$ such that $\overline{V} \subseteq U$ since X is regular. But $\overline{V} \in \mathcal{C}(x) \subseteq \mathcal{C}_r(x)$ and $\mathcal{F} \xrightarrow{rc} x$. So, $\overline{V} \in \mathcal{F}$, and hence $U \in \mathcal{F}$. Therefore, $\mathcal{F} \longrightarrow x$.

(ii) Suppose that $\mathcal{F} \overset{rc}{\propto} x$. Let $U \in \tau(x)$, then there exists $V \in \tau(x)$ such that $\overline{V} \subseteq U$ since X is regular. But $\overline{V} \in \mathcal{C}(x) \subseteq \mathcal{C}_r(x)$ and $\mathcal{F} \overset{rc}{\propto} x$. This implies, $\overline{V} \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. So, $U \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. Therefore, $\mathcal{F} \propto x$.

□

Theorem 5.3.2. Let (X, τ) be an extremally disconnected space, $\mathcal{F}$ be a filter on X and $x \in X$.

(i) If $\mathcal{F} \longrightarrow x$, then $\mathcal{F} \xrightarrow{rc} x$.

(ii) If $\mathcal{F} \propto x$, then $\mathcal{F} \overset{rc}{\propto} x$.

Proof. (i) Suppose that $\mathcal{F} \longrightarrow x$. Let $R \in \mathcal{C}_r(x)$. Then $R \in \mathrm{RO}(x) \subseteq \tau(x)$ since X is extremally disconnected. But $\mathcal{F} \longrightarrow x$ implies $R \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{rc} x$.

(ii) Suppose that $\mathcal{F} \propto x$. Let $R \in [\mathcal{C}_r(x)]$. Then $R \in [\mathrm{RO}(x)] \subseteq \tau(x)$ since X is extremally disconnected. But since $\mathcal{F} \propto x$, then $R \cap F = \emptyset$ for any $F \in \mathcal{F}$. Therefore, $\mathcal{F} \overset{rc}{\propto} x$.

□

Corollary 5.3.1. *Let X be a regular extremally disconnected space, $\mathfrak{F}$ be a filter on X and $x \in X$.*

(i) $\mathfrak{F} \xrightarrow{rc} x$ if and only if $\mathfrak{F} \longrightarrow x$.

(ii) $\mathfrak{F} \overset{rc}{\propto} x$ if and only if $\mathfrak{F} \propto x$.

Proof. This follows directly from Theorems 5.3.1 and 5.3.2. □

5.4 rc-Convergence and Functions

We will investigate the case of rc-limits of filters under the three types of continuity. We will do the same investigation for rc-cluster points of filters.

5.4.1 rc-Continuous Functions

We introduce rc-continuous functions that form a proper subclass of the class of weakly-θ-continuous functions. In studying this new class, we state several characterizations of rc-continuous functions and the notion of a function that has an rc-strongly closed graph.

Definition 5.4.1. [3] A function $f : (X, \tau) \to (Y, \sigma)$ is *rc-continuous* at $x \in X$ if for every regular closed set R in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq R$. Equivalently, for every $R \in \mathfrak{C}_r^\sigma(f(x))$, there exists $U \in \tau(x)$ such that $f(U) \subseteq R$. If this condition is satisfied at each $x \in X$, then f is said to be rc-continuous on X.

Theorem 5.4.1. [3] A function $f : X \to Y$ is rc-continuous if and only if the inverse image of any regular closed set in Y is open in X.

Proof. The proof is a direct consequence of Definition 5.4.1. □

Proposition 5.4.1. If $f : (X, \tau) \to (Y, \sigma)$ is rc-continuous at $x \in X$, then for each $R \in [\mathfrak{C}_r^\sigma(f(x))]$, there exists $U \in \tau(x)$ such that $f(U) \subseteq R$.

Proof. Let $R \in [\mathcal{C}_r^\sigma(f(x))]$. Then $R = \bigcap_{i=1}^{n} R_i$, where $R_i \in \mathcal{C}_r^\sigma(f(x))$ for all $i = 1, \ldots, n$. Since f is rc-continuous at x, then for each $i = 1, \ldots, n$, there exists $U_i \in \tau(x)$ such that $f(U_i) \subseteq R_i$. Let $U = \bigcap_{i=1}^{n} U_i$, then $U \in \tau(x)$ and $f(U) \subseteq \bigcap_{i=1}^{n} f(U_i) \subseteq \bigcap_{i=1}^{n} R_i = R$. □

Theorem 5.4.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is rc-continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \longrightarrow x$, then $f(\mathcal{F}) \overset{rc}{\longrightarrow} f(x)$ in Y.

Proof. Suppose that $\mathcal{F} \longrightarrow x$ in X and $R \in \mathcal{C}_r^\sigma(f(x))$. Since $R \in \mathcal{C}_r^\sigma(f(x))$ and f is rc-continuous at x, then there exists $U \in \tau(x)$ such that $f(U) \subseteq R$. But since $\mathcal{F} \longrightarrow x$ and $U \in \tau(x)$, then $U \in \mathcal{F}$ but $R \supseteq f(U)$, so $R \in f(\mathcal{F})$. Hence, $f(\mathcal{F}) \overset{rc}{\longrightarrow} f(x)$.

Conversely, suppose that for each filter $\mathcal{F}$ on X, $\mathcal{F} \longrightarrow x$ in X implies $f(\mathcal{F}) \overset{rc}{\longrightarrow} f(x)$ in Y. Let $R \in \mathcal{C}_r^\sigma(f(x))$. Since $\mathcal{U}_\tau(x) \subseteq \mathcal{U}_\tau(x)$, then $\mathcal{U}_\tau(x) \longrightarrow x$. By hypothesis, $f(\mathcal{U}_\tau(x)) \overset{rc}{\longrightarrow} f(x)$. That is, $\langle \mathcal{C}_r(f(x)) \rangle \subseteq f(\mathcal{U}_\tau(x))$. So, $R \in f(\mathcal{U}_\tau(x))$. That is, there exists $U \in \mathcal{U}_\tau(x)$ such that $f(U) \subseteq R$. So, there exists $V \in \tau(x)$ such that $V \subseteq U$. This implies $f(V) \subseteq f(U) \subseteq R$. Hence, there exists $V \in \tau(x)$ such that $f(V) \subseteq R$. Therefore, f is rc-continuous at $x \in X$. □

Theorem 5.4.3. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. f is rc-continuous at $x \in X$ if and only if whenever $\mathcal{F}$ is a filter on X with $\mathcal{F} \propto x$, then $f(\mathcal{F}) \overset{rc}{\propto} f(x)$ in Y.

Proof. Suppose that f is rc-continuous at $x \in X$. Let $\mathcal{F}$ be a filter on X such that $\mathcal{F} \propto x$. Let $R \in [\mathcal{C}_r^\sigma(f(x))]$. Then by Proposition 5.4.1, there exists $U \in \tau(x)$ such that $f(U) \subseteq R$. But $\mathcal{F} \propto x$, then $F \cap U \neq \emptyset$ for all $F \in \mathcal{F}$. So, $f(F \cap U) \neq \emptyset$ for all $F \in \mathcal{F}$. Now, $f(F \cap U) \subseteq f(F) \cap f(U) \subseteq f(F) \cap R$ for all $F \in \mathcal{F}$. Thus, $R \cap f(F) \neq \emptyset$ for all $F \in \mathcal{F}$ but $f(\mathcal{F}) = \langle \{f(F) : F \in \mathcal{F}\} \rangle$. Hence, $R \cap G \neq \emptyset$ for all $G \in f(\mathcal{F})$. Therefore, $f(\mathcal{F}) \overset{rc}{\propto} f(x)$.

Conversely, suppose, by the way of contradiction, that f is not rc-continuous at $x \in X$, then there exists $R \in \mathcal{C}_r^\sigma(f(x))$ such that $f(U) \not\subseteq R$ for any $U \in \tau(x)$. So, $U \not\subseteq f^{-1}(R)$ for any $U \in \tau(x)$. This implies, $V \not\subseteq f^{-1}(R)$ for any $V \in \mathcal{U}_\tau(x)$.

Thus, $V \cap F \neq \emptyset$ for any $V \in \mathcal{U}_\tau(x)$ where $F = X - f^{-1}(R)$. By Proposition 1.1.6, $\mathcal{F} = \langle \mathcal{U}_\tau(x)\big|_F \rangle$ is a filter on X such that $F \in \mathcal{F}$ and $\mathcal{U}_\tau(x) \subseteq \mathcal{F}$. This implies $\mathcal{F} \longrightarrow x$ and by Proposition 2.1.1, $\mathcal{F} \propto x$. We claim that $f(\mathcal{F}) \overset{rc}{\not\propto} f(x)$. Since $F \in \mathcal{F}$, then $f(F) \in f(\mathcal{F})$. Now, $R \cap f(F) = R \cap f(X - f^{-1}(R)) = R \cap f(f^{-1}(Y - R)) \subseteq R \cap (Y - R) = \emptyset$. Hence, we have $R \in \mathcal{C}^\sigma_\tau(f(x))$, $f(F) \in f(\mathcal{F})$ and $R \cap f(F) = \emptyset$. Therefore, $f(\mathcal{F}) \overset{rc}{\not\propto} f(x)$. $\square$

Remark 5.5. *The following two examples show that the concepts of rc-continuity and continuity are independent of each other.*

Example 5.4.1. *Consider the function f from the set of real numbers $\mathbb{R}$ with the usual topology onto itself given by $f(x) = x^2$ for any $x \in \mathbb{R}$, then f is continuous but it is not rc-continuous.*

Example 5.4.2. *Consider the function f from the set of real numbers $\mathbb{R}$ with the left ray topology onto itself given by $f(x) = -x$ for any $x \in \mathbb{R}$, then f is rc-continuous but it is not continuous.*

Theorem 5.4.4. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and Y is regular. If f is rc-continuous, then f is continuous.

Proof. Let $x \in X$ and $\mathcal{F}$ be a filter on X such that $\mathcal{F} \longrightarrow x$. Since f is rc-continuous, then $f(\mathcal{F}) \xrightarrow{rc} f(x)$ by Theorem 5.4.2. But Y is regular, so $f(\mathcal{F}) \longrightarrow f(x)$ by Theorem 5.3.1 part (i). Therefore, f is continuous at $x \in X$. Thus, f is continuous since x was arbitrary. $\square$

Theorem 5.4.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and Y is extremally disconnected. If f is continuous, then f is rc-continuous.

Proof. Let $x \in X$ and $\mathcal{F}$ be a filter on X such that $\mathcal{F} \longrightarrow x$. Since f is continuous, then $f(\mathcal{F}) \longrightarrow f(x)$ by Theorem 2.3.1. But Y is extremally disconnected, so $f(\mathcal{F}) \xrightarrow{rc} f(x)$ by Theorem 5.3.2 part (i). Therefore, f is rc-continuous at $x \in X$. Thus, f is rc-continuous since x was arbitrary. $\square$

Corollary 5.4.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and Y is regular extremally disconnected. Then f is continuous if and only if f is rc-continuous.*

5.4.2 θs-Continuous Functions

In [57] is introduced and investigated a new class of continuity in topological spaces called θs-continuous functions, which contains the class of strongly-θ-continuous functions.

Definition 5.4.2. [57] A function $f : (X, \tau) \to (Y, \sigma)$ is *θs-continuous* at $x \in X$ if for every open set V in Y containing $f(x)$, there exists a semi-open set U in X containing x such that $f(\overline{U}) \subseteq V$. Equivalently, for every $V \in \sigma(f(x))$, there exists $H \in \mathcal{C}_r^\tau(x)$ such that $f(H) \subseteq V$. If this condition is satisfied at each $x \in X$, then f is said to be θs-continuous on X.

Theorem 5.4.6. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $\mathcal{F}$ be a filter on X. If θs-continuous at $x \in X$, then $f(\mathcal{F}) \longrightarrow f(x)$ in Y whenever $\mathcal{F} \xrightarrow{rc} x$.

Proof. Assume that $\mathcal{F} \xrightarrow{rc} x$. Let V be an open set in Y containing $f(x)$. Since f is θs-continuous at x, then there exists $H \in \mathcal{C}_r^\tau(x)$ such that $f(H) \subseteq V$. But since $\mathcal{F} \xrightarrow{rc} x$ and $H \in \mathcal{C}_r^\tau(x)$, then $H \in \mathcal{F}$, and hence $V \in f(\mathcal{F})$. Therefore, $f(\mathcal{F}) \longrightarrow f(x)$. □

Theorem 5.4.7. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $\mathcal{F}$ be a filter on X. If f is θs-continuous at $x \in X$, then $f(\mathcal{F}) \propto f(x)$ in Y whenever $\mathcal{F} \overset{rc}{\propto} x$.

Proof. Let $\mathcal{F}$ be a filter on X such that $\mathcal{F} \overset{rc}{\propto} x$. Let $V \in \sigma(f(x))$. Since f is θs-continuous at x, then there exists $H \in \mathcal{C}_r^\tau(x)$ such that $f(H) \subseteq V$. But $\mathcal{F} \overset{rc}{\propto} x$, then $F \cap H \neq \emptyset$ for all $F \in \mathcal{F}$. So, $\emptyset \neq f(F \cap H) \subseteq f(F) \cap f(H) \subseteq f(F) \cap V$ for all $F \in \mathcal{F}$. Thus, V meets each member of $f(\mathcal{F})$. Hence, $f(\mathcal{F}) \propto f(x)$ in Y. □

5.4.3 S-Continuous Functions

Definition 5.4.3. [83] A function $f : (X, \tau) \to (Y, \sigma)$ is *W-almost-open* if for each $V \in \sigma$, $f^{-1}(\overline{V}) \subseteq \overline{f^{-1}(V)}$.

Proposition 5.4.2. [48, 118] Every open function is W-almost-open.

Theorem 5.4.8. [81] If $f : (X, \tau) \to (Y, \sigma)$ is weakly-θ-continuous, then $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$ for each $V \in \sigma$.

Proof. Let $V \in \sigma$ and suppose, by the contradiction, that $\overline{f^{-1}(V)} \nsubseteq f^{-1}(\overline{V})$, then there exists $x \in \overline{f^{-1}(V)}$ such that $x \notin f^{-1}(\overline{V})$. Then $f(x) \notin \overline{V}$, so there exists $W \in \sigma(f(x))$ such that $W \cap V = \emptyset$. Since V is open in Y, then by Theorem 1.2.2, $\overline{W} \cap V \subseteq \overline{W \cap V}$. So, $\overline{W} \cap V = \emptyset$. But f is weakly-θ-continuous and $W \in \sigma(f(x))$, then there exists $U \in \tau(x)$ such that $f(U) \subseteq \overline{W}$. Since $\overline{W} \cap V = \emptyset$, then $f(U) \cap V = \emptyset$. But $x \in \overline{f^{-1}(V)}$ and $U \in \tau(x)$, so $f^{-1}(V) \cap U \neq \emptyset$, and hence $f(U) \cap V \neq \emptyset$, which is a contradiction. Therefore, $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$. □

Proposition 5.4.3. [81] If a function $f : (X, \tau) \to (Y, \sigma)$ is W-almost-open weakly-θ-continuous, then $\overline{f^{-1}(V)} = f^{-1}(\overline{V})$ for all $V \in \sigma$.

Proof. Let $V \in \sigma$. Since f is W-almost-open, then $f^{-1}(\overline{V}) \subseteq \overline{f^{-1}(V)}$. Also, since f is weakly-θ-continuous, then by Theorem 5.4.8, then $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$. Therefore, $\overline{f^{-1}(V)} = f^{-1}(\overline{V})$. □

Corollary 5.4.2. *[48] If a function $f : (X, \tau) \to (Y, \sigma)$ is W-almost-open weakly-θ-continuous, then $(f^{-1}(\overline{V}))^\circ = f^{-1}(\overline{V}^\circ)$ for all $V \in \sigma$.*

Proof. Let $V \in \sigma$ and let $W = Y - \overline{V}$. Then $W \in \sigma$. So, by Proposition 5.4.3, we have $\overline{f^{-1}(W)} = f^{-1}(\overline{W})$. Then, $\overline{f^{-1}(Y - \overline{V})} = f^{-1}(\overline{Y - \overline{V}})$, and so $\overline{X - f^{-1}(\overline{V})} = f^{-1}(Y - \overline{V}^\circ)$, it follows that $X - (f^{-1}(\overline{V}))^\circ = X - f^{-1}(\overline{V}^\circ)$. Therefore, $(f^{-1}(\overline{V}))^\circ = f^{-1}(\overline{V}^\circ)$. □

A W-almost-open weakly-θ-continuous function preserves rc-convergence. We first introduce the following lemma.

Lemma 5.4.1. *If $f : (X, \tau) \to (Y, \sigma)$ is W-almost-open weakly-θ-continuous, then $\langle \mathcal{C}_r^\sigma(f(x)) \rangle \subseteq f(\langle \mathcal{C}_r^\tau(x) \rangle)$ for any $x \in X$.*

Proof. Let $x \in X$ and $f(x) \in \overline{V}$ for some $V \in \sigma$, then $x \in f^{-1}\left(\overline{V}\right)$. By Corollary 5.4.2, $f^{-1}\left(\overline{V}^\circ\right) = \left(f^{-1}(\overline{V})\right)^\circ$. Let $G = f^{-1}\left(\overline{V}^\circ\right)$, then $G \in \tau$ and

$G \subseteq f^{-1}\left(\overline{V}\right)$, so $\overline{G} \subseteq \overline{f^{-1}\left(\overline{V}\right)}$ but $f^{-1}\left(\overline{V}\right) = \overline{f^{-1}(V)}$ by Proposition 5.4.3. Hence, $\overline{G} \subseteq \overline{f^{-1}(V)} \ldots (i)$ but since $V \in \sigma$, then $V \subseteq \overline{V}^{\circ}$, so $f^{-1}(V) \subseteq f^{-1}(\overline{V}^{\circ}) = G$. Hence, $\overline{f^{-1}(V)} \subseteq \overline{G} \ldots (ii)$. From (i) and (ii), we obtain $\overline{G} = \overline{f^{-1}(V)}$. So, $\overline{G} = \overline{f^{-1}(V)} = f^{-1}\left(\overline{V}\right)$. Since $x \in f^{-1}\left(\overline{V}\right) = \overline{G}$ and $G \in \tau$, then $\overline{G} \in \mathcal{C}_r^{\tau}(x)$, and so $f^{-1}\left(\overline{V}\right) \in \mathcal{C}_r^{\tau}(x)$. But $\overline{V} \supseteq f(f^{-1}(\overline{V}))$, so $\overline{V} \in f(\langle \mathcal{C}_r^{\tau}(x) \rangle)$. Therefore, $\langle \mathcal{C}_r^{\sigma}(f(x)) \rangle \subseteq f(\langle \mathcal{C}_r^{\tau}(x) \rangle)$. □

Theorem 5.4.9. [48] Let $f : (X, \tau) \to (Y, \sigma)$ be W-almost-open weakly-θ-continuous at $x \in X$ and $\mathcal{F}$ be a filter on X. If $\mathcal{F} \xrightarrow{rc} x$, then $f(\mathcal{F}) \xrightarrow{rc} f(x)$ in Y.

Proof. Suppose that f is W-almost-open weakly-θ-continuous at $x \in X$ and $\mathcal{F} \xrightarrow{rc} x$ in X. Then $\langle \mathcal{C}_r^{\tau}(x) \rangle \subseteq \mathcal{F}$. So, $f(\langle \mathcal{C}_r^{\tau}(x) \rangle) \subseteq f(\mathcal{F})$. By Lemma 5.4.1, $\langle \mathcal{C}_r^{\sigma}(f(x)) \rangle \subseteq f(\langle \mathcal{C}_r^{\tau}(x) \rangle)$. Hence, $\langle \mathcal{C}_r^{\sigma}(f(x)) \rangle \subseteq f(\mathcal{F})$. Thus, $f(\mathcal{F}) \xrightarrow{rc} f(x)$. □

Theorem 5.4.10. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, $\mathcal{F}$ be a filter on X. If f is W-almost-open weakly-θ-continuous at $x \in X$, then $f(\mathcal{F}) \overset{rc}{\propto} f(x)$ in Y whenever $\mathcal{F} \overset{rc}{\propto} x$.

Proof. Suppose that f is W-almost-open weakly-θ-continuous at $x \in X$ and a filter on X with $\mathcal{F} \overset{rc}{\propto} x$. Then by Lemma 5.4.1, $\langle \mathcal{C}_r^{\sigma}(f(x)) \rangle \subseteq f(\langle \mathcal{C}_r^{\tau}(x) \rangle)$. Now, let $R \in [\mathcal{C}_r^{\sigma}(f(x))]$, then $R \in \langle \mathcal{C}_r^{\sigma}(f(x)) \rangle \subseteq f(\langle \mathcal{C}_r^{\tau}(x) \rangle)$, so there exists $H \in \langle \mathcal{C}_r^{\tau}(x) \rangle$ such that $R \supseteq f(H)$ but $\mathcal{F} \overset{rc}{\propto} x$, then $H \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, so $f(H \cap F) \neq \emptyset$ for all $F \in \mathcal{F}$. Hence, for all $F \in \mathcal{F}$, $\emptyset \neq f(H \cap F) \subseteq f(H) \cap f(F) \subseteq R \cap f(F)$. Thus, $R \cap f(F) \neq \emptyset$ for all $F \in \mathcal{F}$. Therefore, $f(\mathcal{F}) \overset{rc}{\propto} f(x)$ in Y. □

An S-continuous function preserves rc-convergence. We first introduce the following definition.

Definition 5.4.4. [89] A function $f : (X, \tau) \to (Y, \sigma)$ is *S-continuous* at $x \in X$ if for every semi-open set V in Y containing $f(x)$, there exists a semi-open set U in X containing x such that $f(\overline{U}) \subseteq \overline{V}$. Equivalently, for all $G \in \mathcal{C}_r^{\sigma}(f(x))$, there exists $H \in \mathcal{C}_r^{\tau}(x)$ such that $f(H) \subseteq G$. If this condition is satisfied at each $x \in X$, then f is said to be S-continuous on X.

Proposition 5.4.4. If $f:(X,\tau)\to(Y,\sigma)$ is S-continuous at $x\in X$, then for each $R\in[\mathcal{C}_r^\sigma(f(x))]$, there exists $H\in[\mathcal{C}_r^\tau(x)]$ such that $f(H)\subseteq R$.

Proof. Let $R\in[\mathcal{C}_r^\sigma(f(x))]$. Then $R=\bigcap\limits_{i=1}^{n}R_i$, where $R_i\in\mathcal{C}_r^\sigma(f(x))$ for all $i=1,\dots,n$. Since f is S-continuous at x, then for each $i=1,\dots,n$, there exists $H_i\in\mathcal{C}_r^\tau(x)$ such that $f(H_i)\subseteq R_i$. Let $H=\bigcap\limits_{i=1}^{n}H_i$, then $H\in[\mathcal{C}_r^\tau(x)]$ and $f(H)\subseteq\bigcap\limits_{i=1}^{n}f(H_i)\subseteq\bigcap\limits_{i=1}^{n}R_i=R$. □

Theorem 5.4.11. [46] If $f:(X,\tau)\to(Y,\sigma)$ is an almost-continuous almost-open function, then the inverse image of every regular open set in Y is a regular open set in X.

Proof. Suppose that f is an almost-continuous almost-open function. We want to show $f^{-1}(G)=\overline{f^{-1}(G)}^{\circ}$. Since $f^{-1}(G)\in\tau$, then $f^{-1}(G)\subseteq\overline{f^{-1}(G)}^{\circ}$. Next, we show that $\overline{f^{-1}(G)}^{\circ}\subseteq f^{-1}(G)$. Suppose, to the contrary, that there exists $x\in\overline{f^{-1}(G)}^{\circ}$ such that $f(x)\notin G$, then $f(x)\in Y-G=Y-\overline{G}^{\circ}=\overline{Y-\overline{G}}$. Now, $f(x)\in f\Big(\overline{f^{-1}(G)}^{\circ}\Big)$ and $f\Big(\overline{f^{-1}(G)}^{\circ}\Big)$ is open in Y since f is almost-open. As $f(x)\in\overline{Y-\overline{G}}$ and $f\Big(\overline{f^{-1}(G)}^{\circ}\Big)\in\sigma(f(x))$, then $(Y-\overline{G})\cap f\Big(\overline{f^{-1}(G)}^{\circ}\Big)\neq\emptyset$. So, $f\Big(\overline{f^{-1}(G)}^{\circ}\Big)\nsubseteq\overline{G}$. Since f is almost-continuous, then by Proposition 4.4.1, f is weakly-θ-continuous, so by Theorem 5.4.8, $\overline{f^{-1}(G)}\subseteq f^{-1}(\overline{G})$, and hence $f\Big(\overline{f^{-1}(G)}^{\circ}\Big)\subseteq\overline{G}$, which is a contradiction. Therefore, $f^{-1}(G)\in\mathrm{RO}(X)$ for all $G\in\mathrm{RO}(Y)$. □

Theorem 5.4.12. [46] If $f:(X,\tau)\to(Y,\sigma)$ is almost-continuous almost-open, then the inverse image of every regular closed set in Y is a regular closed set in X.

Proof. Let $F\in\mathrm{RC}(Y)$, then $Y-F\in\mathrm{RO}(Y)$. Since f is almost-continuous almost-open, then by Theorem 5.4.11, $f^{-1}(Y-F)=X-f^{-1}(F)\in\mathrm{RO}(X)$, and hence $f^{-1}(F)\in\mathrm{RC}(X)$. □

Theorem 5.4.13. [89] Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then f is S-continuous if and only if for every $F\in\mathrm{RC}(Y)$, $f^{-1}(F)$ is the union of regular closed sets of X.

Corollary 5.4.3. *If f is almost-open almost-continuous, then f is S-continuous.*

Proof. This follows from Theorems 5.4.12 and 5.4.13. $\square$

Theorem 5.4.14. Let $f : (X, \tau) \to (Y, \sigma)$ be S-continuous at $x \in X$ and $\mathcal{F}$ be a filter on X. If $\mathcal{F} \xrightarrow{rc} x$ in X, then $f(\mathcal{F}) \xrightarrow{rc} f(x)$ in Y.

Proof. Assume that $\mathcal{F} \xrightarrow{rc} x$ in X and $G \in \mathcal{C}_r^\sigma(f(x))$. Since f is S-continuous at x, then there exists $H \in \mathcal{C}_r^\tau(x)$ such that $f(H) \subseteq G$. Also, since $\mathcal{F} \xrightarrow{rc} x$ and $H \in \mathcal{C}_r^\tau(x)$, then $H \in \mathcal{F}$. So, $G \in f(\mathcal{F})$. Thus, $f(\mathcal{F}) \xrightarrow{rc} f(x)$. $\square$

Corollary 5.4.4. *[48, Theorem 4.2. p. 316] An almost-open almost-continuous function preserves rc-convergence.*

Proof. This follows from Corollary 5.4.3 and Theorem 5.4.14. $\square$

Theorem 5.4.15. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $\mathcal{F}$ be a filter on X. If f is S-continuous at $x \in X$, then $f(\mathcal{F}) \overset{rc}{\propto} f(x)$ in Y whenever $\mathcal{F} \overset{rc}{\propto} x$.

Proof. Suppose that f is S-continuous at $x \in X$. Let $\mathcal{F}$ be a filter on X such that $\mathcal{F} \overset{rc}{\propto} x$. Let $R \in [\mathcal{C}_r^\sigma(f(x))]$. Then by Proposition 5.4.4, there exists $H \in [\mathcal{C}_r^\tau(x)]$ such that $f(H) \subseteq R$. But since $\mathcal{F} \overset{rc}{\propto} x$ and $H \in [\mathcal{C}_r^\tau(x)]$, then $F \cap H \neq \emptyset$ for all $F \in \mathcal{F}$. So, $f(F \cap H) \neq \emptyset$ for all $F \in \mathcal{F}$. Now, $f(F \cap H) \subseteq f(F) \cap f(H) \subseteq f(F) \cap R$ for all $F \in \mathcal{F}$. Thus, $R \cap f(F) \neq \emptyset$ for all $F \in \mathcal{F}$ but $f(\mathcal{F}) = \langle \{f(F) : F \in \mathcal{F}\} \rangle$. Hence, $R \cap G \neq \emptyset$ for all $G \in f(\mathcal{F})$. Therefore, $f(\mathcal{F}) \overset{rc}{\propto} f(x)$. $\square$

5.4.4 More on Functions and rc-Convergence

Theorem 5.4.16. If $f : (X, \tau) \to (Y, \sigma)$ is S-continuous and X is extremally disconnected, then f is rc-continuous.

Proof. Let $x \in X$ and $\mathcal{F}$ be a filter on X such that $\mathcal{F} \longrightarrow x$ but since X is extremally disconnected and Theorem 5.3.2 part (i), then $\mathcal{F} \xrightarrow{rc} x$. Since f is

S-continuous and by Theorem 5.4.14, then $f(\mathcal{F}) \xrightarrow{rc} f(x)$ in Y. Therefore, f is rc-continuous at $x \in X$. Therefore, f is rc-continuous since x was arbitrary. $\square$

Theorem 5.4.17. If $f : (X, \tau) \to (Y, \sigma)$ is rc-continuous and X is regular, then f is S-continuous.

Proof. Let $x \in X$ and $R \in \mathcal{C}_r^\sigma(f(x))$. Since f is rc-continuous, then there exists an open set U in X containing x such that $f(U) \subseteq R$. But X is regular, so there exists $H \in \mathcal{C}(x)$ such that $H \subseteq U$. Since $\mathcal{C}(x) \subseteq \mathcal{C}_r^\tau(x)$, then $H \in \mathcal{C}_r^\tau(x)$. Also, we have $f(H) \subseteq f(U) \subseteq R$. Therefore, f is S-continuous at $x \in X$. Thus, f is S-continuous since x was arbitrary. $\square$

Theorem 5.4.18. If $f : (X, \tau) \to (Y, \sigma)$ is θs-continuous and Y is extremally disconnected, then f is S-continuous.

Proof. Let $x \in X$ and $R \in \mathcal{C}_r^\sigma(f(x))$, then R is open in Y containing $f(x)$ since Y is extremally disconnected. But f is θs-continuous, so there exists $H \in \mathcal{C}_r^\tau(x)$ such that $f(H) \subseteq R$. Therefore, f is S-continuous at x. Thus, f is S-continuous since x was arbitrary. $\square$

Corollary 5.4.5. *Let X and Y be extremally disconnected spaces. If $f : (X, \tau) \to (Y, \sigma)$ is θs-continuous, then f is rc-continuous.*

Proof. If f is θs-continuous, then by Theorem 5.4.18, f is S-continuous since Y is extremally disconnected but then by Theorem 5.4.16 and since f is S-continuous, we have f is rc-continuous. $\square$

Theorem 5.4.19. If $f : (X, \tau) \to (Y, \sigma)$ is S-continuous and Y is regular, then f is θs-continuous.

Proof. Let $x \in X$ and V be an open set in Y containing $f(x)$. Since Y is regular, so there exists $R \in \mathcal{C}(f(x))$ such that $R \subseteq V$. But $\mathcal{C}(f(x)) \subseteq \mathcal{C}_r^\sigma(f(x))$, then $R \in \mathcal{C}_r^\sigma(f(x))$. By S-continuity of f, there exists $H \in \mathcal{C}_r^\tau(x)$ such that $f(H) \subseteq R$. But $R \subseteq V$, so $f(H) \subseteq V$. Hence, there exists $H \in \mathcal{C}_r^\tau(x)$ such that $f(H) \subseteq V$. Therefore, f is θs-continuous at x. Thus, f is θs-continuous since x was arbitrary. $\square$

Corollary 5.4.6. *Let X and Y be regular spaces. If $f : (X, \tau) \to (Y, \sigma)$ is rc-continuous, then f is θs-continuous.*

Proof. This follows from Theorems 5.4.17 and 5.4.19. $\square$

Corollary 5.4.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function, where X and Y are regular extremally disconnected spaces. Then the following are equivalent:*

(i) f is S-continuous.

(ii) f is rc-continuous.

(iii) f is θs-continuous.

Proof.

(i) $\Longrightarrow$ (ii) Follows from Theorem 5.4.16.

(ii) $\Longrightarrow$ (iii) Follows from Corollary 5.4.6.

(iii) $\Longrightarrow$ (i) Follows from Theorem 5.4.18.

$\square$

Theorem 5.4.20. Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of topological spaces, let $\mathcal{F}$ be a filter on $X = \prod\limits_{\alpha \in \Delta} X_\alpha$ and $x \in X$.

(i) If $\mathcal{F} \xrightarrow{rc} x$ in X, then $\pi_\alpha(\mathcal{F}) \xrightarrow{rc} \pi_\alpha(x)$ in X_α for all $\alpha \in \Delta$.

(ii) If $\mathcal{F} \overset{rc}{\propto} x$ in X, then $\pi_\alpha(\mathcal{F}) \overset{rc}{\propto} \pi_\alpha(x)$ in X_α for all $\alpha \in \Delta$.

Proof. (i) Suppose that $\mathcal{F} \xrightarrow{rc} x$ in X. Since π_α is open continuous for all $\alpha \in \Delta$, then by Propositions 5.4.2 and 4.4.1, π_α is W-almost-open weakly-θ-continuous for all $\alpha \in \Delta$, so by Theorem 5.4.9, $\pi_\alpha(\mathcal{F}) \xrightarrow{rc} \pi_\alpha(x)$ for all $\alpha \in \Delta$.

(ii) Assume that $\mathcal{F} \overset{rc}{\propto} x$. Since for all $\alpha \in \Delta$, π_α is open continuous for all $\alpha \in \Delta$, then by Propositions 5.4.2 and 4.4.1, π_α is W-almost-open weakly-θ-continuous. So, by Theorem 5.4.10, $\pi_\alpha(\mathcal{F}) \overset{rc}{\propto} \pi_\alpha(x)$ for all $\alpha \in \Delta$.

$\square$

5.5 S-Closed Spaces

5.5.1 Characterizations of S-closed Spaces

Definition 5.5.1. [111] A topological space (X, τ) is called *S-closed* if every semi-open cover $\mathcal{U}$ of X contains a finite subfamily $\{U_1, \ldots, U_n\}$ such that $X = \bigcup\limits_{i=1}^{n} \overline{U_i}$.

Definition 5.5.2. [85] A subset A of a topological space X is said to be

(a) an *S-closed subspace* if the space (A, τ_A) is S-closed.

(b) an *S-closed relative* to X if for every cover $\{V_\alpha : \alpha \in \Delta\}$ of A by semi-open sets in X, there exists a finite subset Ω of Δ such that $A \subseteq \bigcup\limits_{\alpha \in \Omega} \overline{V}_\alpha$.

Remark 5.6. *A topological space (X, τ) is said to be S-closed if it is an S-closed relative to itself.*

Theorem 5.5.1. [48] Let (X, τ) be a topological space. Then X is S-closed if and only if every regular closed cover of X has a finite subcover.

Proof. Let $\mathcal{C} \subseteq \mathrm{RC}(X)$ be a cover of X. By Proposition 1.2.4 part (iii), $\mathrm{RC}(X) \subseteq \mathrm{SO}(X)$. So, $\mathcal{C} \subseteq \mathrm{SO}(X)$ is a cover of X. But X is S-closed, then there exist $C_1, \ldots, C_n \in \mathcal{C}$ such that $X = \bigcup\limits_{i=1}^{n} \overline{C}_i$ but since each C_i is regular closed, then each C_i is closed, and hence for all $i = 1, \ldots, n$, $\overline{C}_i = C_i$, then $X = \bigcup\limits_{i=1}^{n} C_i$. Therefore, $\mathcal{C}' = \{C_1, \ldots, C_n\}$ is a finite subcover of $\mathcal{C}$.

Conversely, suppose that every cover $\mathcal{C} \subseteq \mathrm{RC}(X)$ of X has a finite subcover. Let $\mathcal{V} \subseteq \mathrm{SO}(X)$ be a cover of X. Let $\mathcal{C} = \{\overline{V} : V \in \mathcal{V}\}$. Then by Proposition 1.2.7 part (iii), $\mathcal{C} \subseteq \mathrm{RC}(X)$. Since $X = \bigcup\limits_{V \in \mathcal{V}} V \subseteq \bigcup\limits_{V \in \mathcal{V}} \overline{V} \subseteq X$, then $\mathcal{C}$ is a cover of X. By hypothesis, $\mathcal{C}$ has a finite subcover, that is, there exist $V_1, V_2, \ldots, V_n \in \mathcal{V}$ such that $X = \bigcup\limits_{i=1}^{n} \overline{V}_i$. Therefore, X is S-closed. $\square$

Corollary 5.5.1. *Let A be a subset of a topological space X. Then A is S-closed subspace if and only if every cover $\mathcal{C} \subseteq \mathrm{RC}(A)$ of A has a finite subcover.*

Theorem 5.5.2. [18] The property of a topological space being S-closed is a semi-regular property. That is, (X, τ) is S-closed if and only if (X, τ_s) is S-closed.

Proof. We know that $\mathrm{RC}(X, \tau) = \mathrm{RC}(X, \tau_s)$. Now, (X, τ) is S-closed if and only if every cover $\mathcal{C} \subseteq \mathrm{RC}(X, \tau)$ has a finite subcover if and only if every cover $\mathcal{C} \subseteq \mathrm{RC}(X, \tau_s)$ has a finite subcover if and only if (X, τ_s) is S-closed. $\square$

Theorem 5.5.3. Let (X, τ) be a topological space and $A \subseteq X$. Then A is an S-closed relative to X if and only if every cover of A by regular closed sets of X has a finite subcover.

Proof. Let $\mathcal{C} \subseteq \mathrm{RC}(X)$ be a cover of A. Since $\mathrm{RC}(X) \subseteq \mathrm{SO}(X)$, then $\mathcal{C} \subseteq \mathrm{SO}(X)$ is a cover of A. But A is an S-closed relative to X, then $\mathcal{C}$ has a finite subfamily, say $\mathcal{C}' = \{C_1, C_2, \ldots, C_n\}$ such that $A \subseteq \bigcup_{i=1}^{n} \overline{C}_i$ but since each C_i is regular closed (and hence closed), then $\overline{C}_i = C_i$ for all $i = 1, \ldots, n$, so $A \subseteq \bigcup_{i=1}^{n} C_i$. Therefore, $\mathcal{C}'$ is a finite subcover of $\mathcal{C}$.

Conversely, suppose that every cover of A by regular closed sets of X has a finite subcover. Let $\mathcal{V}$ be a cover of A by semi-open sets of X. Then by Proposition 1.2.7 part (iii), $\{\overline{V} : V \in \mathcal{V}\} \subseteq \mathrm{RC}(X)$. Since $A \subseteq \bigcup_{V \in \mathcal{V}} V$ and $V \subseteq \overline{V}$ for all $V \in \mathcal{V}$, then $A \subseteq \bigcup_{V \in \mathcal{V}} \overline{V}$. So, $\{\overline{V} : V \in \mathcal{V}\}$ is a cover of A by regular closed sets of X. By hypothesis, it has a finite subcover, that is, there exist $V_1, V_2, \ldots, V_n \in \mathcal{V}$ such that $A \subseteq \bigcup_{i=1}^{n} \overline{V}_i$. Therefore, A is an S-closed relative to X. $\square$

Remark 5.7. *[85] An S-closed relative to a topological space X is not necessarily an S-closed subspace in X, even if it is closed in X, as the following example shows.*

Example 5.5.1. *[85] Let τ be the co-countable topology on $\mathbb{R}$ and let $\mathbb{N}$ be the set of all natural numbers. Then $\mathbb{N}$ is an S-closed relative to $(\mathbb{R}, \tau)$ and closed in $(\mathbb{R}, \tau)$, but not an S-closed subspace.*

Proof. Let $\mathcal{C}$ be a cover of $\mathbb{N}$ by regular closed sets in $(\mathbb{R}, \tau)$. Note that $\overline{U} = \mathbb{R}$ for each nonempty $U \in \tau$. So, $\mathrm{RC}(\mathbb{R}, \tau) = \{\emptyset, \mathbb{R}\}$. Thus, $\mathcal{C}$ is a finite subcover of itself. Hence, by Theorem 5.5.3, $\mathbb{N}$ is an S-closed relative to $(\mathbb{R}, \tau)$. Next, we show that $\mathbb{N}$ is not an S-closed subspace in $(\mathbb{R}, \tau)$. Note that for each $n \in \mathbb{N}$, $\{n\} = \mathbb{N} \cap U$, where $U = \mathbb{R} - (\mathbb{N} - \{n\}) \in \tau$. Thus, for each $n \in \mathbb{N}$, $\{n\} \in \tau_{\mathbb{N}}$. That is, the subspace topology $(\mathbb{N}, \tau_{\mathbb{N}})$ of $(\mathbb{R}, \tau)$ is the discrete topology on $\mathbb{N}$. As $\mathcal{A} = \{\{n\} : n \in \mathbb{N}\} \subseteq \mathrm{RC}(\mathbb{N}, \tau_{\mathbb{N}})$ is a cover of $\mathbb{N}$ which has no finite subcover. Thus, by Corollary 5.5.1, $\mathbb{N}$ is not an S-closed subspace of $(\mathbb{R}, \tau)$. $\square$

Theorem 5.5.4. [84] An open set G of a topological space X is an S-closed subspace if and only if G is an S-closed relative to X.

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a cover of G by semi-open sets of X. Since G is open in X and $U_\alpha \in \mathrm{SO}(X)$ for each $\alpha \in \Delta$, then by Proposition 1.2.2, $G \cap U_\alpha \in \mathrm{SO}(X)$ for each $\alpha \in \Delta$. Hence, by Proposition 1.2.3, $G \cap U_\alpha$ is semi-open in G for each $\alpha \in \Delta$. Now, since $G \subseteq \bigcup\limits_{\alpha \in \Delta} U_\alpha$, then $G = G \cap \Big(\bigcup\limits_{\alpha \in \Delta} U_\alpha\Big) = \bigcup\limits_{\alpha \in \Delta} (G \cap U_\alpha)$. So, $\{G \cap U_\alpha : \alpha \in \Delta\} \subseteq \mathrm{SO}(G)$ is a cover of G. But G is S-closed, then there exists a finite subset Ω of Δ such that $G = \bigcup\limits_{\alpha \in \Omega} \mathrm{Cl}_G(G \cap U_\alpha)$. Hence, $G \subseteq \bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha$. Therefore, G is an S-closed relative to X.

Conversely, suppose that G is an S-closed relative to X. Let $\{U_\alpha : \alpha \in \Delta\} \subseteq \mathrm{SO}(G)$ be a cover of G. Since G is open in X, then G is semi-open in X. So, by Theorem 1.2.3, U_α is semi-open in X for each $\alpha \in \Delta$. Since G is an S-closed relative to X. Then there exists a finite subset Ω of Δ such that $G \subseteq \bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha$. Thus, $G = G \cap (\bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha) = \bigcup\limits_{\alpha \in \Omega} (G \cap \overline{U}_\alpha) = \bigcup\limits_{\alpha \in \Omega} \mathrm{Cl}_G(U_\alpha)$. Therefore, G is an S-closed subspace of X. $\square$

Theorem 5.5.5. [84] A topological space X is S-closed if and only if every proper regular open set of X is an S-closed subspace.

Proof. Suppose that X is S-closed and let G be a proper regular open set of X. By Theorem 5.5.4, we show that G is an S-closed relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be a cover of G by semi-open sets of X. Since $X - G$ is regular closed, then by Proposition 1.2.4 part (iii), $X - G \in \mathrm{SO}(X)$. Now, $X = (X - G) \cup \bigcup\limits_{\alpha \in \Delta} U_\alpha$. So, $\{X - G\} \cup \{U_\alpha : \alpha \in \Delta\}$ is a semi-open cover of X but X is S-closed, then there exists a finite subset Ω of Δ such that $X = \overline{X - G} \cup \bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha$. So, $X = (X - G) \cup \bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha$ since

$X - G$ is closed in X. Thus, $G = G \cap X = G \cap \Big((X - G) \cup \bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha\Big) = G \cap (\bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha)$. Therefore, we obtain $G \subseteq \bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha$.

Conversely, let $\mathfrak{U} = \{U_\alpha : \alpha \in \Delta\}$ be a semi-open cover of X. Take $U_{\alpha_\circ} \in \mathfrak{U}$ such that $X \neq \overline{U}_{\alpha_\circ}$ and $U_{\alpha_\circ} \neq \emptyset$. Then since $U_{\alpha_\circ} \in \mathrm{SO}(X)$, $\overline{U}_{\alpha_\circ} \in \mathrm{RC}(X)$, and hence $X - \overline{U}_{\alpha_\circ}$ is a proper regular open set of X. Thus, by hypothesis, $X - \overline{U}_{\alpha_\circ}$ is an S-closed subspace in X, and so by Thoerem 5.5.4, $X - \overline{U}_{\alpha_\circ}$ is an S-closed relative to X. Hence, there exists a finite subset Ω of Δ such that $X - \overline{U}_{\alpha_\circ} \subseteq \bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha$. Therefore, we obtain $X = \bigcup\limits_{\alpha \in \Omega \cup \{\alpha_\circ\}} \overline{U}_\alpha$. □

Theorem 5.5.6. [85] Let A be a subset of a topological space X. If A is an S-closed relative to X, then $\overline{A}$ is an S-closed relative to X.

Proof. Let $\mathfrak{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $\overline{A}$ by semi-open sets in X, then $\mathfrak{U}$ is a cover of A. But A is an S-closed relative to X. So, there exists a finite subset Ω of Δ such that $A \subseteq \bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha$. Thus, we have $\overline{A} \subseteq \overline{\bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha} = \bigcup\limits_{\alpha \in \Omega} \overline{\overline{U}}_\alpha = \bigcup\limits_{\alpha \in \Omega} \overline{U}_\alpha$. Therefore, $\overline{A}$ is an S-closed relative to X. □

Proposition 5.5.1. Let F be a subset of a topological space X. If X is S-closed and F is regular closed in X, then F is an S-closed relative to X.

Proof. Let F be regular closed in X, then F° is regular open in X. If $F^\circ = X$, then $F = \overline{F^\circ} = X$ is an S-closed relative to X. If $F^\circ \neq X$, then by Theorems 5.5.4 and 5.5.5, F° is an S-closed relative to X. By Theorem 5.5.6, $\overline{F^\circ}$ is an S-closed relative to X. But since F is regular closed in X, then $\overline{F^\circ} = F$. Therefore, F is an S-closed relative to X. □

We are now ready to make characterizations of S-closed spaces using our main concept, the rc-convergence of filters.

Theorem 5.5.7. [48] For a topological space (X, τ), the following are equivalent:

(i) X is S-closed.

(ii) Every regular closed cover of X has a finite subcover.

(iii) For every family $\mathcal{A}$ of $\mathrm{RO}(X)$ such that $\bigcap\limits_{A\in\mathcal{A}} A = \emptyset$, there exists a finite subfamily $\mathcal{A}^{'}$ of $\mathcal{A}$ such that $\bigcap\limits_{A\in\mathcal{A}'} A = \emptyset$.

(iv) Every filter on X rc-accumulates at some point of X.

(v) Every ultrafilter on X rc-converges to some point of X.

Proof.

(i) $\Longleftrightarrow$ (ii) This follows from Theorem 5.5.1.

(ii) $\Longrightarrow$ (iii) Let $\mathcal{A} \subseteq \mathrm{RO}(X)$ be a family of subsets X such that $\bigcap\limits_{A\in\mathcal{A}} A = \emptyset$. Let $\mathcal{C} = \{X - A : A \in \mathcal{A}\}$. Then $\mathcal{C} \subseteq \mathrm{RC}(X)$. Since $\bigcap\limits_{A\in\mathcal{A}} A = \emptyset$, then $X = X - \emptyset = X - \bigcap\limits_{A\in\mathcal{A}} A = \bigcup\limits_{A\in\mathcal{A}} (X - A)$. Thus, $\mathcal{C} \subseteq \mathrm{RC}(X)$ is a cover of X. But X is S-closed, so there exist $A_1, A_2, \ldots, A_n \in \mathcal{A}$ such that $\bigcup\limits_{i=1}^{n} (X - A_i) = X$, this implies $X - \bigcap\limits_{i=1}^{n} A_i = X$, and hence $\bigcap\limits_{i=1}^{n} A_i = \emptyset$. Therefore, $\mathcal{A}^{'} = \{A_1, \ldots, A_n\} \subseteq \mathcal{A}$ and $\mathcal{A}^{'} = \emptyset$.

(iii) $\Longrightarrow$ (ii) Let $\mathcal{C} \subseteq \mathrm{RC}(X)$ be a cover of X. Let $\mathcal{A} = \{X - C : C \in \mathcal{C}\}$, then $\mathcal{A} \subseteq \mathrm{RO}(X)$ and $\bigcap\limits_{C\in\mathcal{C}} (X - C) = X - \bigcup\limits_{C\in\mathcal{C}} C = X - X = \emptyset$. So, by hypothesis, there exist $C_1, C_2, \ldots, C_n \in \mathcal{C}$ such that $\bigcap\limits_{i=1}^{n} (X - C_i) = \emptyset$. Since $X - \bigcup\limits_{i=1}^{n} C_i = \bigcap\limits_{i=1}^{n} (X - C_i) = \emptyset$, then $\bigcup\limits_{i=1}^{n} C_i = X$. Hence, $\mathcal{C}^{'} = \{C_1, C_2, \ldots, C_n\}$ is a finite subcover of $\mathcal{C}$.

(iv) $\Longrightarrow$ (v) Let $\mathcal{F}$ be an ultrafilter on X. Then by (iv), $\mathcal{F} \overset{rc}{\propto} x$ for some $x \in X$. But since $\mathcal{F}$ is an ultrafilter, then by Theorem 5.1.12, $\mathcal{F} \xrightarrow{rc} x$.

(v) $\Longrightarrow$ (iv) Let $\mathcal{F}$ be a filter on X. Then by Theorem 1.1.3, there exists an ultrafilter $\mathcal{M}$ on X such that $\mathcal{F} \subseteq \mathcal{M}$. By hypothesis, $\mathcal{M} \xrightarrow{rc} x$ for some $x \in X$, so $\mathcal{M} \overset{rc}{\propto} x$ but since $\mathcal{F} \subseteq \mathcal{M}$, then by Theorem 5.1.11, $\mathcal{F} \overset{rc}{\propto} x$.

(iv) $\Longrightarrow$ (i) Suppose that X is not S-closed, then there exists a semi-open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ such that for any finite subset Ω of Δ, $X \neq \bigcup\limits_{\alpha\in\Omega} \overline{U}_\alpha$. Let $\mathcal{B} = \{\bigcap\limits_{\alpha\in\Omega} (X - \overline{U}_\alpha) : \Omega \text{ is a finite subset of } \Delta\}$, then $\mathcal{B}$ is a filter base in X. Let $\mathcal{F} = \langle\mathcal{B}\rangle_X$, then by hypothesis, $\mathcal{F} \overset{rc}{\propto} x \in X$. Since $x \in X = \bigcup\limits_{\alpha\in\Delta} U_\alpha$, then $x \in U_{\alpha_\circ}$ for some $\alpha_\circ \in \Delta$. Since $U_{\alpha_\circ} \in \mathrm{SO}(X)$ and $x \in U_{\alpha_\circ}$, then $U_{\alpha_\circ} \in \mathrm{SO}(x)$, and so $\overline{U}_{\alpha_\circ} \in \mathcal{C}^\tau_r(x) \subseteq [\mathcal{C}^\tau_r(x)]$ but $X - \overline{U}_{\alpha_\circ} \in \mathcal{B} \subseteq \mathcal{F}$ and $\mathcal{F} \overset{rc}{\propto} x$, then $(X - \overline{U}_{\alpha_\circ}) \cap \overline{U}_{\alpha_\circ} \neq \emptyset$, which is a contradiction. Therefore, X is S-closed.

(ii) $\implies$ (v) Suppose on the contrary that there is an ultrafilter $\mathcal{M}$ on X such that $\mathcal{M} \overset{rc}{\not\to} x$ for all $x \in X$. Then for all $x \in X$, there is $R_x \in \mathcal{C}_r^\tau(x)$ such that $R_x \notin \mathcal{M}$. Let $\mathcal{R} = \{R_x : x \in X\}$, then $\mathcal{R}$ is a regular closed cover of X. By (ii), $\mathcal{R}$ has a finite subcover, that is, there exist $x_1, \ldots, x_n \in X$ such that $X = \bigcup\limits_{i=1}^{n} R_{x_i}$. Now, for each $i = 1, \ldots, n$, we have $R_{x_i} \notin \mathcal{M}$ but $\mathcal{M}$ is an ultrafilter on X, then $X - R_{x_i} \in \mathcal{M}$ for all $i = 1, \ldots, n$. Hence, $\bigcap\limits_{i=1}^{n}(X - R_{x_i}) \in \mathcal{M}$. But $\bigcap\limits_{i=1}^{n}(X - R_{x_i}) = X - \bigcup\limits_{i=1}^{n} R_{x_i} = X - X = \emptyset$. So, $\emptyset \in \mathcal{M}$, which is a contradiction. Therefore, every ultrafilter on X rc-converges to some point in X.

□

Theorem 5.5.8. [68] For a subset A of a topological space (X, τ), the following are equivalent:

(i) A is an S-closed relative to X.

(ii) Every cover of A by regular closed sets of X has a finite subcover.

(iii) For every family $\mathcal{C}$ of regular open sets of X such that $\Big(\bigcap\limits_{C \in \mathcal{C}} C\Big) \cap A = \emptyset$, there exists a finite subfamily $\mathcal{C}'$ of $\mathcal{C}$ such that $\Big(\bigcap\limits_{C \in \mathcal{C}'} C\Big) \cap A = \emptyset$.

(iv) Every filter on X which meets A rc-accumulates at some point of A.

(v) Every ultrafilter on X which meets A rc-converges to some point of A.

Proof. Similar to the proof of Theorem 5.5.7. □

Lemma 5.5.1. *Let $\mathcal{M}$ be an ultrafilter on a topological space X and $x \in X$. Then $\mathcal{M} \xrightarrow{s} x$ if and only if $\mathcal{M} \overset{s}{\propto} x$.*

Proof. If $\mathcal{M} \xrightarrow{s} x$, then $\mathcal{M} \xrightarrow{rc} x$, so $\mathcal{M} \overset{rc}{\propto} x$, and hence $\mathcal{M} \overset{s}{\propto} x$.

Conversely, suppose that $\mathcal{M} \overset{s}{\propto} x$. If $\mathcal{M} \overset{s}{\not\to} x$, then there is $R \in \mathcal{C}_r^\tau(x)$ such that $R \notin \mathcal{M}$ but $\mathcal{M}$ is an ultrafilter on X, so $X - R \in \mathcal{M}$. Since $R \in \mathcal{C}_r^\tau(x)$ and $X - R \in \mathcal{M}$, then $\emptyset = R \cap (X - R) \in \mathcal{M}$, which is a contradiction. Therefore, $\mathcal{M} \xrightarrow{s} x$. □

Theorem 5.5.9. For a topological space (X, τ), the following are equivalent:

(i) X is S-closed.

(ii) Every filter on X s-accumulates at some point of X.

(iii) Every ultrafilter on X s-converges to some point of X.

Proof.

(i) $\Longrightarrow$ (ii) Let $\mathfrak{F}$ be a filter on X. Since X is S-closed, then by Theorem 5.5.7, $\mathfrak{F} \overset{rc}{\propto} x$ for some $x \in X$. But by Proposition 5.1.5, $\mathfrak{F} \overset{s}{\propto} x$.

(ii) $\Longrightarrow$ (iii) Let $\mathfrak{F}$ be an ultrafilter on X. Then by (ii), $\mathfrak{F} \overset{s}{\propto} x$ for some in $x \in X$. But $\mathfrak{F}$ is an ultrafilter on X, then by Lemma 5.5.1, $\mathfrak{F} \xrightarrow{s} x$.

(iii) $\Longrightarrow$ (i) Let $\mathfrak{F}$ be an ultrafilter on X. Then by (iii), $\mathfrak{F} \xrightarrow{s} x$ for some $x \in X$. But then by Proposition 5.1.2, $\mathfrak{F} \xrightarrow{rc} x$. Hence, every ultrafilter rc-converges to some point $x \in X$. Therefore, X is S-closed by Theorem 5.5.7.

□

Theorem 5.5.10. [10] A topological space (X, τ) is S-closed if and only if (X, τ_{rc}) is compact.

Proof. (X, τ) is S-closed if and only if every ultrafilter on X rc-converges if and only if every ultrafilter on X τ_{rc}-converges if and only if (X, τ_{rc}) is compact. □

Theorem 5.5.11. [84] Let X be a regular space. If X is S-closed, then X is compact.

Proof. Assume that X is S-closed. Let $\mathfrak{F}$ be an ultrafilter on X, then by Theorem 5.5.7, $\mathfrak{F} \xrightarrow{rc} x$ for some $x \in X$. Since X is regular and by Theorem 5.3.1 part (i), $\mathfrak{F} \longrightarrow x$. Therefore, by Theorem 2.4.2, X is a compact space. □

Theorem 5.5.12. Let X be an extremally disconnected space. If X is compact, then X is S-closed.

Proof. Assume that X is compact. Let $\mathfrak{F}$ be an ultrafilter on X, then by Theorem 2.4.2, $\mathfrak{F} \longrightarrow x$ for some $x \in X$. Since X is extremally disconnected and by Theorem 5.3.2 part (i), $\mathfrak{F} \xrightarrow{rc} x$. Therefore, by Theorem 5.5.7, X is an S-closed space. □

Corollary 5.5.2. *Let X be a regular extremally disconnected space. Then X is S-closed if and only if X is compact.*

Proof. This follows from Theorems 5.5.11 and 5.5.12. □

Theorem 5.5.13. [89] A θ-semiclosed subset A of an S-closed space X is an S-closed relative to X.

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A by regular closed sets of X. So, $A \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$. Since A is θ-semiclosed, then $X - A$ is θ-semiopen. So, for all $x \in X - A$, there exists a regular closed set F_x in X such that $x \in F_x \subseteq X - A$. Hence, $X - A = \bigcup_{x\in X-A} F_x$. So, $X = A\cup(X-A) \subseteq \bigcup_{\alpha\in\Delta} U_\alpha \cup \bigcup_{x\in X-A} F_x$. This implies, $\mathcal{U} \cup \{F_x : x \in X - A\}$ is a regular closed cover of X. But X is S-closed, then there exist $\alpha_1, \ldots, \alpha_n \in \Delta$ and $x_1, \ldots, x_m \in X - A$ such that $X = \bigcup_{i=1}^{n} U_{\alpha_i} \cup \bigcup_{j=1}^{m} F_{x_j}$. Since $X - A = \bigcup_{x\in X-A} F_x$ and $A \cap (X - A) = \emptyset$, then $A \cap F_x = \emptyset$ for all $x \in X - A$, hence $A \cap F_{x_j} = \emptyset$ for all $j = 1, \ldots, m$. Thus,

$$A = A \cap X = \Big(A \cap \bigcup_{i=1}^{n} U_{\alpha_i}\Big) \cup \Big(A \cap \bigcup_{j=1}^{m} F_{x_j}\Big) = \Big(A \cap \bigcup_{i=1}^{n} U_{\alpha_i}\Big) \cup \emptyset = A \cap \bigcup_{i=1}^{n} U_{\alpha_i}.$$

So, $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Therefore, A is an S-closed relative to X. □

5.5.2 S-Closedness and Functions

Theorem 5.5.14. Let $f : (X, \tau) \to (Y, \sigma)$ be an S-continuous function. If $A \subseteq X$ is an S-closed relative to X, then $f(A) \subseteq Y$ is an S-closed relative to Y.

Proof. Let f be S-continuous. Let $A \subseteq X$ be an S-closed relative to X. Let $\mathcal{G}$ be a filter on Y which meets $f(A)$. Then $f^{-1}(\mathcal{G})$ is a filter on X which meets A. But A is an S-closed relative to X. Then by Theorem 5.5.8, $f^{-1}(\mathcal{G}) \overset{rc}{\propto} a$ for some $a \in A$. But f is S-continuous, then Theorem 5.4.15, $ff^{-1}(\mathcal{G}) \overset{rc}{\propto} f(a)$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$. So, by Theorem 5.1.10, $\mathcal{G} \overset{rc}{\propto} f(a)$. Therefore, $f(A)$ is an S-closed relative to Y by Theorem 5.5.8. □

Corollary 5.5.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be an almost-open almost-continuous function. If $A \subseteq X$ is an S-closed relative to X, then $f(A) \subseteq Y$ is an S-closed relative to Y.*

Proof. This follows from Corollary 5.4.3 and Theorem 5.5.14. $\square$

Theorem 5.5.15. Let $f : (X, \tau) \to (Y, \sigma)$ be an rc-continuous function. If $A \subseteq X$ is compact, then $f(A) \subseteq Y$ is an S-closed relative to Y.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be rc-continuous. Let $A \subseteq X$ be compact. Let $\mathcal{G}$ be a filter on Y which meets $f(A)$. Then $f^{-1}(\mathcal{G})$ is a filter on X which meets A. But A is compact in X. Then by Theorem 2.4.3, $f^{-1}(\mathcal{G}) \propto a$ for some $a \in A$. But f is rc-continuous, then by Theorem 5.4.3, $ff^{-1}(\mathcal{G}) \overset{rc}{\propto} f(a)$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$. So, by Theorem 5.1.10, $\mathcal{G} \overset{rc}{\propto} f(a)$. Therefore, $f(A)$ is an S-closed relative to Y by Theorem 5.5.8. $\square$

Theorem 5.5.16. Let $f : (X, \tau) \to (Y, \sigma)$ be a θs-continuous function. If $A \subseteq X$ is an S-closed relative to X, then $f(A) \subseteq Y$ is compact.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be θs-continuous. Let $A \subseteq X$ be an S-closed relative to X. Let $\mathcal{G}$ be a filter on Y which meets $f(A)$. Then $f^{-1}(\mathcal{G})$ is a filter on X which meets A. But A is an S-closed relative to X. Then by Theorem 4.5.3, $f^{-1}(\mathcal{G}) \overset{rc}{\propto} a$ for some $a \in A$. But f is θs-continuous, then by Theorem 5.4.7, $ff^{-1}(\mathcal{G}) \propto f(a)$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$. So, by Theorem 2.1.9, $\mathcal{G} \propto f(a)$. Therefore, $f(A)$ is compact by Theorem 2.4.3. $\square$

Theorem 5.5.17. Let $f : (X, \tau) \to (Y, \sigma)$ be a W-almost-open weakly-θ-continuous function. If $A \subseteq X$ is an S-closed relative to X, then $f(A) \subseteq Y$ is an S-closed relative to Y.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be W-almost-open weakly-θ-continuous. Let $A \subseteq X$ be an S-closed relative to X. Let $\mathcal{G}$ be a filter on Y which meets $f(A)$. Then $f^{-1}(\mathcal{G})$ is a filter on X which meets A. But A is an S-closed relative to X. Then by Theorem 5.5.8, $f^{-1}(\mathcal{G}) \overset{rc}{\propto} a$ for some $a \in A$. But f is W-almost-open weakly-θ-continuous, then by Theorem 5.4.10, $ff^{-1}(\mathcal{G}) \overset{rc}{\propto} f(a)$ but $\mathcal{G} \subseteq ff^{-1}(\mathcal{G})$. So, by

Theorem 5.1.10, $\mathcal{G} \overset{rc}{\propto} f(a)$. Therefore, $f(A)$ is an S-closed relative to Y by Theorem 5.5.8. □

Theorem 5.5.18. [10] A topological space (X,τ) is S-closed if and only if it is an rc-continuous image of a compact space.

Proof. Let (X,τ) be S-closed. Then by Theorem 5.5.10, (X,τ_{rc}) is compact. Consider the identity function $\mathrm{id}_X : (X,\tau_{rc}) \to (X,\tau)$. Then id_X is obviously rc-continuous. Therefore, there exist a compact space and an rc-continuous function such that the S-closed space (X,τ) is the rc-continuous image of a compact space. The converse follows from Theorem 5.5.15. □

Theorem 5.5.19. [84] If $X = \prod_{\alpha\in\Delta} X_\alpha$ is S-closed, then each space X_α, $\alpha \in \Delta$, is S-closed.

Proof. Since π_α is an onto W-almost-open weakly-θ-continuous for all $\alpha \in \Delta$ and X is S-closed, then by Corollary 5.5.3, $\pi_\alpha(X) = X_\alpha$ is S-closed for all $\alpha \in \Delta$. □

5.5.3 rc-Strongly Closed Graphs

Definition 5.5.3. [3] A function $f : (X,\tau) \to (Y,\sigma)$ has an *rc-strongly closed* graph Γ_f if whenever $(x,y) \in X \times Y$ and $(x,y) \notin \Gamma_f$, there exist $U \in \tau(x)$ and $R \in \mathcal{C}_r^\sigma(y)$ such that $(U \times R) \cap \Gamma_f = \emptyset$.

Theorem 5.5.20. Let $f : (X,\tau) \to (Y,\sigma)$ be a function, then f has an rc-strongly closed graph if and only if for each $x \in X$ and each $y \in Y$, with $(x,y) \notin \Gamma_f$, there exist $U \in \tau(x)$ and $R \in \mathcal{C}_r^\sigma(y)$ such that $f(U) \cap R = \emptyset$.

Proof. The straightforward proof follows from Definition 5.5.3 and is omitted. □

The following two examples show that the concepts of closed graph and rc-strongly closed graph are independent of each other.

Example 5.5.2. *In Example 3.5.2, the function f has a closed graph but it doesn't have an rc-strongly closed graph.*

Example 5.5.3. *Consider the topological space (X, τ) given in Example 5.1.1. Define a function $f : (X, \tau) \to (X, \tau)$ by $f(x) = c$ for any $x \in X$. Then f has an rc-strongly closed graph but it doesn't have a closed graph.*

Our first characterization of functions that have rc-strongly closed graph is in terms of rc-convergence.

Theorem 5.5.21. [3] A function $f : (X, \tau) \to (Y, \sigma)$ has an rc-strongly closed graph if and only if for any filter $\mathcal{F}$ on X such that $\mathcal{F} \longrightarrow x \in X$ and $f(\mathcal{F}) \xrightarrow{rc} y \in Y$, then $(x, y) \in \Gamma_f$.

Proof. Suppose, by the way of contradiction, that f has an rc-strongly closed graph. Let $\mathcal{F} \longrightarrow x \in X$ and $f(\mathcal{F}) \xrightarrow{rc} y \in Y$, then we show that $(x, y) \in \Gamma_f$. Suppose on the contrary that $(x, y) \notin \Gamma_f$. Since Γ_f is rc-strongly closed and by Theorem 5.5.20, then there exist $U \in \tau(x)$ and $R \in \mathcal{C}_r^\sigma(y)$ such that $f(U) \cap R = \emptyset$. But since $\mathcal{F} \longrightarrow x$ and $U \in \tau(x)$, then $U \in \mathcal{F}$, and hence $f(U) \in f(\mathcal{F})$. On the other hand, $f(\mathcal{F}) \xrightarrow{rc} y$ and $R \in \mathcal{C}_r^\sigma(y)$, so $R \in f(\mathcal{F})$. Thus, $f(U) \cap R \neq \emptyset$, which is a contradiction. Therefore, $(x, y) \in \Gamma_f$.

Conversely, suppose, by the way of contradiction, that f does not have an rc-strongly closed graph. Then there exists $(x, y) \notin \Gamma_f$ such that for all $U \in \tau(x)$ and all $R \in \mathcal{C}_r^\sigma(y)$, we have $f(U) \cap R \neq \emptyset$. This implies that $U \cap f^{-1}(R) \neq \emptyset$ for all $U \in \tau(x)$ and all $R \in \mathcal{C}_r^\sigma(y)$. Let $\mathcal{F} = \{F \subseteq X : F \supseteq U \cap f^{-1}(R), U \in \tau(x), R \in \mathcal{C}_r^\sigma(y)\}$, then $\mathcal{F}$ is a filter on X. We claim that $\mathcal{F} \longrightarrow x$ and $f(\mathcal{F}) \xrightarrow{rc} y$. First, let $U_\circ \in \tau(x)$. Then $U_\circ \supseteq U_\circ \cap f^{-1}(R)$ for each $R \in \mathcal{C}_r^\sigma(y)$. Hence, $U_\circ \in \mathcal{F}$. Next, let $R_\circ \in \mathcal{C}_r^\sigma(y)$. Then $R_\circ \supseteq f(f^{-1}(R_\circ)) \supseteq f(U \cap f^{-1}(R_\circ))$ for each $U \in \tau(x)$, but $U \cap f^{-1}(R_\circ) \in \mathcal{F}$ for each $U \in \tau(x)$. So, $R_\circ \in f(\mathcal{F})$. Therefore, we have constructed a filter $\mathcal{F} \longrightarrow x$ in X for which $f(\mathcal{F}) \xrightarrow{rc} y$ in Y. By hypothesis, $(x, y) \in \Gamma_f$, which is a contradiction. Therefore, Γ_f is rc-strongly closed. $\square$

Corollary 5.5.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be any function, where Y is a regular extremally disconnected space. Then the following are equivalent:*

(i) f *has an rc-strongly closed graph.*

(ii) *For each filter* $\mathcal{F}$ *on* X*,* $\mathcal{F} \longrightarrow x$ *in* X *and* $f(\mathcal{F}) \xrightarrow{rc} y$ *in* Y *implies* $y = f(x)$*.*

(iii) *For each filter* $\mathcal{F}$ *on* X*,* $\mathcal{F} \longrightarrow x$ *in* X *and* $f(\mathcal{F}) \longrightarrow y$ *in* Y *implies* $y = f(x)$*.*

(iv) f *has a closed graph.*

Proof.

$(i) \implies (ii)$ Follows from Theorem 5.5.21.

$(ii) \implies (iii)$ Follows from Theorem 5.3.2 part (i) and the fact that Y is extremally disconnected.

$(iii) \implies (iv)$ Follows from Theorem 2.4.9.

(iv) $\implies$ (i) Suppose that f has a closed graph. Let $\mathcal{F}$ be a filter on X with $\mathcal{F} \longrightarrow x$ and $f(\mathcal{F}) \xrightarrow{rc} y$. Since Y is regular, then by Theorem 5.3.1 part (i), $f(\mathcal{F}) \longrightarrow y$. But f has a closed graph, so by Theorem 2.4.9, $(x, y) \in \Gamma_f$. Therefore, f has an rc-strongly closed graph by Theorem 5.5.21.

□

The graph of an rc-continuous function need not be rc-strongly closed as it is shown in the next example.

Example 5.5.4. *Consider the identity function* $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$*, where* τ *and* σ *are the usual and left ray topologies on* $\mathbb{R}$*, respectively. Since* $\sigma \subsetneq \tau$*, then* f *is continuous but* $(\mathbb{R}, \sigma)$ *is extremally disconnected, so by Theorem 5.4.5,* f *is rc-continuous. However, the graph* Γ_f *is not rc-strongly closed graph since* $(0, 1) \notin \Gamma_f$ *but for any* $U \in \tau(0)$ *and* $R \in \mathcal{C}_r^{\sigma}(1) = \{\mathbb{R}\}$*, we have* $(0, 0) \in (U \times R) \cap \Gamma_f$*.*

We are now ready to give a sufficient condition on the codomain of an rc-continuous function f to insure that it has an rc-strongly closed graph.

Theorem 5.5.22. [88] Let $f : (X, \tau) \to (Y, \sigma)$ be rc-continuous and (Y, σ) be semi-Urysohn. Then f has an rc-strongly closed graph.

Proof. Suppose that $\mathcal{F}$ is a filter on X with $\mathcal{F} \longrightarrow x \in X$ and $f(\mathcal{F}) \xrightarrow{rc} y \in Y$. Since f is rc-continuous, then by Theorem 5.4.2, $f(\mathcal{F}) \xrightarrow{rc} f(x)$ in Y. But Y is semi-Urysohn implies $f(x) = y$ by Theorem 5.2.2. So, $(x, y) \in \Gamma_f$. Hence, by Theorem 5.5.21, f has an rc-strongly closed graph. $\square$

Note that in example 5.5.4, the topological space $(\mathbb{R}, \sigma)$ is not a semi-Urysohn space. The remaining results of this section relate functions with rc-strongly closed graph to S-closed spaces.

Example 5.5.5. *Consider the identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$, where τ and σ are the usual and discrete topologies on $\mathbb{R}$, respectively. Then f has an rc-strongly closed graph but f is not rc-continuous.*

We are now ready to give a sufficient condition on the codomain of a function f has an rc-strongly closed graph to insure that it is rc-continuous.

Theorem 5.5.23. [3] Let (Y, σ) be an S-closed space. For every topological space (X, τ), each function $f : (X, \tau) \to (Y, \sigma)$ with an rc-strongly closed graph is rc-continuous.

Proof. Let $x \in X$ and $R \in \mathcal{C}_r^\sigma(f(x))$. For each $y \in Y - R$, $y \neq f(x)$, then for each $y \in Y - R$, $(x, y) \notin \Gamma_f$. But f has an rc-strongly closed graph, then by Theorem 5.5.20, there exist $U_y \in \tau(x)$ and $R_y \in \mathcal{C}_r^\sigma(y)$ such that $f(U_y) \cap R_y = \emptyset$. Let $\mathcal{R} = \{R\} \cup \{R_y : y \in Y - R\}$, then $\mathcal{R}$ is a regular closed cover of Y since $Y = R \cup (Y - R) \subseteq R \cup \Big(\bigcup\limits_{y \in Y - R} R_y\Big) \subseteq Y$. But Y is S-closed, then $\mathcal{R}$ has a finite subcover, say $\mathcal{R}' = \{R, R_{y_1}, \ldots, R_{y_n} : y_1, \ldots, y_n \in Y - R\}$. So, $Y = R \cup \bigcup\limits_{i=1}^{n} R_{y_i}$. Let $U = \bigcap\limits_{j=1}^{n} U_{y_j}$. Then $U \in \tau(x)$ and $f(U) = f\Big(\bigcap\limits_{j=1}^{n} U_{y_j}\Big) \subseteq \bigcap\limits_{j=1}^{n} f(U_{y_j}) \subseteq f(U_{y_i})$ for all $i = 1, \ldots, n$. Now, $f\Big(U\Big) \cap \Big(\bigcup\limits_{i=1}^{n} R_{y_i}\Big) = \bigcup\limits_{i=1}^{n} \Big(f(U) \cap R_{y_i}\Big) \subseteq \bigcup\limits_{i=1}^{n} \Big(f(U_{y_i}) \cap R_{y_i}\Big) = \bigcup\limits_{i=1}^{n} \emptyset = \emptyset$. This implies that, $f(U) \subseteq Y - (\bigcup\limits_{i=1}^{n} R_{y_i}) \subseteq R$. Hence, f is rc-continuous at the arbitrary point $x \in X$ and therefore, f is rc-continuous. $\square$

Theorem 5.5.24. Let $\mathcal{F}$ be a filter on a topological space (Y, σ) and $X = Y \cup \{p\}$ with $p \notin Y$. If θ-$s\mathrm{Adh}_\sigma(\mathcal{F}) = \emptyset$, then the prime space (X, τ_p) is Hausdorff.

Proof. Similar to the proof of Theorem 2.4.12. $\square$

Lemma 5.5.2. *Let $X = Z \cup \{p\}$ where Z is a set with $p \notin Z$, (Y, σ) be a topological space and $y \in Y$. Let $g : Z \to (Y, \sigma)$ be a function and $\mathcal{F}$ be a filter on Z. Define a function $\widetilde{g} : (X, \tau_p) \to (Y, \sigma)$ by $\widetilde{g}(z) = g(z)$ for any $z \in Z$ and $\widetilde{g}(p) = y$. Then $g(\mathcal{F}) \xrightarrow{s} y$ in (Y, σ) if and only if $\widetilde{g}$ is rc-continuous on X.*

Proof. Similar to the proof of Lemma 2.4.1. □

Our final result shows that the condition of theorem 5.5.23 characterizes S-closed spaces if the spaces X are chosen from the class $\mathcal{S}$ to obtain the following characterization of S-closed spaces.

Theorem 5.5.25. Let (Y, σ) be a Hausdorff space. Then (Y, σ) is an S-closed space if and only if for any space $(X, \tau) \in \mathcal{S}$, each function $f : (X, \tau) \to (Y, \sigma)$, that has an rc-strongly closed graph, is rc-continuous.

Proof. The first direction follows by Theorem 5.5.23. Conversely, suppose, by the way of contradiction, that Y is not S-closed, then by Theorem 5.5.9, there is a filter $\mathcal{F}$ on Y such that θ-sAdh$_\sigma(\mathcal{F}) = \emptyset$. Let $X = Y \cup \{p\}$ where $p \notin Y$. Consider the topological space (X, τ_p). Then by Theorem 5.5.24, (X, τ_p) is Hausdorff. Also, by Theorems 1.4.2 and 1.4.4, (X, τ_p) is completely normal and fully normal. This implies that $(X, \tau_p) \in \mathcal{S}$. Fix a point $b \in Y$ and define $\widetilde{\mathrm{id}}_Y : (X, \tau_p) \to (Y, \sigma)$ by $\widetilde{\mathrm{id}}_Y(x) = \mathrm{id}_Y(x) = x$ for any $x \in Y$ and $\widetilde{\mathrm{id}}_Y(p) = b$. Let $(x, y) \in X \times Y$ and $(x, y) \notin \Gamma_{\widetilde{\mathrm{id}}_Y}$. Consider the case when $x \neq p$. Since $\widetilde{\mathrm{id}}_Y(x) \neq y$ and (Y, σ) is Hausdorff, then there exists $V_y \in \sigma(y)$ such that $\widetilde{\mathrm{id}}_Y(x) \notin \overline{V}_y$. Hence, $U_x = \{x\} \in \tau_p(x)$, $W_y = \overline{V}_y \in \mathcal{C}^\sigma_r(y)$ and $\widetilde{\mathrm{id}}_Y(U_x) \cap W_y = \widetilde{\mathrm{id}}_Y(\{x\}) \cap \overline{V}_y = \{\widetilde{\mathrm{id}}_Y(x)\} \cap \overline{V}_y = \emptyset$. Consider the case when $x = p$. Then $b = \widetilde{\mathrm{id}}_Y(p) \neq y$. Again, since (Y, σ) is Hausdorff, then there exists $V_y \in \sigma(y)$ such that $b \notin \overline{V}_y$. Moreover, since θ-sAdh$_\sigma(\mathcal{F}) = \emptyset$, then by Theorem 5.1.4, we have $\mathcal{F} \overset{s}{\not\to} y$, so there exist $R_y \in \mathcal{C}^\sigma_r(y)$ and $F \in \mathcal{F}$ such that $F \cap R_y = \emptyset$. Take $W_y = \overline{V_y \cap R_y}$. Then $y \in W_y$, $b \notin W_y$ and $W_y \subseteq R_y$. Since $R_y \in \mathrm{RC}(Y)$, then by Proposition 1.2.4 part (iii), $R_y \in \mathrm{SO}(Y)$. But since $V_y \in \sigma$, then by Theorem 1.2.2, $V_y \cap R_y \in \mathrm{SO}(Y)$. Thus, by Proposition 1.2.7 part (iii), $W_y = \overline{V_y \cap R_y} \in \mathrm{RC}(Y)$. So, $W_y \in \mathcal{C}^\sigma_r(y)$. Hence, $U_x = F \cup \{p\} \in \tau_p(x)$ and $W_y \in \mathcal{C}^\sigma_r(y)$, so by Theorem 5.5.20, $\widetilde{\mathrm{id}}_Y(U_x) \cap F_y = \widetilde{\mathrm{id}}_Y(F \cup \{p\}) \cap W_y = (\mathrm{id}_Y(F) \cup \{b\}) \cap W_y = G \cap W_y \subseteq G \cap R_y = \emptyset$. We have shown, in both cases, that for each $(x, y) \in (X \times Y) - \Gamma_{\widetilde{\mathrm{id}}_Y}$, there exist $U_x \in \tau_p(x)$ and $W_y \in \mathcal{C}^\sigma_r(y)$ such that $\widetilde{\mathrm{id}}_Y(U_x) \cap W_y = \emptyset$. Thus, by Theorem 5.5.20, $\widetilde{\mathrm{id}}_Y$ has an rc-strongly closed graph. By hypothesis, $\widetilde{\mathrm{id}}_Y$ is rc-continuous, so by Lemma 5.5.2, $\mathrm{id}_Y(\mathcal{F}) \xrightarrow{s} b$ implies $\mathcal{F} \xrightarrow{s} b$ in (Y, σ) but by Proposition 5.1.2, we have $\mathcal{F} \xrightarrow{rc} b$ and by Proposition 5.1.6, $\mathcal{F} \overset{rc}{\propto} b$, and so by Proposition 5.1.5, $\mathcal{F} \overset{s}{\propto} b$, hence Theorem 5.1.4, θ-$s\mathrm{Adh}_\sigma(\mathcal{F}) \neq \emptyset$, which is a contradiction. Therefore, (Y, σ) is S-closed. $\square$

Chapter 6

Various Types of Convergence Structures

Since the topological structure on a topological space is determined by the data of the convergence of filters on the space, the convergence structure has been introduced to generalize the topological structure [36].

It is the purpose of this chapter to investigate the relationship among δ, θ, and rc-convergence structures. Also, the relationship among some types of compactness: nearly compact, quasi-H-closed, and S-closed spaces is also investigated. We explore some information about a filter convergence structure and a filter convergence space as q-closure and q-adherence of sets, q-closed sets, a topology induced by a convergence structure q, a q-neighborhood filter, a topological concept, a topological modification of q, a pretopological concept, and a pretopological modification of q. We also obtain some results about the aforesaid concepts and provide basic ideas of a convergence theory, which would enable one to tackle θ, δ, and rc-convergence structures.

6.1 Relationship among δ, θ and rc-Convergence

Theorem 6.1.1. Let $\mathcal{F}$ be a filter on a topological space (X, τ) and $x \in X$.

(i) If $\mathcal{F} \xrightarrow{\delta} x$, then $\mathcal{F} \xrightarrow{\theta} x$.

(ii) If $\mathcal{F} \xrightarrow{rc} x$, then $\mathcal{F} \xrightarrow{\theta} x$.

Proof. (i) Let $U \in \tau(x)$, then $\overline{U}^{\circ} \in \mathcal{F}$ since $\mathcal{F} \xrightarrow{\delta} x$ but $\overline{U}^{\circ} \subseteq \overline{U}$. So, $\overline{U} \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{\theta} x$.

(ii) If $\mathcal{F} \xrightarrow{rc} x$, then $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}$ but $\mathcal{C}(x) \subseteq \mathcal{C}_r(x)$, and so $\langle \mathcal{C}(x) \rangle \subseteq \langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{\theta} x$.

□

Theorem 6.1.2. Let (X, τ) be an almost-regular space, $\mathcal{F}$ be a filter on X and $x \in X$. Then $\mathcal{F} \xrightarrow{\delta} x$ if and only if $\mathcal{F} \xrightarrow{\theta} x$.

Proof. If $\mathcal{F} \xrightarrow{\delta} x$, then $\mathcal{F} \xrightarrow{\theta} x$ by Theorem 6.1.1 part (i). Conversely, let $G \in \mathrm{RO}(x)$, then there exists $H \in \mathrm{RO}(x)$ such that $\overline{H} \subseteq G$. Since $H \in \tau(x)$, then $\overline{H} \in \mathcal{C}(x)$ but since $\mathcal{F} \xrightarrow{\theta} x$, then $\overline{H} \in \mathcal{F}$, and so $G \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{\delta} x$. □

Theorem 6.1.3. Let X be an extremally disconnected space, $\mathcal{F}$ be a filter on X and $x \in X$. Then the following are equivalent:

(i) $\mathcal{F} \xrightarrow{\delta} x$.

(ii) $\mathcal{F} \xrightarrow{\theta} x$.

(iii) $\mathcal{F} \xrightarrow{rc} x$.

Proof.

(i) $\Longrightarrow$ (ii) Follows from Theorem 6.1.1 part (i).

(ii) $\Longrightarrow$ (iii) Since $\mathcal{F} \xrightarrow{\theta} x$, then $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}$ but $\mathcal{C}(x) = \mathcal{C}_r(x)$ since X is extremally disconnected. So $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{rc} x$.

(iii) $\Longrightarrow$ (i) Since $\mathcal{F} \xrightarrow{rc} x$, then $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}$ but $\mathrm{RO}(x) = \mathrm{RC}(x) = \mathcal{C}_r(x)$ since X is extremally disconnected. Hence, $\langle \mathrm{RO}(x) \rangle \subseteq \mathcal{F}$. That is, $\mathcal{U}_s(x) \subseteq \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{\delta} x$.

□

Theorem 6.1.4. [48] For a topological space (X, τ), the following are equivalent:

(i) X is extremally disconnected.

(ii) For each $x \in X$ and each filter $\mathcal{F}$ on X, if $\mathcal{F} \xrightarrow{\delta} x$, then $\mathcal{F} \xrightarrow{rc} x$.

(iii) For each $x \in X$ and each filter $\mathcal{F}$ on X, $\mathcal{F} \xrightarrow{\theta} x$ if and only if $\mathcal{F} \xrightarrow{rc} x$.

(iv) For each $x \in X$ and each filter $\mathcal{F}$ on X, if $\mathcal{F} \longrightarrow x$, then $\mathcal{F} \xrightarrow{rc} x$.

Proof.

(i) $\implies$ (ii) Follows from Theorem 6.1.3.

(ii) $\implies$ (i) Let $U \in \tau$. We show that $\overline{U} \in \tau$. Let $x \in \overline{U}$ be arbitrary. Since $\tau_s \subseteq \tau$, then $\mathcal{U}_s(x) \subseteq \mathcal{U}(x)$, so $\mathcal{U}(x) \xrightarrow{\delta} x$. By hypothesis, $\mathcal{U}(x) \xrightarrow{rc} x$. So, $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{U}(x)$. But since $\overline{U} \in \mathcal{C}_r(x)$, then $\overline{U} \in \mathcal{U}(x)$. Thus, for all $x \in \overline{U}$, $\overline{U} \in \mathcal{U}(x)$, and hence $\overline{U} \in \tau$. Therefore, X is extremally disconnected.

(i) $\implies$ (iii) Follows from Theorem 6.1.3.

(iii) $\implies$ (ii) Let $x \in X$ and $\mathcal{F}$ be any filter on X and $\mathcal{F} \xrightarrow{\delta} x$, then by Theorem 6.1.1 part (i), $\mathcal{F} \xrightarrow{\theta} x$. So, by hypothesis, $\mathcal{F} \xrightarrow{rc} x$.

(iii) $\implies$ (iv) Let $x \in X$ and $\mathcal{F}$ be any filter on X and $\mathcal{F} \longrightarrow x$, then by Proposition 4.1.2 part (ii), $\mathcal{F} \xrightarrow{\theta} x$. So, by hypothesis, $\mathcal{F} \xrightarrow{rc} x$.

(iv) $\implies$ (i) Let $U \in \tau$. We show that $\overline{U} \in \tau$. Let $x \in \overline{U}$ be arbitrary. Since $\mathcal{U}(x) \longrightarrow x$, then by hypothesis, $\mathcal{U}(x) \xrightarrow{rc} x$. So, $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{U}(x)$. But $\overline{U} \in \mathcal{C}_r(x)$, then $\overline{U} \in \mathcal{U}(x)$. Thus, for all $x \in \overline{U}$, $\overline{U} \in \mathcal{U}(x)$. Hence, $\overline{U} \in \tau$. Therefore, X is extremally disconnected.

□

6.2 Relationship between Types of Compactness

Theorem 6.2.1. [7] Let X be a topological space.

(i) If X is nearly compact, then X is quasi-H-closed.

(ii) If X is S-closed, then X is quasi-H-closed.

Proof.

(i) Assume that X is nearly compact. Let $\mathcal{F}$ be an ultrafilter on X, then $\mathcal{F} \xrightarrow{\delta} x$ for some $x \in X$. By Theorem 6.1.1 part (i), $\mathcal{F} \xrightarrow{\theta} x$. Therefore, by Theorem 4.5.2, X is a quasi-H-closed space.

(ii) Assume that X is S-closed. Let $\mathcal{F}$ be an ultrafilter on X, then $\mathcal{F} \xrightarrow{rc} x$ for some $x \in X$. By Theorem 6.1.1 part (ii), $\mathcal{F} \xrightarrow{\theta} x$. Therefore, by Theorem 4.5.2, X is a quasi-H-closed space.

□

Theorem 6.2.2. [19, 48] Let X be an almost-regular space.

(i) X is nearly compact if and only if X is quasi-H-closed.

(ii) If X is S-closed, then X is nearly compact.

Proof.

(i) If X is almost-regular, then by Theorem 6.1.2, a filter $\mathfrak{F} \xrightarrow{\delta} x$ if and only if $\mathfrak{F} \xrightarrow{\theta} x$. Thus, X is nearly compact if and only if every ultrafilter δ-converges if and only if every ultrafilter θ-converges if and only if X is quasi-H-closed.

(ii) If X is S-closed, then by Theorem 6.2.1 part (ii), X is quasi-H-closed. So, by part (i) and since X is almost-regular, X is nearly compact.

□

Theorem 6.2.3. [89] Let X be an extremally disconnected space and $A \subseteq X$. Then the following are equivalent:

(i) A is an N-closed relative to X.

(ii) A is a quasi-H-closed relative to X.

(iii) A is an S-closed relative to X.

Proof. This follows from Theorems 3.5.4, 4.5.3, 5.5.8 and 6.1.3. □

Corollary 6.2.1. *[48] Let X be an extremally disconnected space. Then the following are equivalent:*

(i) X is nearly compact.

(ii) X is quasi-H-closed.

(iii) X is S-closed.

Proof. Follows from Theorem 6.2.3 with $A = X$. □

Theorem 6.2.4. [115] Let X be a regular space. Then for any $A \subseteq X$, $\overline{A} = \delta\text{-Cl}(A) = \theta\text{-Cl}(A)$.

Proof. Let X be a regular space and $A \subseteq X$. Then by Theorem 4.3.2, $\overline{A} = \theta\text{-Cl}(A)$. So, by Proposition 4.1.6, we have $\overline{A} \subseteq \delta\text{-Cl}(A) \subseteq \theta\text{-Cl}(A)$ and since X is regular and by Theorem 4.3.2, then $\theta\text{-Cl}(A) = \overline{A}$. Therefore, $\overline{A} = \delta\text{-Cl}(A) = \theta\text{-Cl}(A)$. $\square$

Theorem 6.2.5. Let X be an extremally disconnected space and $A \subseteq X$. Then $\delta\text{-Cl}(A) = \theta\text{-Cl}(A) = \theta\text{-sCl}(A)$.

Proof. Follows from Definitions 3.1.3, 4.1.4, 5.1.7 and Propositions 1.3.6, 5.1.1. $\square$

Corollary 6.2.2. *[89] Let X be an extremally disconnected space and $A \subseteq X$. Then the following are equivalent:*

(i) A is δ-closed.

(ii) A is θ-closed.

(iii) A is θ-semiclosed.

Proof. Follows from Theorem 6.2.5. $\square$

Lemma 6.2.1. *Let (X, τ) be a topological space. If X is not extremally disconnected, then there is a regular open set G in X such that G is neither closed nor dense in X.*

Proof. Suppose for each $G \in \text{RO}(X)$, either $\overline{G} = G$ or $\overline{G} = X$. Let $U \in \tau$, then $\overline{U}^{\circ} \in \text{RO}(X)$, and so by above $\overline{\overline{U}^{\circ}} = \overline{U}^{\circ}$ or $\overline{\overline{U}^{\circ}} = X$. If $\overline{\overline{U}^{\circ}} = X$, then $X = \overline{\overline{U}^{\circ}} \subseteq \overline{\overline{U}} = \overline{U}$, and hence $\overline{U} = X \in \tau$. If $\overline{\overline{U}^{\circ}} = \overline{U}^{\circ}$, then since U is open, we have $U \subseteq \overline{U}^{\circ}$, and so $\overline{U} \subseteq \overline{\overline{U}^{\circ}} = \overline{U}^{\circ}$. But $\overline{U}^{\circ} \subseteq \overline{U}$. Thus, $\overline{U} = \overline{U}^{\circ} \in \tau$. Hence, for each $U \in \tau$, $\overline{U} \in \tau$. Therefore, X is extremally disconnected. $\square$

Theorem 6.2.6. [48] If (X, τ) is almost-regular and S-closed, then X is extremally disconnected and nearly compact.

Proof. Let X be almost-regular and S-closed. First, we show that X is extremally disconnected. Suppose, by the way of contradiction, that X is not extremally disconnected. Then by Lemma 6.2.1, there exists $G \in \text{RO}(X)$ such that $\overline{G} \neq G$ and $X \neq \overline{G}$. Let $x \in \overline{G} - G$, then there exists a filter $\mathcal{F}$ on X such that $G \in \mathcal{F}$

and $\mathcal{F} \longrightarrow x$. But $G \subseteq \overline{G}$, then $\overline{G} \in \mathcal{F}$, so $\mathcal{F}$ meets $\overline{G}$. Since G is a proper regular open in X and X is S-closed, then by Theorems 5.5.4 and 5.5.5, G is an S-closed relative to X. By Theorem 5.5.6, $\overline{G}$ is an S-closed relative to X, so by Theorem 5.5.8, $\mathcal{F} \overset{rc}{\propto} p$ for some $p \in \overline{G}$. That is, $\mathcal{F}(\cap)[\mathcal{C}_r(p)]$. If $p \notin G$, then $p \in X - G$, but $X - G \in \mathrm{RC}(X)$ since $G \in \mathrm{RO}(X)$. So, $X - G \in \mathcal{C}_r(p)$. But $G \in \mathcal{F}$ and $\mathcal{F}(\cap)[\mathcal{C}_r(p)]$, so $G \cap (X - G) \neq \emptyset$, which is a contradiction. Hence, $p \in G$ but $G \in \mathrm{RO}(X)$, thus $G \in \mathrm{RO}(p)$. But X is almost-regular, then by Theorem 1.3.2, there exists a regular open set H in X such that $p \in H \subseteq \overline{H} \subseteq G$. Since $p \in H \in \tau$, then $\overline{H} \in \mathcal{C}(p)$. As $x \notin G$, then $x \notin \overline{H}$. Thus, $x \in X - \overline{H}$, and hence $X - \overline{H} \in \mathcal{U}(x)$. But $\overline{H} \cap (X - \overline{H}) = \emptyset$, so we have $\mathcal{C}(p) \perp \mathcal{U}(x)$ but $\mathcal{C}(p) \subseteq \mathcal{C}_r(p) \subseteq [\mathcal{C}_r(p)]$ and $\mathcal{U}(x) \subseteq \mathcal{F}$, so by Proposition 1.1.1 part (ii), so $[\mathcal{C}_r(p)] \perp \mathcal{F}$ but this contradicts the fact that $[\mathcal{C}_r(p)](\cap)\mathcal{F}$. Therefore, X is extremally disconnected. Next, since X is almost-regular and S-closed, then by Theorem 6.2.2, X is nearly compact. □

Corollary 6.2.3. *[48] Let X be almost-regular. Then the following are equivalent:*

(i) X is S-closed.

(ii) X is nearly-compact and extremally disconnected.

(iii) X_s is regular, compact and extremally disconnected.

Proof.

(i) $\Longrightarrow$ (ii) Since X is almost-regular and S-closed, then X is nearly compact and extremally disconnected by Theorem 6.2.6.

(ii) $\Longrightarrow$ (i) Follows from Corollary 6.2.1.

(ii) $\Longleftrightarrow$ (iii) This follows from Theorems 1.3.3, 3.5.5 and Proposition 1.3.5.

□

Lemma 6.2.2. *[46] A topological space (X, τ) is weakly-T_2 if and only if for each $x \neq y$ in X, there exist $G \in \mathrm{RO}(X)$ and $F \in \mathrm{RC}(X)$ such that $x \in G$, $y \in F$ and $G \cap F = \emptyset$.*

Proof. Assume that X is weakly-T_2 and $x \neq y$ in X, then $y \neq x$, so by Proposition 1.3.7, there exists a regular closed set F such that $y \in F$ but $x \notin F$. Let $G = X - G$, then G is regular open in X and $x \in G$. Also, $G \cap F = (X - F) \cap F = \emptyset$.

Conversely, let $x \neq y$ in X. Then $y \neq x$, so by hypothesis, there exist $G \in \mathrm{RO}(y)$ and $F \in \mathrm{RC}(x)$ such that $G \cap F = \emptyset$. Since $y \in G$ and $G \cap F = \emptyset$, then $y \notin F$. So, for each $x \neq y$ in X, there exists a regular closed set F such that $x \in F$ but $y \notin F$. Therefore, by Proposition 1.3.7, X is weakly-T_2. □

Lemma 6.2.3. *Let (X, τ) be a topological space. Then X is Hausdorff if and only if for each $x \neq y$ in X, $\mathcal{U}_s(x) \perp \mathcal{C}(y)$.*

Proof. Suppose that X is Hausdorff. Let $x \neq y$ in X, then there exist open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since U is open in X and $x \in U$, then $\overline{U}^{\circ} \in \mathrm{RO}(x) \subseteq \mathcal{U}_s(x)$. Also, since $y \in V \in \tau$, then $\overline{V} \in \mathcal{C}(y)$. Moreover, by Theorem 1.2.2 and since $\overline{U}^{\circ}$ and V are open, $\overline{U}^{\circ} \cap \overline{V} \subseteq \overline{\overline{U}^{\circ} \cap V} \subseteq \overline{\overline{U} \cap V} \subseteq \overline{\overline{U \cap V}} = \overline{U \cap V} = \overline{\emptyset} = \emptyset$. Hence, $\mathcal{U}_s(x) \perp \mathcal{C}(y)$. Conversely, let $x \neq y$ in X. Then $\mathcal{U}_s(x) \perp \mathcal{C}(y)$. So, there exist $G \in \mathcal{U}_s(x)$ and $H \in \mathcal{C}(y)$ such that $G \cap H = \emptyset$. But then there exist open sets U and V in X such that $x \in U \subseteq \overline{U}^{\circ} \subseteq G$ and $y \in V \subseteq H$. Moreover, $U \cap V \subseteq G \cap H = \emptyset$. Thus, X is Hausdorff. □

Theorem 6.2.7. [48] If (X, τ) is weakly-T_2 and S-closed, then X is H-closed and extremally disconnected.

Proof. Let X be weakly-T_2 and S-closed. First, we show that X is extremally disconnected. Suppose on the contrary that X is not extremally disconnected. Then by Lemma 6.2.1, there exists $G \in \mathrm{RO}(X)$ such that $\overline{G} - G \neq \emptyset$ and $X - \overline{G} \neq \emptyset$. Let $x \in \overline{G} - G$. Then there exists a filter $\mathcal{F}$ on X such that $G \in \mathcal{F}$ and $\mathcal{F} \longrightarrow x$, so $\mathcal{F}$ meets $\overline{G}$. But since $\overline{G}$ is an S-closed relative to X, so by Theorem 5.5.8, $\mathcal{F} \overset{rc}{\propto} y$ for some $y \in \overline{G}$. That is, $\mathcal{F}(\cap)[\mathcal{C}_r(y)]$. If $y \in \overline{G} - G$, then $y \in X - G$, but $X - G \in \mathrm{RC}(X)$ since $G \in \mathrm{RO}(X)$. So $X - G \in \mathcal{C}_r(y)$. But $G \in \mathcal{F}$ and $\mathcal{F}(\cap)[\mathcal{C}_r(y)]$, so $G \cap (X - G) \neq \emptyset$, which is a contradiction. Hence, $y \in G$ but G is open in X since $G \in \mathrm{RO}(X)$, thus $G \in \mathcal{U}(y)$. Since $x \notin G$ and $y \in G$, then $x \neq y$, so by Lemma 6.2.2 and since X is weakly-T_2, there exist $G \in \mathrm{RO}(x)$ and $F \in \mathcal{C}_r(y)$ such that $G \cap F = \emptyset$. But $\mathcal{C}_r(y) \subseteq [\mathcal{C}_r(y)]$. So, $F \in [\mathcal{C}_r(y)]$. Thus, $\mathcal{U}_s(x) \perp [\mathcal{C}_r(y)]$. But since $\mathcal{F} \longrightarrow x$, then by Proposition 3.1.2 part (ii), $\mathcal{F} \xrightarrow{\delta} x$, so $\mathcal{U}_s(x) \subseteq \mathcal{F}$, thus by Proposition 1.1.1 part (ii), $\mathcal{F} \perp [\mathcal{C}_r(y)]$, which is a contradiction since $\mathcal{F}(\cap)[\mathcal{C}_r(y)]$. Hence, X is extremally disconnected. Next, since X is weakly T_2, then as above, we have for each $x \neq y$, $\mathcal{U}_s(x) \perp [\mathcal{C}_r(y)]$ but X is extremally disconnected, so $\mathcal{C}_r(y) = \mathcal{C}(y)$. Thus, for each $x \neq y$, $\mathcal{U}_s(x) \perp [\mathcal{C}(y)]$ but $\mathcal{C}(y)$ is a filter base in X, so we have for each $x \neq y$, $\mathcal{U}_s(x) \perp \mathcal{C}(y)$. Thus, by Lemma 6.2.3, X is Hausdorff. Finally, since X is S-closed, then by Theorem 6.2.1 part (ii), X is quasi-H-closed. Thus, X is a Hausdorff quasi-H-closed. Therefore, X is H-closed. □

Corollary 6.2.4. *If X is weakly-T_2 and extremally disconnected, then X is Hausdorff.*

Proof. Obvious, by the proof of Theorem 6.2.7. □

Corollary 6.2.5. *[48] Let X be a weakly-T_2 space.*

(i) If X is nearly-compact and extremally disconnected, then X is S-closed and Hausdorff.

(ii) If X is quasi-H-closed and extremally disconnected, then X is S-closed and Hausdorff.

Proof.

(i) If X is nearly compact and extremally disconnected, then by Corollary 6.2.1, X is S-closed. Also, if X is weakly-T_2 and extremally disconnected, then by Corollary 6.2.4, X is Hausdorff.

(ii) If X is quasi-H-closed and extremally disconnected, then by Corollary 6.2.1, X is S-closed. Also, if X is weakly-T_2 and extremally disconnected, then by Corollary 6.2.4, X is Hausdorff.

□

Corollary 6.2.6. *Let X be weakly-T_2. Then the following are equivalent:*

(i) X is S-closed.

(ii) X is quasi-H-closed and extremally disconnected.

(iii) X_s is compact Hausdorff and extremally disconnected.

Proof.

(i) $\Longrightarrow$ (ii) Since X is weakly-T_2 and S-closed, then by Theorem 6.2.7, X is H-closed and extremally disconnected, and so X is quasi-H-closed and extremally disconnected.

(ii) $\Longrightarrow$ (iii) Since X is weakly-T_2 and extremally disconnected, then by Corollary 6.2.4, X is Hausdorff. So, by Proposition 1.3.3, X_s is Hausdorff. Also, by Proposition 1.3.5, X_s is extremally disconnected. Since X is quasi-H-closed and extremally disconnected, then by Corollary 6.2.1, X is nearly compact, and so X_s is compact.

(iii) $\Longrightarrow$ (i) Assume that X_s is compact and extremally disconnected, then by Theorem 3.5.5 and Proposition 1.3.5, X is nearly compact and extremally disconnected, respectively. So, by Corollary 6.2.1, X is S-closed.

□

6.3 Relationship between Various Functions

In this section we investigate relationships between types of functions.

Theorem 6.3.1. If $f : (X, \tau) \to (Y, \sigma)$ is almost-continuous, then f is weakly-θ-continuous.

Proof. Let $x \in X$ and $\mathcal{F}$ be a filter on X such that $\mathcal{F} \longrightarrow x$. Since f is almost-continuous and by Theorem 3.4.2, $f(\mathcal{F}) \xrightarrow{\delta} f(x)$ in Y, then $f(\mathcal{F}) \xrightarrow{\theta} f(x)$ by Theorem 6.1.1 part (i). Therefore, f is weakly-θ-continuous at $x \in X$ by Theorem 4.4.2. Thus, f is weakly-θ-continuous since x was arbitrary. □

Theorem 6.3.2. If $f : (X, \tau) \to (Y, \sigma)$ is weakly-θ-continuous and Y is almost-regular, then f is almost-continuous.

Proof. Let $x \in X$ and $\mathcal{F}$ be a filter on X such that $\mathcal{F} \longrightarrow x$. Since f is weakly-θ-continuous and by Theorem 4.4.2, then $f(\mathcal{F}) \xrightarrow{\theta} f(x)$ in Y but Y is almost-regular and by Theorem 6.1.2, we have $f(\mathcal{F}) \xrightarrow{\delta} f(x)$. Therefore, f is almost-continuous at $x \in X$. Thus, f is almost-continuous since x was arbitrary. □

Corollary 6.3.1. *Let $f : X \to Y$ be a function and Y be almost-regular. Then f is weakly-θ-continuous if and only if f is almost-continuous.*

Theorem 6.3.3. [8] Let $f : X \to Y$ be a function and X be almost-regular. If f is super-continuous, then f is strongly-θ-continuous.

Theorem 6.3.4. [48] If $f : X \to Y$ is W-almost-open weakly-θ-continuous, then f is almost-continuous.

Proof. Let $x \in X$ and V be an open neighborhood of $f(x)$ in Y. Since f is weakly-θ-continuous, then there exists an open neighborhood U of x such that $f(U) \subseteq \overline{V}$. Then $U \subseteq f^{-1}\left(\overline{V}\right)$, so $U^{\circ} \subset \left(f^{-1}\left(\overline{V}\right)\right)^{\circ}$ but U is open and $\left(f^{-1}\left(\overline{V}\right)\right)^{\circ} = f^{-1}\left(\overline{V}^{\circ}\right)$ by Corollary 5.4.2. So $U \subseteq f^{-1}\left(\overline{V}^{\circ}\right)$. Thus, $f(U) \subseteq ff^{-1}\left(\overline{V}^{\circ}\right) \subseteq \overline{V}^{\circ}$. Therefore, f is almost-continuous. □

Theorem 6.3.5. If $f : (X, \tau) \to (Y, \sigma)$ is rc-continuous, then f is weakly-θ-continuous.

Proof. Let $x \in X$ and $\mathcal{F}$ be a filter on X such that $\mathcal{F} \longrightarrow x$. Since f is rc-continuous and by Theorem 5.4.2, $f(\mathcal{F}) \xrightarrow{rc} f(x)$ in Y, then $f(\mathcal{F}) \xrightarrow{\theta} f(x)$ by Theorem 6.1.1 part (ii). Therefore, f is weakly-θ-continuous at $x \in X$ by Theorem 4.4.2. Thus, f is weakly-θ-continuous since x was arbitrary. □

Theorem 6.3.6. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, where Y is extremally disconnected. Then the following are equivalent:

(i) f is weakly-θ-continuous.

(ii) f is almost-continuous.

(iii) f is rc-continuous.

Proof. This follows from Theorems 6.1.3, 3.4.2, 4.4.2, 5.4.2, Propositions 1.3.6, 5.1.1 part (ii) and the fact that Y is extremally disconnected. □

Theorem 6.3.7. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, where X and Y are extremally disconnected spaces. Then the following are equivalent:

(i) f is θ-continuous.

(ii) f is δ-continuous.

(iii) f is S-continuous.

Proof. This follows from Definitions 3.4.3, 4.4.3, 5.4.4, Propositions 1.3.6, 5.1.1 part (ii) and the fact that X and Y are extremally disconnected. □

Theorem 6.3.8. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, where X is extremally disconnected. Then the following are equivalent:

(i) f is strongly-θ-continuous.

(ii) f is super-continuous.

(iii) f is θs-continuous.

Proof. This follows from Definitions 3.4.2, 4.4.2, 5.4.2, Propositions 1.3.6, 5.1.1 part (ii) and the fact that X is extremally disconnected. □

Theorem 6.3.9. Let $f : (X, \tau) \to (Y, \sigma)$ be a function.

(i) If f has a strongly closed graph, then f has an almost-strongly closed graph.

(ii) If f has a strongly closed graph, then f has an rc-strongly closed graph.

Proof. (i) This follows from Theorems 3.5.15, 4.5.12 and 6.1.1 part (i).

(ii) This follows from Theorems 4.5.12, 5.5.21 and 6.1.1 part (ii).

□

Theorem 6.3.10. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, where Y is almost-regular. Then f has a strongly closed graph if and only if f has an almost-strongly closed graph.

Proof. This follows from Theorems 3.5.15, 4.5.12 and 6.1.2. □

Theorem 6.3.11. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and Y be extremally disconnected. The following are equivalent:

(i) f has a strongly closed graph.

(ii) f has an almost-strongly closed graph.

(iii) f has an rc-strongly closed graph.

Proof. This follows from Theorems 3.5.15, 4.5.12, 5.5.21 and 6.1.3. □

6.4 θ, δ and rc-Convergence Structures

In this section, we introduce the definition of a filter convergence structure and related concepts. Very roughly speaking, a convergence space is a set together with a designated collection of convergent filters. The theory of convergence structures was developed in order to handle non-topological convergences.

Definition 6.4.1. [11, 91] Let X be a set and $\mathbf{F}(X)$ be the set of all filters on X. A mapping $q : X \to \mathcal{P}(\mathbf{F}(X))$ is called a *filter convergence structure* on X and (X, q) a *filter convergence space* if the following hold for all $x \in X$:

(i) $\langle x \rangle \in q(x)$;

(ii) For all filters $\mathcal{F}, \mathcal{G} \in q(x)$, $\mathcal{F} \cap \mathcal{G} \in q(x)$;

(iii) For each $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$, if $\mathcal{F} \in q(x)$ and $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G} \in q(x)$.

Instead of $\mathcal{F} \in q(x)$, we shall usually write $\mathcal{F} \xrightarrow{q} x$.

Definition 6.4.2. [11] Given a set X, the set of all convergence structures on X is denoted by $\mathbf{C}(X)$.

Example 6.4.1 (The natural convergence structure of a topology)**.** *[11] Every topological space (X, τ) yields a convergence space. Let (X, τ) be a topological space and $\mathcal{U}_\tau(x)$ be the neighborhood system at $x \in X$ with respect to τ. Then the convergence structure $c(\tau)$ induced by τ is defined as follows:*

$$\mathcal{F} \in c(\tau)(x) \text{ iff } \mathcal{U}_\tau(x) \subseteq \mathcal{F} \text{ for each } \mathcal{F} \in \mathbf{F}(X).$$

The convergence structure $c(\tau)$ is called the natural convergence structure of τ.

Example 6.4.2 (θ, δ and rc-convergence structures)**.** *Let (X, τ) be a topological space.*

(i) The δ-convergence structure on X: For $x \in X$ and $\mathcal{F} \in \boldsymbol{F}(X)$, $\mathcal{F} \in \delta(x)$ iff $\mathcal{U}_s(x) \subseteq \mathcal{F}$. This follows from Remark 3.2 and Theorem 3.1.7.

(ii) The θ-convergence structure on X: For $x \in X$ and $\mathcal{F} \in \boldsymbol{F}(X)$, $\mathcal{F} \in \theta(x)$ iff $\langle \mathcal{C}(x) \rangle \subseteq \mathcal{F}$. This follows from Remark 4.3 and Theorem 4.1.8.

(iii) The rc-convergence structure on X: For $x \in X$ and $\mathcal{F} \in \boldsymbol{F}(X)$, $\mathcal{F} \in rc(x)$ iff $\langle \mathcal{C}_r(x) \rangle \subseteq \mathcal{F}$. This follows from Remark 5.4 and Theorem 5.1.8.

Remark 6.1. *[11] On $\mathbf{C}(X)$, define a relation $\leq$ on $\mathbf{C}(X)$ by $q_1 \leq q_2 \iff q_2(x) \subseteq q_1(x)$ for all $x \in X$. Then $(\mathbf{C}(X), \leq)$ is a poset.*

Definition 6.4.3. [11] Let q_1 and q_2 be two convergence structures on X. If $q_1 \leq q_2$, then we say that q_1 is *coarser* than q_2 and q_2 is *finer* than q_1.

Proposition 6.4.1. For any topological space (X, τ), we have $\theta \leq \delta \leq c(\tau)$.

Proof. By Theorem 6.1.1 part (i), $\theta \leq \delta$ and by Proposition 3.1.2 part (ii), $\delta \leq c(\tau)$. Thus, $\theta \leq \delta \leq c(\tau)$. □

Definition 6.4.4. [11] Let (X, q) be a convergence space and $A \subseteq X$. Then

$$\mathrm{Cl}_q(A) = \{x \in X : \exists\, \mathcal{F} \in q(x) \text{ such that } A \in \mathcal{F}\} \text{ is the } \textit{q-closure} \text{ of } A.$$

Proposition 6.4.2. For a subset A of a topological space (X, τ), we have

(i) $\mathrm{Cl}_{c(\tau)}(A) = \overline{A}$.

(ii) $\mathrm{Cl}_\delta(A) = \delta\text{-}\mathrm{Cl}(A)$.

(iii) $\mathrm{Cl}_\theta(A) = \theta\text{-}\mathrm{Cl}(A)$.

(iv) $\mathrm{Cl}_{rc}(A) = \mathrm{Cl}_{\tau_{rc}}(A)$ but $\mathrm{Cl}_{rc}(A) \neq \theta\text{-}\mathrm{sCl}(A)$.

Proof. (i) This follows from Theorem 2.1.2.

(ii) By Theorem 3.1.3, $\mathrm{Cl}_\delta(A) = \delta\text{-}\mathrm{Cl}(A)$.

(iii) By Theorem 4.1.4, $\mathrm{Cl}_\theta(A) = \theta\text{-}\mathrm{Cl}(A)$.

(iv) By Theorem 2.1.3 and Proposition 5.1.3, we have $\mathrm{Cl}_{rc}(A) = \mathrm{Cl}_{\tau_{rc}}(A)$. Finally, by Example 5.1.7 and Thereom 5.1.5, $\mathrm{Cl}_{rc}(A) \neq \theta\text{-sCl}(A)$.

$\square$

Proposition 6.4.3. [11] Let (X, q) be a convergence space. Then the following hold:

(i) $\mathrm{Cl}_q(\emptyset) = \emptyset$.

(ii) $A \subseteq \mathrm{Cl}_q(A)$ for all $A \subseteq X$.

(iii) If $A \subseteq B$, then $\mathrm{Cl}_q(A) \subseteq \mathrm{Cl}_q(B)$.

(iv) $\mathrm{Cl}_q(A \cup B) = \mathrm{Cl}_q(A) \cup \mathrm{Cl}_q(B)$ for all $A, B \subseteq X$.

Proof.

(i) If there is an $x \in X$ such that $x \in \mathrm{Cl}_q(\emptyset)$, then $\emptyset \in \mathcal{F}$ for some filter $\mathcal{F} \in q(x)$, which is impossible since $\emptyset \notin \mathcal{F}$. Thus, for all $x \in X$, $x \notin \mathrm{Cl}_q(\emptyset)$. That is, $\mathrm{Cl}_q(\emptyset) = \emptyset$.

(ii) Let $x \in A$, then $A \in \langle x \rangle$ and $\langle x \rangle \in q(x)$. So, $x \in \mathrm{Cl}_q(A)$.

(iii) Let $x \in \mathrm{Cl}_q(A)$, then there exists $\mathcal{F} \in q(x)$ such that $A \in \mathcal{F}$ but $A \subseteq B$. So, $B \in \mathcal{F}$ since $\mathcal{F}$ is a filter on X. Therefore, $x \in \mathrm{Cl}_q(B)$.

(iv) It is easy to check that $\mathrm{Cl}_q(A) \cup \mathrm{Cl}_q(B) \subseteq \mathrm{Cl}_q(A \cup B)$. Next, let $x \in \mathrm{Cl}_q(A \cup B)$. Then there is a filter $\mathcal{F} \in q(x)$ such that $A \cup B \in \mathcal{F}$. If $A \in \mathcal{F}$, then $x \in \mathrm{Cl}_q(A)$. If $A \notin \mathcal{F}$, then for all $F \in \mathcal{F}$, $F \cap (X - A) \neq \emptyset$. Let $\mathcal{G} = \langle \mathcal{F}\big|_{X-A} \rangle$, then by Proposition 1.1.6, $\mathcal{G}$ is a filter on X such that $X - A \in \mathcal{G}$ and $\mathcal{F} \subseteq \mathcal{G}$. Since $\mathcal{F} \in q(x)$, then $\mathcal{G} \in q(x)$. Also, since $A \cup B \in \mathcal{F} \subseteq \mathcal{G}$ and $X - A \in \mathcal{G}$, then $B - A = (A \cup B) \cap (X - A) \in \mathcal{G}$, so $B \in \mathcal{G}$. Hence, we have $\mathcal{G} \in q(x)$ and $B \in \mathcal{G}$. Thus, $x \in \mathrm{Cl}_q(B)$.

$\square$

In general, the q-closure operator is not idempotent as the following example.

Example 6.4.3. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $A = \{a\}$. Consider the convergence structure θ on X. Then we show that $\mathrm{Cl}_\theta(A) = \{a, c\}$ and $\mathrm{Cl}_\theta(\mathrm{Cl}_\theta(A)) = X$. Clearly, $a \in \mathrm{Cl}_\theta(A)$ and since the only open set containing c is X, then $c \in \mathrm{Cl}_\theta(A)$. But $b \notin \mathrm{Cl}_\theta(A)$ since $\{b\}$ is open in X containing b*

and $\overline{\{b\}} \cap A = \emptyset$. Next, clearly, $a, c \in \mathrm{Cl}_\theta\left(\mathrm{Cl}_\theta(A)\right) = \mathrm{Cl}_\theta(\{a, c\})$. Now, we show that $b \in \mathrm{Cl}_\theta(\{a, c\})$. The open sets in X containing b are $\{b\}$, $\{a, b\}$ and X. As $\overline{\{b\}} = \{b, c\}$, $\overline{\{a, b\}} = X$ and $\overline{X} = X$, then $\overline{U} \cap \{a, c\} \neq \emptyset$ for every $U \in \tau(b)$. So, $b \in \mathrm{Cl}_\theta(\{a, c\})$. Hence, $\mathrm{Cl}_\theta(\{a, c\}) = X$. Therefore, $\mathrm{Cl}_\theta(\mathrm{Cl}_\theta(A)) \neq \mathrm{Cl}_\theta(A)$. That is, Cl_θ is not idempotent.

Definition 6.4.5. [11] Let (X, q) be a convergence space and $\mathcal{F} \in \mathbf{F}(X)$. Then

$$a_q(\mathcal{F}) = \{x \in X : \exists\ \mathcal{G} \in q(x) \text{ such that } \mathcal{F} \subseteq \mathcal{G}\}$$

is the set of all *q-adherent* points for $\mathcal{F}$.

Proposition 6.4.4. Let (X, q) be a convergence space. Then $a_q(\mathcal{F}) = \bigcap\limits_{F \in \mathcal{F}} \mathrm{Cl}_q(F)$ for any $\mathcal{F} \in \mathbf{F}(X)$.

Proof. Let $\mathcal{F} \in \mathbf{F}(X)$. Then

$$\begin{aligned} a_q(\mathcal{F}) &= \{x \in X : \exists\ \mathcal{G} \in q(x) \text{ such that } \mathcal{F} \subseteq \mathcal{G}\} \\ &= \{x \in X : \exists\ \mathcal{G} \in q(x) \text{ such that } F \in \mathcal{G},\ \forall\ F \in \mathcal{F}\} \\ &= \{x \in X : x \in \mathrm{Cl}_q(F), \forall\ F \in \mathcal{F}\} \\ &= \bigcap_{F \in \mathcal{F}} \mathrm{Cl}_q(F). \end{aligned}$$

□

Proposition 6.4.5. For a filter $\mathcal{F}$ on a topological space (X, τ), we have

(i) δ-$\mathrm{Adh}(\mathcal{F}) = a_\delta(\mathcal{F})$.

(ii) θ-$\mathrm{Adh}(\mathcal{F}) = a_\theta(\mathcal{F})$.

(iii) $\mathrm{Adh}_{\tau_{rc}}(\mathcal{F}) = a_{rc}(\mathcal{F})$.

Proof. (i) δ-$\mathrm{Adh}(\mathcal{F}) = \bigcap\limits_{F \in \mathcal{F}} \delta$-$\mathrm{Cl}(F)$ but by Propsition 6.4.2 part (ii), δ-$\mathrm{Cl}(F) = \mathrm{Cl}_\delta(F)$ for any $F \in \mathcal{F}$. By Remark 3.1, δ-$\mathrm{Adh}(\mathcal{F}) = a_\delta(\mathcal{F})$.

(ii) θ-$\mathrm{Adh}(\mathcal{F}) = \bigcap\limits_{F \in \mathcal{F}} \theta$-$\mathrm{Cl}(F)$ but by Proposition 6.4.2 part (iii), θ-$\mathrm{Cl}(F) = \mathrm{Cl}_\theta(F)$ for any $F \in \mathcal{F}$. By Remark 4.2, θ-$\mathrm{Adh}(\mathcal{F}) = a_\theta(\mathcal{F})$.

(iii) $\mathrm{Adh}_{\tau_{rc}}(\mathcal{F}) = \bigcap_{F\in\mathcal{F}} \mathrm{Cl}_{\tau_{rc}}(F)$ but by Proposition 6.4.2 part (iv), $\mathrm{Cl}_{\tau_{rc}}(F) = \mathrm{Cl}_{rc}(F)$ for any $F \in \mathcal{F}$. Hence, $\mathrm{Adh}_{\tau_{rc}}(\mathcal{F}) = \bigcap_{F\in\mathcal{F}} \mathrm{Cl}_{rc}(F) = a_{rc}(\mathcal{F})$.

$\square$

Remark 6.2. *For a filter $\mathcal{F}$ on a topological space (X, τ), θ-sAdh$(\mathcal{F}) \neq a_{rc}(\mathcal{F})$. This follows from Example 5.1.8 and Proposition 6.4.5 part (iii).*

Definition 6.4.6. [11] A subset A of a convergence space (X, q) is said to be *q-closed* if $A = \mathrm{Cl}_q(A)$.

Definition 6.4.7. [11] A subset A of a convergence space (X, q) is said to be *q-open* if $X - A$ is q-closed.

Let (X, q) be a convergence space. Then $\tau(q) = \{U \subseteq X : U \text{ is } q\text{-open}\}$ is a topology on X called the *topology induced* by a convergence structure q [11].

Example 6.4.4. *Let (X, τ) be a topological space, then we have $\tau(\theta) = \tau_\theta$, $\tau(\delta) = \tau_\delta$ and $\tau(rc) = \tau_{rc}$.*

Theorem 6.4.1. [11] Let (X, q) be a convergence space and $A \subseteq X$. Then A is q-open if and only if whenever $\mathcal{F} \in \mathbf{F}(X)$ and $\mathcal{F} \xrightarrow{q} x \in A$, then $A \in \mathcal{F}$.

Proof. Assume that A is q-open. Let $\mathcal{F} \in \mathbf{F}(X)$ be such that $\mathcal{F} \xrightarrow{q} x \in A$. Suppose on the contrary that $A \notin \mathcal{F}$. Let $\mathcal{G} = \langle \mathcal{F}\big|_{(X-A)} \rangle$, then by Proposition 1.1.6, $\mathcal{G}$ is a filter on X such that $X - A \in \mathcal{G}$ and $\mathcal{F} \subseteq \mathcal{G}$. Since $\mathcal{F} \xrightarrow{q} x$, then $\mathcal{G} \xrightarrow{q} x$. So, $x \in \mathrm{Cl}_q(X - A) = X - A$ since $X - A$ is q-closed. This implies $x \notin A$, which is a contradiction. Therefore, $A \in \mathcal{F}$.

Conversely, suppose that whenever $\mathcal{F} \xrightarrow{q} x \in A$, we have $A \in \mathcal{F}$. Let $x \in \mathrm{Cl}_q(X - A)$, then there exists $\mathcal{F} \in \mathbf{F}(X)$ such that $\mathcal{F} \xrightarrow{q} x$ and $X - A \in \mathcal{F}$. If $x \in A$, then by hypothesis, $A \in \mathcal{F}$ but then $\emptyset = A \cap (X - A) \in \mathcal{F}$, which is a contradiction. So, $x \notin A$, hence $x \in X - A$. Therefore, $\mathrm{Cl}_q(X - A) = X - A$. So, $X - A$ is q-closed and therefore, A is q-open in X. $\square$

Definition 6.4.8. [11] Let (X, q) be a convergence space. For all $x \in X$, the filter $\mathcal{U}_q(x) = \bigcap\{\mathcal{F} : \mathcal{F} \xrightarrow{q} x\}$ is called the *q-neighborhood filter* of x and its elements the q-neighborhoods of x.

Theorem 6.4.2. [11] Let (X, q) be a convergence space and $U \subseteq X$. Then U is q-open if and only if it is a q-neighborhood of each of its points.

Proof. Let U be q-open and let $x \in U$. If $\mathcal{F} \in \mathbf{F}(X)$ and $\mathcal{F} \xrightarrow{q} x$, then by Theorem 6.4.1, $U \in \mathcal{F}$. Therefore, $U \in \bigcap\limits_{\mathcal{F} \in q(x)} \mathcal{F} = \mathcal{U}_q(x)$. Conversely, suppose that for each $x \in U$, $U \in \mathcal{U}_q(x)$. Suppose that $\mathcal{F} \in \mathbf{F}(X)$ and $\mathcal{F} \xrightarrow{q} x \in U$. Then by hypothesis, $U \in \mathcal{U}_q(x)$ but $\mathcal{U}_q(x) \subseteq \mathcal{F}$. So, $U \in \mathcal{F}$, hence by Theorem 6.4.1, U is q-open. $\square$

Theorem 6.4.3. [11] Let (X, q) be a convergence space, $U \subseteq X$ and $x \in X$. Then U is a q-neighborhood of x if and only if $x \notin \mathrm{Cl}_q(X - U)$.

Proof. If $U \notin \mathcal{U}_q(x)$, then there is a filter $\mathcal{F} \in q(x)$ such that $U \notin \mathcal{F}$. Let $\mathcal{G} = \langle \mathcal{F}\big|_{X-U} \rangle$, then by Proposition 1.1.6, $\mathcal{G}$ is a filter on X such that $X - U \in \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{G}$, so $\mathcal{G} \in q(x)$ and $X - U \in \mathcal{G}$. Consequently, $x \in \mathrm{Cl}_q(X - U)$. Conversely, if $x \in \mathrm{Cl}_q(X - U)$, then there is a filter $\mathcal{F} \in q(x)$ such that $X - U \in \mathcal{F}$. Hence, $U \notin \mathcal{F}$ but $\mathcal{U}_q(x) \subseteq \mathcal{F}$ since $\mathcal{F} \in q(x)$. So, $U \notin \mathcal{U}_q(x)$. $\square$

Theorem 6.4.4. Let (X, q) be a convergence space, $A \subseteq X$ and $x \in X$. Then $x \in \mathrm{Cl}_q(A)$ if and only if $U \cap A \neq \emptyset$ for all $U \in \mathcal{U}_q(x)$.

Proof. Let $x \in \mathrm{Cl}_q(A)$ and $U \in \mathcal{U}_q(x)$. So, there exists $\mathcal{F} \in \mathbf{F}(X)$ such that $A \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{q} x$. Since $\mathcal{F} \xrightarrow{q} x$, then $\mathcal{U}_q(x) \subseteq \mathcal{F}$, and so $U \in \mathcal{F}$. Hence, $U \cap A \in \mathcal{F}$. Thus, $U \cap A \neq \emptyset$.

Conversely, suppose that for all $U \in \mathcal{U}_q(x)$, $U \cap A \neq \emptyset$. We show that $x \in \mathrm{Cl}_q(A)$. Suppose on the contrary that $x \notin \mathrm{Cl}_q(A)$. Then $x \notin \mathrm{Cl}_q(X - (X - A))$. By Theorem 6.4.3, $X - A \in \mathcal{U}_q(x)$, and so by hypothesis, $(X - A) \cap A \neq \emptyset$ which is a contradition. Therefore, $x \in \mathrm{Cl}_q(A)$. $\square$

Proposition 6.4.6. Let X be a set and $q_1, q_2 \in \mathbf{C}(X)$. If $q_1 \leq q_2$. Then

(i) $\mathcal{U}_{q_1}(x) \subseteq \mathcal{U}_{q_2}(x)$ for all $x \in X$.

(ii) $\mathrm{Cl}_{q_2}(A) \subseteq \mathrm{Cl}_{q_1}(A)$ for all $A \subseteq X$.

(iii) $\tau(q_1) \subseteq \tau(q_2)$.

Proof. (i) Let $x \in X$ and $U \in \mathcal{U}_{q_1}(x)$. Let $\mathcal{F}$ be any filter on X such that $\mathcal{F} \xrightarrow{q_2} x$. Then $\mathcal{F} \xrightarrow{q_1} x$ since $q_1 \leq q_2$. But then $\mathcal{U}_{q_1}(x) \subseteq \mathcal{F}$, so $U \in \mathcal{F}$. Thus, $U \in \mathcal{F}$ for any $\mathcal{F} \in q_2(x)$. Hence, $U \in \mathcal{U}_{q_2}(x)$. Therefore, $\mathcal{U}_{q_1}(x) \subseteq \mathcal{U}_{q_2}(x)$.

(ii) Let $x \in \mathrm{Cl}_{q_2}(A)$. Then by Theorem 6.4.4, $U \cap A \neq \emptyset$ for all $U \in \mathcal{U}_{q_2}(x)$. But $\mathcal{U}_{q_1}(x) \subseteq \mathcal{U}_{q_2}(x)$, so $U \cap A \neq \emptyset$ for all $U \in \mathcal{U}_{q_1}(x)$. Again, by Theorem 6.4.4, $x \in \mathrm{Cl}_{q_1}(A)$. Therefore, $\mathrm{Cl}_{q_2}(A) \subseteq \mathrm{Cl}_{q_1}(A)$.

(iii) Let $U \in \tau(q_1)$. Then $U \in \mathcal{U}_{q_1}(x)$ for all $x \in U$ by Theorem 6.4.2. But by part (i), $\mathcal{U}_{q_1}(x) \subseteq \mathcal{U}_{q_2}(x)$ for all $x \in X$. So $U \in \mathcal{U}_{q_2}(x)$ for all $x \in U$. Again, by Theorem 6.4.2, $U \in \tau(q_2)$. Therefore, $\tau(q_1) \subseteq \tau(q_2)$.

□

Definition 6.4.9. [11] A convergence space is called *topological* if the convergence structure is the natural convergence structure of a topology.

Example 6.4.5. *By Propositions 3.1.1 and 5.1.3, δ and rc are topological convergence structures, respectively.*

Definition 6.4.10. [11] Let (X, q) be a convergence space and $\tau(q)$ be the topology induced by q. The natural convergence structure of $\tau(q)$ is called the *topological modification* of q and is denoted by $\lambda(q)$.

Proposition 6.4.7. [11] Let (X, q) be a convergence space. Then

(i) $\lambda(q) \leq q$.

(ii) $\mathcal{U}_{\lambda(q)}(x) \subseteq \mathcal{U}_q(x)$ for all $x \in X$.

(iii) $\mathrm{Cl}_q(A) \subseteq \mathrm{Cl}_{\lambda(q)}(A)$ for all $A \subseteq X$.

(iv) $\tau(\lambda(q)) = \tau(q)$.

Proof.

(i) Let $x \in X$ and assume that $\mathcal{F} \xrightarrow{q} x$. If $A \in \tau(q)$ and $x \in A$, then A is a q-open containing x. So, $A \in \mathcal{U}_q(x)$ but $\mathcal{U}_q(x) \subseteq \mathcal{F}$, so $A \in \mathcal{F}$. Thus, $\mathcal{F} \longrightarrow x$ in $(X, \tau(q))$, that is, $\mathcal{F} \xrightarrow{\lambda(q)} x$. Therefore, $\lambda(q) \leq q$.

(ii) This follows from part (i) and Proposition 6.4.6 part (i).

(iii) This follows from part (i) and Proposition 6.4.6 part (iii).

(iv) By part (i) and Proposition 6.4.6 part (ii), we have $\tau(\lambda(q)) \subseteq \tau(q)$. Now, let $U \in \tau(q)$. Assume $\mathcal{F} \xrightarrow{\lambda(q)} x \in U$, then since $U \in \tau(q)(x)$, we have $U \in \mathcal{F}$. Thus, by Theorem 6.4.1, U is $\lambda(q)$-open, that is, $U \in \tau(\lambda(q))$. Therefore, $\tau(\lambda(q)) = \tau(q)$.

□

Definition 6.4.11. [11] A convergence space (X, q) is called *pretopological* if $\mathcal{U}_q(x)$ q-converges to x for every $x \in X$.

Proposition 6.4.8. [11] Every topological convergence space (X, q) is pretopological.

Proof. Assume that (X, q) is topological, then $q = c(\tau)$ for some topology τ on X. Let $x \in X$, then

$$\begin{aligned}
\mathcal{U}_q(x) &= \bigcap\{\mathcal{F} \subseteq \mathbf{F}(X) : \mathcal{F} \xrightarrow{q} x\} \\
&= \bigcap\{\mathcal{F} \subseteq \mathbf{F}(X) : \mathcal{F} \xrightarrow{c(\tau)} x\} \\
&= \bigcap\{\mathcal{F} \subseteq \mathbf{F}(X) : \mathcal{U}_\tau(x) \subseteq \mathcal{F}\} = \mathcal{U}_\tau(x).
\end{aligned}$$

Since $\mathcal{U}_\tau(x) \xrightarrow{c(\tau)} x$, then $\mathcal{U}_q(x) \xrightarrow{q} x$. Therefore, q is pretopological. □

Example 6.4.6. *The θ-convergence structure is pretopological since $\mathcal{U}_\theta(x) = \langle \mathcal{C}(x) \rangle \xrightarrow{\theta} x$ for each $x \in X$.*

One can associate to each convergence space (X, q) a pretopological convergence space $(X, \pi(q))$ in a natural way. Let (X, q) be a convergence space. We define a new convergence structure $\pi(q)$ on X as follows: $\mathcal{F} \in \pi(q)(x)$ if and only if $\mathcal{U}_q(x) \subseteq \mathcal{F}$. It is easy to show that $\pi(q)$ is a convergence structure and it is pretopological. $\pi(q)$ is called the *pretopological modification* of q [11, 30].

Proposition 6.4.9. [11] Let (X, q) be a convergence space. Then

(i) $\pi(q) \leq q$.

(ii) $\mathcal{U}_{\pi(q)}(x) = \mathcal{U}_q(x)$ for all $x \in X$.

(iii) $\mathrm{Cl}_{\pi(q)}(A) = \mathrm{Cl}_q(A)$ for all $A \subseteq X$.

(iv) $\tau(\pi(q)) = \tau(q)$.

Proof.

(i) Assume $\mathcal{F} \xrightarrow{q} x$. Then $\mathcal{U}_q(x) = \bigcap\{\mathcal{G} : \mathcal{G} \xrightarrow{q} x\} \subseteq \mathcal{F}$, that is, $\mathcal{F} \xrightarrow{\pi(q)} x$. Therefore, $\pi(q) \leq q$.

(ii) $\mathcal{U}_{\pi(q)}(x) = \bigcap\{\mathcal{F} : \mathcal{F} \xrightarrow{\pi(q)} x\} = \bigcap\{\mathcal{F} : \mathcal{U}_q(x) \subseteq \mathcal{F}\} = \mathcal{U}_q(x)$.

(iii) This follows from part (ii) and Theorem 6.4.4.

(iv) This follows from part (ii) and Theorem 6.4.2.

□

Theorem 6.4.5. [50] Let (X, q) be a convergence space. Then

(i) (X, q) is topological if and only if $q = \lambda(q)$.

(ii) (X, q) is pretopological if and only if $q = \pi(q)$.

Proof.

(i) Assume that (X, q) is topological, then $q = c(\tau)$ for some topology τ on X. Clearly, $\mathcal{U}_q(x) = \mathcal{U}_\tau(x)$. This implies $\tau(q) = \tau$. Thus, $q = c(\tau(q)) = \lambda(q)$. The converse is clear.

(ii) Assume that (X, q) is pretopological. Then $\mathcal{U}_q(x) \xrightarrow{q} x$ for any $x \in X$. By Proposition 6.4.9, $\pi(q) \leq q$. Now, let $x \in X$ and $\mathcal{F} \xrightarrow{\pi(q)} x$. Then $\mathcal{U}_q(x) \subseteq \mathcal{F}$. As $\mathcal{U}_q(x) \xrightarrow{q} x$, then $\mathcal{F} \xrightarrow{q} x$. So, $q \leq \pi(q)$. Thus, $q = \pi(q)$. The converse is clear.

□

Proposition 6.4.10. Let τ_1 and τ_2 be two topologies on X. Then $\tau_1 \subseteq \tau_2$ if and only if $c(\tau_1) \leq c(\tau_2)$.

Proof. Let $\mathcal{F} \xrightarrow{c(\tau_2)} x$. We want to show that $\mathcal{F} \xrightarrow{c(\tau_1)} x$. Let $U \in \tau_1$ and $x \in U$, then $U \in \tau_2$ since $\tau_1 \subseteq \tau_2$. Hence, $U \in \tau_2$ and $x \in U$ but since $\mathcal{F} \xrightarrow{c(\tau_2)} x$, then $U \in \mathcal{F}$. Therefore, $\mathcal{F} \xrightarrow{c(\tau_1)} x$.

Conversely, suppose that $c(\tau_1) \leq c(\tau_2)$. Assume that $U \in \tau_1$. Let $\mathcal{F} \xrightarrow{c(\tau_2)} x \in U$, then we show $U \in \mathcal{F}$. Since $c(\tau_2)(x) \subseteq c(\tau_1)(x)$, then $\mathcal{F} \xrightarrow{c(\tau_1)} x \in U$, and hence $U \in \mathcal{F}$. Therefore, $U \in \tau_2$. That is, $\tau_1 \subseteq \tau_2$.

□

Lemma 6.4.1. *Let X be a set and $q_1, q_2 \in \mathbf{C}(X)$. If $q_1 \leq q_2$, then*

(i) $\lambda(q_1) \leq \lambda(q_2)$.

(ii) $\pi(q_1) \leq \pi(q_2)$.

Proof. (i) Since $q_1 \leq q_2$, then by Proposition 6.4.6 part (iii), $\tau(q_1) \subseteq \tau(q_2)$. Thus, by Proposition 6.4.10, $c(\tau(q_1)) \leq c(\tau(q_2))$. Therefore, $\lambda(q_1) \leq \lambda(q_2)$.

(ii) Since $q_1 \leq q_2$, then $\mathcal{U}_{q_1}(x) \subseteq \mathcal{U}_{q_2}(x)$ for all $x \in X$. Let $x \in X$ and let $\mathcal{F} \in \pi(q_2)(x)$, then $\mathcal{U}_{q_2}(x) \subseteq \mathcal{F}$. So, $\mathcal{U}_{q_1}(x) \subseteq \mathcal{F}$. Hence, $\mathcal{F} \in \pi(q_1)(x)$. Thus, $\pi(q_2)(x) \subseteq \pi(q_1)(x)$. Therefore, $\pi(q_1) \leq \pi(q_2)$.

□

Proposition 6.4.11. [11] Let (X, q) be a convergence space. Then

(i) $\lambda(q)$ is the finest topological convergence structure coarser than q.

(ii) $\pi(q)$ is the finest pretopological convergence structure coarser than q.

(iii) $\lambda(q) \leq \pi(q) \leq q$.

Proof.

(i) We know that $\lambda(q)$ is topological and $\lambda(q) \leq q$. Let p be a topological convergence structure on X such that $p \leq q$. Then by Lemma 6.4.1 part (i), $\lambda(p) \leq \lambda(q)$. But $p = \lambda(p)$ since p is topological, so $p \leq \lambda(q)$. Thus, $\lambda(q)$ is the finest topological convergence structure coarser than q.

(ii) We know that $\pi(q)$ is pretopological and $\pi(q) \leq q$. Let p be a pretopological convergence structure on X such that $p \leq q$. Then by Lemma 6.4.1 part (ii),

$\pi(p) \leq \pi(q)$. But $p = \pi(p)$ since p is pretopological, so $p \leq \pi(q)$. Thus, $\pi(q)$ is the finest pretopological convergence structure coarser than q.

(iii) Since every topologcal convergence structure is pretopological, then $\lambda(q)$ is pretopological but $\lambda(q) \leq q$. So, by part (i), $\lambda(q) \leq \pi(q)$ and by Proposition 6.4.9 part (i), $\pi(q) \leq q$. Therefore, $\lambda(q) \leq \pi(q) \leq q$.

□

The following proposition provides a characterization of topological convergence spaces.

Proposition 6.4.12. [11] A convergence space (X, q) is topological if and only if (X, q) is pretopological and the q-closure operator is idempotent.

Proof. Assume that (X, q) is topological, then by Proposition 6.4.8, (X, q) is pretopological. Since $q = \lambda(q)$, then $\mathrm{Cl}_q = \mathrm{Cl}_{\lambda(q)} = \mathrm{Cl}_{c(\tau(q))} = \mathrm{Cl}_{\tau(q)}$, and hence Cl_q is idempotent.

Conversely, suppose that (X, q) is a pretopological convergence space and Cl_q is idempotent, then we show $q = \lambda(q)$. Now, by Proposition 6.4.7 part (i), $\lambda(q) \leq q$. We show $q \leq \lambda(q)$. Suppose that $\mathcal{F} \xrightarrow{\lambda(q)} x$. Then $\mathcal{U}_{\tau(q)}(x) \subseteq \mathcal{F}$. Let $A \in \mathcal{U}_q(x)$, then $x \notin \mathrm{Cl}_q(X - A)$ by Theorem 6.4.3. Since Cl_q is idempotent, then $\mathrm{Cl}_q(X - A)$ is q-closed. Let $U = X - \mathrm{Cl}_q(X - A)$, then U is q-open and $x \in U$. Thus, $U \in \tau(q)(x)$. But since $X - A \subseteq \mathrm{Cl}_q(X - A)$, then $X - \mathrm{Cl}_q(X - A) \subseteq A$, and so $U \subseteq A$. So we have $U \in \tau(q)$ and $x \in U \subseteq A$. Hence, $A \in \mathcal{U}_{\tau(q)}(x)$. So, $\mathcal{U}_q(x) \subseteq \mathcal{U}_{\tau(q)}(x)$. But $\mathcal{U}_{\tau(q)}(x) \subseteq \mathcal{F}$, so $\mathcal{U}_q(x) \subseteq \mathcal{F}$. Hence, $\mathcal{F} \xrightarrow{\pi(q)} x$. But q is pretopological implies $\pi(q) = q$. Thus, $\mathcal{F} \xrightarrow{q} x$. Hence, $q \leq \lambda(q)$. Therefore, $q = \lambda(q)$. That is, (X, q) is topological. □

As the following example shows, the conditions in proposition 6.4.12 are independent.

Example 6.4.7. *Consider the topological space given in Example 6.4.3. Then by Example 6.4.6, (X, θ) is pretopological but the θ-closure operator is not idempotent by Example 6.4.3. Therefore, by Proposition 6.4.12, (X, θ) is not topological.*

Theorem 6.4.6. [50] Let (X, τ) be a topological space. Then θ is a topological convergence structure on X if and only if (X, τ) is almost-regular.

Proof. If (X, τ) is almost-regular, then by Theorem 6.1.2, $\theta = \delta$, and so θ is topological by Example 6.4.5. Conversely, suppose that θ is topological. Then $\theta\text{-Cl} = \mathrm{Cl}_\theta$ is idempotent. Let U be a regular open set in X. Then $F = X - U$ is regular closed in X. So, $F = \overline{V}$ for some open V in X. By Theorem 4.1.1, $\mathrm{Cl}_\theta(V) = \overline{V}$. Thus, $\mathrm{Cl}_\theta(F) = \mathrm{Cl}_\theta(\overline{V}) = \mathrm{Cl}_\theta(\mathrm{Cl}_\theta(V)) = \mathrm{Cl}_\theta(V) = \overline{V} = F$. Hence, F is θ-closed, and so $U = X - F$ is θ-open. Hence, $\tau_s \subseteq \tau_\theta$ but $\tau_\theta \subseteq \tau_s$. So, $\tau_s = \tau_\theta$. By Theorem 4.1.2, (X, τ) is almost-regular. □

Theorem 6.4.7. [48] For a topological space (X, τ), the following are equivalent:

(i) X is extremally disconnected.
(ii) $rc \leq \delta$.
(iii) $rc = \theta$.
(iv) $rc \leq c(\tau)$.

Proof. This follows from Theorem 6.1.4. □

Theorem 6.4.8. Let (X, τ) be a topological space. Then $\delta = c(\tau)$ if and only if (X, τ) is a semi-regular space.

Proof. This follows from Theorem 3.3.4 and Proposition 3.1.2 part (ii). □

Theorem 6.4.9. Let (X, τ) be a topological space. Then $\theta = c(\tau)$ if and only if (X, τ) is a regular space.

Proof. This follows from Theorem 4.3.4 and Proposition 4.1.2 part (ii). □

Theorem 6.4.10. Let (X, τ) be a topological space. Then $rc = c(\tau)$ if and only if (X, τ) is a regular extremally disconnected space.

Proof. If $rc = c(\tau)$, then $rc \leq c(\tau)$. So, by Theorem 6.4.7, X is extremally disconnected and $rc = \theta$. So, $\theta = c(\tau)$. Hence, by Theorem 6.4.9, X is regular. The converse follows from Corollary 5.3.1 part (i). □

Index of Symbols and Notations

Symbols

$\delta\text{-Cl}(A)$ The δ-closure of a set A.

$(X, \leq)$ A partially ordered set.

(X, τ) A topological space.

(X, τ_s) The semi-regularization topological space of (X, τ).

(X, τ_p) A prime space.

(X, q) A convergence space.

$[\mathcal{S}]_X$ The filter base generated by the filter subbase $\mathcal{S}$ with respect to X.

$\delta\text{-Adh}(\mathcal{F})$ The set of all δ-adherent points of $\mathcal{F}$.

$\lambda(q)$ The topological modification of q.

$\langle x \rangle$ The principal filter generated by x.

$\langle \mathcal{B} \rangle_X$ The filter generated by the filter base $\mathcal{B}$ with respect to X.

$\langle \mathcal{S} \rangle_X$ The filter generated by the filter subbase $\mathcal{S}$ with respect to X.

$\mathbf{C}(X)$ The set of all convergence structures on X.

$\mathcal{C}^\tau(x)$ The set of all closures of open neighborhoods of x with respect to τ.

$\mathcal{C}^\tau_r(x)$ The set of all regular closed sets containing x with respect to τ.

$\mathcal{P}(X)$ The power set of X.

$\mathcal{U}_{\tau_s}(x)$ The τ_s-neighborhood filter of x.

$\mathcal{U}_\tau(x)$ The τ-neighborhood filter of x.

$\mathcal{U}_q(x)$ The q-neighborhood filter of x.

$\mathrm{Adh}_\tau(\mathcal{F})$ The set of all adherent points of a filter $\mathcal{F}$.

$\mathrm{Cl}_{\tau_{rc}}(\,\cdot\,)$ The closure operator in the τ_{rc}-topology.

$\mathrm{I}(A)$ The set of all isolated points of A.

$\mathrm{RC}(X, \tau)$ The set of all regular closed sets of (X, τ).

$\mathrm{RC}_\tau(x)$ The set of all regular closed sets containing x with respect to τ.

$\mathrm{RO}(X, \tau)$ The set of all regular open sets of (X, τ).

$\mathrm{RO}_\tau(x)$ The set of all regular open sets containing x with respect to τ.

$\mathrm{SO}(X, \tau)$ The set of all semi-open sets of (X, τ).

$\mathrm{SO}_\tau(x)$ The set of all semi-open sets containing x with respect to τ.

$\mathrm{Cl}_\tau(\,\cdot\,)$ A closure operator with respect to τ.

$\mathrm{Cl}_q(\,\cdot\,)$ A q-closure operator with respect to a convergence space.

$\mathrm{Int}_\tau(\,\cdot\,)$ An interior operator with respect to τ.

$\overline{A}$ The closure of a set A.

$\Phi(X)$ A class of filters on a set X.

$\pi(q)$ The pretopological modification of q.

$\tau(q)$ The topology induced by a convergence structure q.

τ^+ The family of all θ-semiopen sets in (X, τ).

τ_δ The family of all δ-open sets in (X, τ) is a new topology on X.

τ_θ The family of all θ-open sets in (X, τ) is a new topology on X.

$\mathbf{F}(X)$ The set of all filters on X.

θ-$\mathrm{Adh}(\mathcal{F})$ The set of all θ-adherent points of a filter $\mathcal{F}$.

θ-$\mathrm{Cl}(A)$ The θ-closure of a set A.

θ-$\mathrm{sAdh}(\mathcal{F})$ The set of all θ-semi-adherent points of a filter $\mathcal{F}$.

θ-$\mathrm{sCl}(A)$ The θ-semiclosure of a set A.

A'	The set of all cluster points of A.
A°	The interior of a set A.
$a_q(\mathcal{F})$	The set of all q-adherent points for a filter $\mathcal{F}$.
$c(\tau)$	The natural convergence structure.
q	A convergence structure.
$q\text{-}\mathrm{Cl}(\,\cdot\,)$	A q-closure operator with respect to a topological space.

Abbreviations

F.I.P	The finite intersection property.
poset	A partially ordered set.

Index

Bibliography

[1] Handbook of the history of general topology, 1998.

[2] I. T. Adamson. *A general topology workbook*. Springer Science & Business Media, 1995.

[3] B. Al-Nashef. RC-continuous functions and functions with rc-strongly closed graph. *International Journal of Mathematics and Mathematical Sciences*, 2003(72):4547–4555, 2003.

[4] Z. Altawallbeh. More on almost countably compact spaces. *Italian Journal of Pure and Applied Mathematics*, 43:177–184, 2020.

[5] D. Andrijević. Semi-preopen sets. *Matematički Vesnik*, 38(93):24–32, 1986.

[6] C. E. Aull and R. Lowen. *Handbook of the history of general topology*, volume 3. Springer Science & Business Media, 1997.

[7] K. B. Bagchi, A. Mukharjee, and M. Paul. A new approach to nearly compact spaces. *Acta et Commentationes Universitatis Tartuensis de Mathematica*, 22(1):137–147, 2018.

[8] C. Baker. On super continuous functions. *Bulletin of the Korean Mathematical Society*, 22(1):17–22, 1985.

[9] M. Bassam Ahmad. *Topology: An Easy Approach, preprint*.

[10] C. Basu. A class of spaces and their anti spaces. *Tamkang Journal of Mathematics*, 34(4):357–364, 2003.

[11] R. Beattie and H.-P. Butzmann. *Convergence structures and applications to functional analysis*. Springer Science & Business Media, 2013.

[12] C. Berge. *Topological Spaces: including a treatment of multi-valued functions, vector spaces, and convexity*. Courier Corporation, 1997.

[13] G. Birkhoff. A new definition of limit. *Bull. Amer. Math. Soc*, 41(636):1, 1935.

[14] G. Birkhoff. Moore-Smith convergence in general topology. *Annals of Mathematics*, pages 39–56, 1937.

[15] N. Bourbaki. *Les structures fondamentales de l'analyse: Topologie générale. Structures topologiques. Structures uniformes*. Hermann, 1940.

[16] N. Bourbaki. *Topologie générale: Chapitres 1 à 4*, volume 3. Springer Science & Business Media, 2007.

[17] T.-D. Bradley, T. Bryson, and J. Terilla. *Topology: a categorical approach*. MIT Press, 2020.

[18] D. E. Cameron. Properties of $\mathbb{S}$-closed spaces. *Proceedings of the American mathematical society*, 72(3):581–586, 1978.

[19] F. Cammaroto and T. Noiri. On $\mathbb{R}$-compact spaces. *Matematički Vesnik*, 41(106):141–147, 1989.

[20] H. Cartan. Filtres et ultrafiltres. *Comptes Rendus Hebdomadaires des Seances de l'Academie des Sciences, Paris*, 205:777–779, 1937.

[21] H. Cartan. Théorie des filtres. *CR Acad. Sci. Paris*, 205:595–598, 1937.

[22] K. Chandrasekhararao. On almost regular spaces. *Int. Journal of Math. Analysis*, 7(38):702–710, 2013.

[23] K. Chariklia. $\mathbb{S}$-bounded and countably $\mathbb{S}$-bounded sets. *Mathematica japonicae*, 41(2):453–461, 1995.

[24] J. Činčura. Heredity and coreflective subcategories of the category of topological spaces. *Applied categorical structures*, 9(2):131–138, 2001.

[25] S. Crossley. Semi-closure. *Semi-closure, Texas J. Sci*, 22:99–112, 1971.

[26] A. Csaszar. *General topology*. Number 9. Akadémiai Kiadó, 1978.

[27] R. Dickman and R. Krystock. S-sets and S-perfect mappings. *Proceedings of the American Mathematical Society*, 80(4):687–692, 1980.

[28] R. Dickman Jr and J. R. Porter. θ-perfect and θ-absolutely closed functions. *Illinois Journal of Mathematics*, 21(1):42–60, 1977.

[29] K. Dlaska and M. Ganster. S-sets and co-S-closed topologies. *Indian journal of pure and applied mathematics*, 23:731–731, 1992.

[30] S. Dolecki and F. Mynard. *Convergence foundations Of topology*. World Scientific Publishing Company, 2016.

[31] J. Dontchev and M. Ganster. More on semi-Urysohn spaces. *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, 25(2):129–136, 2002.

[32] J. Dontchev and T. Noiri. N-closed subsets of nearly compact spaces. *Acta Mathematica Hungarica*, 86(1-2):117–125, 2000.

[33] J. Dugundji. Topology, series in advanced mathematics. *Boston: Allyn and Bacon, Inc*, 16:447, 1966.

[34] Z. Duszyński. On quasi-H-closed subspaces. *Ricerche di matematica*, 59(2):187–197, 2010.

[35] R. Engelking. Sigma series in pure mathematics. In *General topology*. Heldermann Berlin, 1989.

[36] H. FISCHER. Limesräume. *Mathematische Annalen*, 137:269–303, 1959.

[37] S. Fomin. Extensions of topological spaces, R.(Doklady) Akad. Sci. USSR (N.S.). pages 114–116, 1941.

[38] S. Franklin and M. Rajagopalan. On subsequential spaces. *Topology and its Applications*, 35(1):1–19, 1990.

[39] R. Fuller. Relations among continuous and various non-continuous functions. *Pacific Journal of Mathematics*, 25(3):495–509, 1968.

[40] A. H. Guide. *Infinite dimensional analysis*. Springer, 2006.

[41] A. Gupta and K. Kishore. On functions with strongly closed graph. *International Journal of Pure and Applied Mathematics*, 110(2):383–387, 2016.

[42] P. R. Halmos. *Lectures on Boolean Algebras*. Van Nostrand, 1967.

[43] T. Hamlett. H-closed spaces and the associated θ-convergence space. *Math. Chronicle*, 8:83–88, 1979.

[44] T. R. Hamlett and L. L. Herrington. *The closed graph and* P*-closed graph properties in general topology*, volume 3. American Mathematical Society, 1981.

[45] K. P. Hart, J.-i. Nagata, and J. E. Vaughan. *Encyclopedia of general topology*. Elsevier, 2003.

[46] L. L. Herrington. Some properties preserved by the almost-continuous function. *Boll. Un. Mat. Ital.(4)*, 10:556–568, 1974.

[47] L. L. Herrington and P. E. Long. Characterizations of H-closed spaces. *Proceedings of the American Mathematical Society*, pages 469–475, 1975.

[48] R. A. Herrmann. RC-convergence. *Proceedings of the American Mathematical Society*, 75(2):311–317, 1979.

[49] R. A. Herrmann. Point monads and P-closed spaces. *Notre Dame Journal of Formal Logic*, 20(2):395–400, 1979.

[50] R. A. Herrmann. Preconvergence compactness and P-closed spaces. *International Journal of Mathematics and Mathematical Sciences*, 7(2):303–310, 1984.

[51] J. E. Joseph. Characterizations of nearly-compact spaces. *Boll. Un. Mat. Ital*, 5(13-B):311–321, 1976.

[52] J. E. Joseph. On H-closed spaces. *Proceedings of the American Mathematical Society*, 55(1):223–226, 1976.

[53] J. E. Joseph and M. H. Kwack. On S-closed spaces. *Proceedings of the American Mathematical Society*, 80(2):341–348, 1980.

[54] S. Kasahara. Characterizations of compactness and countable compactness. *Proceedings of the Japan Academy*, 49(7):523–524, 1973.

[55] J. L. Kelley. Convergence in topology. *Duke Mathematical Journal*, 17(3):277–283, 1950.

[56] D. Kent and N. Rath. Filter spaces. *Applied categorical structures*, 1(3):297–309, 1993.

[57] A. B. Khalaf and F. H. Easif. θs-continuous functions. *Dohuk Univ*, pages 1–6, 1999.

[58] J.-R. Kim. Theta topology and its application to the family of all topologies on X. *Journal of the chungcheong mathematical society*, 28(3):431–441, 2015.

[59] C. Konstadilaki. $\mathbb{S}$-bounded and countably $\mathbb{S}$-bounded sets. *Mathematica japonicae*, 41(2):453–461, 1995.

[60] G. Köthe. *Topological vector spaces. translated by djh garling*. Springer-Verlag, 1969.

[61] P. T. Lambrinos. On almost compact and nearly compact spaces. *Rendiconti del Circolo Matematico di Palermo*, 24(1-2):14–18, 1975.

[62] N. Levine. Semi-open sets and semi-continuity in topological spaces. *The American mathematical monthly*, 70(1):36–41, 1963.

[63] P. E. Long. Functions with closed graphs. *The American Mathematical Monthly*, 76(8):930–932, 1969.

[64] P. E. Long. *An introduction to general topology*. Merrill, 1971-01-01.

[65] P. E. Long and L. L. Herrington. Functions with strongly closed graphs. *Boll. Un. Mat. Ital.(4)*, 12(3):381–384, 1975.

[66] P. E. Long and L. L. Herrington. Strongly θ-continuous functions. *Journal of the Korean Mathematical Society*, 18(1):21–28, 1981.

[67] P. E. Long and L. L. Herrington. The T_θ-topology and faintly continuous functions. *Kyungpook Mathematical Journal*, 22(1):7–14, 1982.

[68] G. D. Maio. $\mathbb{S}$-closed spaces, $\mathbb{S}$-sets and $\mathbb{S}$-continuous functions. *Acad. Sci. Torino*, 118:125–134, 1984.

[69] V. Malykhin and A. Bella. Single non-isolated point spaces. 1996.

[70] A. Mathur. Some weaker forms of compactness. *General Topology and Its Relations to Modern Analysis and Algebra*, pages 181–191, 1971.

[71] E. H. Moore. Definition of limit in general integral analysis. *Proceedings of the National Academy of Sciences of the United States of America*, 1(12):628, 1915.

[72] E. H. Moore and H. L. Smith. A general theory of limits. *American Journal of Mathematics*, 44(2):102–121, 1922.

[73] M. Mršević, I. Reilly, and M. Vamanamurthy. On semi-regularization topologies. *Journal of the Australian Mathematical Society*, 38(1):40–54, 1985.

[74] M. Mukherjee and A. Deb Ray. On H-closed spaces and grills. *Analele Ştiinţifice ale Universităţii Al. I. Cuza din Iaşi. Serie Nouă. Matematică*, 44:147–160, 1998.

[75] M. Mukherjee and A. Debray. On S-cluster sets and S-closed spaces. *International Journal of Mathematics and Mathematical Sciences*, 23(9):597–603, 2000.

[76] B. Munshi. Super-continuous mappings. *Indian J. Pure Appl. Math.*, 13:229–236, 1982.

[77] F. Mynard and E. Pearl. *Beyond topology*, volume 486. American Mathematical Society, 2009.

[78] F. Nagel. Set theory and topology. an introduction to the foundations of analysis. part I: Sets, relations, numbers. 2013.

[79] F. Nagel. Set theory and topology. an introduction to the foundations of analysis. part II: Topology-fundamental notions. 2013.

[80] T. Noiri. On semi-continuous mappings. *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, 54(2):210–214, 1973.

[81] T. Noiri. On weakly continuous mappings. *Proceedings of the American Mathematical Society*, 46(1):120–124, 1974.

[82] T. Noiri. Properties of θ-continuous functions. *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, 58(6):887–891, 1975.

[83] T. Noiri. Semi-T_2 spaces. *Ann. Soc. Sci. Bruxelles Ser.I*, 90(2):215–220, 1976.

[84] T. Noiri. On S-closed spaces. *Annales de la Société Scientifique de Bruxelles. Série I*, 91(4):189–194, 1977.

[85] T. Noiri. On S-closed subspaces. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur (8)*, 64:157–162, 1978.

[86] T. Noiri. On δ-continuous functions. *Journal of the Korean Mathematical Society*, 16(2):161–166, 1980.

[87] T. Noiri. Characterizations of S-closed Hausdorff spaces. *Journal of the Australian Mathematical Society*, 51(2):300–304, 1991.

[88] T. Noiri and S. Jafari. Properties of (θ, s)-continuous functions. *Topology and its Applications*, 123(1):167–179, 2002.

[89] T. Noiri and S. On. S-closed and S-perfect functions. *Atti della Acad. Delle Sci. Torino*, pages 71–79, 1986.

[90] C. F. Nourani. *A Functorial Model Theory: Newer Applications to Algebraic Topology, Descriptive Sets, and Computing Categories Topos*. CRC Press, 2014.

[91] M. O'Brien, V. Troitsky, and J. van der Walt. Net convergence structures with applications to vector lattices. *Quaestiones Mathematicae*, pages 1–38, 2022.

[92] W. J. Pervin. *Foundations of general topology*. Academic Press, 2014.

[93] J. Porter and J. Thomas. On H-closed and minimal Hausdorff spaces. *Transactions of the American Mathematical Society*, 138:159–170, 1969.

[94] G. Preuss. Foundations of topology: an approach to convenient topology. 2002.

[95] J. D. Price. Convergence properties of filters and nets. *Thesis, University of North Texas Libraries*, 1968.

[96] I. Reilly and M. Vamanamurthy. On super-continuous mappings. *Indian J. Pure. Appl. Math*, 14(6):767–772, 1983.

[97] J. Reynolds. H-closed spaces and H-sets in the convergence setting. *arXiv preprint arXiv:1510.08044*, 2015.

[98] E. Schechter. *Handbook of Analysis and its Foundations*. Academic Press, 1996.

[99] J. M. Schmidt. Beiträge zur filtertheorie. II. *Mathematische Nachrichten*, 10:197–232, 1952.

[100] M. Ó. Searcóid and M. Searcoid. *Elements of abstract analysis*. Springer, 2002.

[101] J. Sharma. *Krishna's Topology: (For Honours and Post Graduate Students of All Indian Universities)*. Krishna Prakashan, 2014.

[102] M. Singal and S. P. Arya. On almost-regular spaces. *Glasnik Mat*, 4(24):89–99, 1969.

[103] M. Singal and A. Mathur. On nearly compact spaces. *Boll. Un. Mat. Ital*, 4(2):702–710, 1969.

[104] M. Singal and A. Mathur. On nearly compact spaces. II. *Boll. Un. Math. Ital.(4)*, 9:670–678, 1974.

[105] M. Singal and A. R. Singal. On almost M-compact spaces. *Annales de la Société Scientifique de Bruxelles. Série I*, (3):233–242, 1968.

[106] M. Singal, A. R. Singal, et al. Almost-continuous mappings. *Yokohama Mathematical Journal*, 16(2):63–73, 1968.

[107] T. B. Singh. *Introduction to topology*. Springer, 2019.

[108] T. Soundararajan. Weakly Hausdorff spaces and the cardinality of topological spaces. *General topology and its relations to modern analysis and algebra*, pages 301–306, 1971.

[109] L. A. Steen. *Counterexamples in topology*, volume 18. Springer.

[110] M. H. Stone. Applications of the theory of Boolean rings to general topology. *Transactions of the American Mathematical Society*, 41(3):375–481, 1937.

[111] T. Thompson. $\mathbb{S}$-closed spaces. *Proceedings of the American Mathematical Society*, 60(1):335–338, 1976.

[112] T. Thompson. Characterizations of nearly compact spaces. *Kyungpook Mathematical Journal*, 17(1):37–41, 1977.

[113] D. J. T.R. Hamlett and C. Konstandilaki. On some properties weaker than $\mathbb{S}$-closed. *Mathematica japonicae*, 46(2):297–304, 1997.

[114] J. Tukey. Convergence and uniformity in topology. *Ann. Math. Studies*, 1940.

[115] N. Velicko. $\mathbb{H}$-closed topological spaces. *Amer. Math. Soc. Transl.*, 78:103–118, 1968.

[116] L. Vietoris. Stetige mengen. *Monatshefte für Mathematik und Physik*, 31(1):173–204, 1921.

[117] A. Weil. *Sur les espaces à structure uniforme et sur la topologie générale*. Hermann & Cie Editeurs, first edition edition, 1937.

[118] A. Wilansky. *Topics in functional analysis*, volume 45. Springer, 2006.

[119] A. Wilansky. *Topology for analysis*. Courier Corporation, 2008.

[120] S. Willard. *General topology*. Addison-Wesely Publishing Company, 1970.